Generalizations of Mazurkiewicz traces

This Master's Thesis represents the final part of my computer science study in Leiden. I would like to thank my supervisor Jetty Kleijn for her support.

Contents

1 Abstract

To describe the semantics of 1-safe Petri Nets Mazurkiewicz trace theory is sufficient. But for the semantics of more general concurrency systems Mazurkiewicz traces are not enough. In this thesis several generalizations of Mazurkiewicz traces are compared in a framework based on prefixes and partial orders. The so called crop traces, a generalization described by Biermann and Rozoy, are the most general traces. This generalization is based on right-congruences, just like the local traces described by Hoogers. However Bauget and Gastin describe traces based on congruences. The relations, by which the congruences are induced, can be restricted to context commutations relations, leading to cc traces, left- or right-context commutations relations, leading to lcc and rcc traces, left-context commutations with a limited left-context, leading to k-context and $\leq$k-context traces. Properties of these generalizations have been investigated, which lead to a hierarchy of generalized traces, as shown in figure 72.

2 Introduction

In a concurrent system events do not necessarily take place in a sequential fashion. Events, which are not causally related and do not share common resources, may occur independently of one another. The trace theory developed by Mazurkiewicz, see [M87], provides a means to describe the behaviour of concurrent systems using a globally defined binary independence relation between events. Petri nets are a well-known model of concurrent systems and Mazurkiewicz' trace theory has been successfully used to describe the semantics of so-called 1-safe Petri nets in terms of non-conflicting runs of the net. A Petri net is *1-safe* if it has the property that no execution of it according to the firing rule leads to a place with two or more tokens.

For more general nets however Mazurkiewicz traces are too restricted. When a place may contain more than one token, concurrency and conflict between transitions depend on the current state of the net. Thus the concurrency can no longer be captured in a global binary independence relation. Mazurkiewicz traces have been generalized in various ways and in this thesis we will investigate some of these generalizations.

The framework within which we carry out our investigations and comparisons, consists of partial orders and a specific kind of edge labelled event-preserving graphs associated with equivalence classes of words (traces). All the notions and terminology concerning words, sets, relations, and equivalences are defined in section 3 and those concerning the graphs are defined in section 4. The last preliminary section contains all the notions and terminology concerning the partial orders. After these preliminary sections 3 through 5, which fix our notions and terminology concerning equivalences, graphs and partial orders, we describe in section 6 the Mazurkiewicz trace theory. In the next section 7 two kinds of generalizations, crop traces from [BR95] and cop traces from [BG95], of the Mazurkiewicz trace theory are handled. Other generalizations, local traces from [H94], cc traces, lcc traces from [BG95], rcc traces, and k-context traces from [BR95], are considered in section 8.

3 Preliminaries

In this section we fix general notations and conventions used throughout the paper. This section is divided into four subsections. In the first subsection we define all the notions and terminology concerning words of an alphabet. There are also some functions on words given. In the next subsection we give the notations and conventions for a set (or an alphabet) and functions on sets. Then the subsection for relations, binary relations and labelled binary relations, follows. The last subsection concerns the equivalences.

3.1 Words

Let A be an alphabet. By ϵ we denote the empty word. The concatenation between two words $w, v \in A^*$ is denoted by $w \cdot v$. We usually write wv. If $u = wv$ then we call w a *prefix* of u. The *length of a word* $w \in A^*$ is denoted by $|w|$ and defined in the following way: $|\epsilon| = 0$ and $|wa| = |w| + 1$ if $a \in A$. For a word $w \in A^*$ we write $|w|_a$ if we mean the *number of occurrences of a letter* $a \in A$ *in* w. We define $|\epsilon|_a = 0$ and for all $x \in A^*$ and $b \in A$ $|xb|_a = |x|_a + 1$ if $b = a$ and $|xb|_a = |w|_a$ otherwise. The word v is a *permutation of* the word w if $|v|_a = |w|_a$ for all $a \in A$. The *set of all letters occurring in a word* $w \in A^*$, denoted by $alph(w)$, is defined in the following way: $alph(\epsilon) = \emptyset$ and, for all $x \in A^*$ and $a \in A$, $alph(xa) = alph(x) \cup \{a\}$. The *set of events of* A, denoted by E_A, is defined as $E_A = \{(a, i) |\ a \in A \text{ and } i \in \mathbb{N}\}$. The *enumeration function* $ev : A^* \to E_A$ is defined by $ev(\epsilon) = \emptyset$ and $ev(wa) = ev(w) \cup \{(a, |w|_a + 1)\}$ for all $w \in A^*$ and $a \in A$. The *raise function* $t_u : E_A \to E_A$, where $u \in A^*$, is defined by $t_u((a, i)) = (a, i + |u|_a)$ for all $(a, i) \in E_A$. The *labelling function* $l_A : E_A \to A$ is the function defined by $l_A((a, i)) = a$ for all $(a, i) \in E_A$.

Let $f : A \to B$ be a total function from A to some alphabet B. Let $\cdot$ be the concatenation in B. The *homomorphic extension of* f *to* A^* is also denoted by f and is defined in the following way $f(a_1 \dots a_n) = f(a_1) \cdot \dots \cdot f(a_n)$ for $a_i \in A$ where $1 \leq i \leq n \in \mathbb{N}$. In particular the labelling function l_A and raise function t_u will be used as homomorphisms.

3.2 Sets

Let S be a set. Let $S_1, S_2 \subseteq S$. The *difference of two sets* S_1 and S_2 is denoted by $S_1 \backslash S_2$. The *number of elements of the set* S is denoted by $|S|$. The *set of finite subsets of the set* S is denoted by $P_f(S)$ and defined as $\{S' \subseteq S |\ S' \text{ is finite}\}$.

Let A be an alphabet. A non-empty set $S \subseteq A^*$ is *event-preserving* if $|v|_a = |w|_a$

for all $a \in A$ and $v, w \in S$. We extend ev to event-preserving sets V by setting $ev(V) = ev(w)$, where $w \in V$. Note that $ev(V)$ is well-defined in this way. $Prefix(S)$ is the *set of prefixes of* S and is defined as $Prefix(S) = \{w' \in A^* | w'x = w \text{ for some } x \in A^* \text{ and } w \in S\}$.

Let A be an alphabet and let $S \subseteq A$. The *set of linearisations* of S, denoted by $Lin(S)$, is defined as $Lin(S) = \{w \in S^* || w|_a = 1 \text{ for all } a \in S\}$.

Let S, S' be sets. Let $f : S \to S'$ be a function. We denoted by $f|_{S_1}$ the *restriction of* f *to* S_1. f is a *total function* if $f(s)$ is defined for all $s \in S$.

Let $f : S \to S$ be a function. The function f is *injective* if whenever $f(s_1) = t$ and $f(s_2) = t$ for some $s_1, s_2, t \in S$ then $s_1 = s_2$. f is *surjective* if for all $t \in S$ there exists $s \in S$ such that $f(s) = t$.

3.3 Relations

3.3.1 Binary relations

Let N be a set and let $R \subseteq N \times N$ be a binary relation over N. Instead of $(p, q) \in R$ for $p, q \in N$ we may also write pRq. R is *reflexive* if $(p, p) \in R$ for all $p \in N$. R is *irreflexive* if $(p, p) \notin R$ for all $p \in N$. R is *injective* if for all $p_1, p_2, q \in N$ whenever $(p_1, q) \in R$ and $(p_2, q) \in R$ then $p_1 = p_2$. R is *surjective* if for all $p_2 \in N$ there exists a $p_1 \in N$ such that $(p_1, p_2) \in R$. R is *bijective* if R is injective and surjective. R is *transitive* if for all $p_1, p_2, p_3 \in N$ whenever $(p_1, p_2), (p_2, p_3) \in R$ then $(p_1, p_3) \in R$. R is *symmetric* if $(p_1, p_2) \in R$ implies $(p_2, p_1) \in R$ for all $p_1, p_2 \in N$. R is *anti-symmetric* if $(p_1, p_2) \in R$ and $(p_2, p_1) \in R$ implies $p_1 = p_2$ for all $p_1, p_2 \in N$. R is an *equivalence relation (over* N*)* if it is reflexive, symmetric, and transitive. And R is a *partial ordering relation (over* N*)* if it is reflexive, anti-symmetric, and transitive.

The *inverse relation* of R, denoted by R^{-1}, is defined by $R^{-1} = \{(p, q) | (q, p) \in R\}$. The *restriction of* R *to* V, for some $V \subseteq N$ is denoted by $R|_V$ and defined as $R|_V = \{(p, q) | (p, q) \in R \text{ and } p \in V\}$.

3.3.2 Labelled binary relations

Let L be a set of labels, N be a set, and $R \subseteq N \times L \times N$ be a binary relation over N labelled by L. All notions for binary relations carry over to labelled binary relations $R \subseteq N \times L \times N$ through the underlying binary relation R' defined by $R' = \{(p_1, p_2) | (p_1, l, p_2) \in R \text{ for some } l \in L\}$. Thus R is *reflexive* if for all $p \in N$ there exists a $l \in L$ such that $(p, l, p) \in R$. R is *transitive* if for all $p_1, p_2, p_3 \in N$ and $l_1, l_2 \in L$ whenever $(p_1, l_1, p_2) \in R$ and $(p_2, l_2, p_3) \in R$, there exists a $l_3 \in L$ such that $(p_1, l_3, p_3) \in R$. And the relation R is *anti-symmetric* if $(p_1, l_1, p_2) \in R$ and $(p_2, l_2, p_1) \in R$ implies $p_1 = p_2$ for all $p_1, p_2 \in N$ and $l_1, l_2 \in L$. The *restriction of* R *to* S, for some $S \subseteq N \times L \times N$ is denoted by $R|_S$ and defined as $R|_S = \{(p, l, q) | (p, l, q) \in S \text{ and } (p, l, q) \in R\}$.

Now let $R \subseteq N \times A \times N$, where A is an alphabet. The *reflexive and transitive closure* of R, denoted by R^*, is defined in the following way:

$$R^0 = \{(p, \epsilon, p) | p \in N\},$$
$$R^{i+1} = \{(p, xa, p') | \exists p'' \in N : (p, x, p'') \in R^i \text{ and } (p'', a, p') \in R\}$$
$$\text{for all } i \geq 0, \text{ and}$$
$$R^* = \bigcup_{i \geq 0} R^i.$$

The reflexive and transitive closure of R has some interesting aspects.

Lemma 1 *Let A be an alphabet, N a set, and $R \subseteq N \times A \times N$. Let $p, q \in N$ and $w \in A^*$. If $(p, w, q) \in R^i$ then $|w| = i$.*

Proof:

We prove the statement by induction on i.

Let $i = 0$. Then we have $(p, w, q) \in R^0$. By definition $w = \epsilon$. Thus $|w| = 0$.

Suppose the statement is proven for $i = k$ for some $k \geq 0$.

Assume $i = k + 1$, thus $(p, w, q) \in R^{k+1}$. By the definition of the reflexive and transitive closure of R there exist $w' \in A^*$, $p' \in N$, and $a \in A$ such that $w = w'a$, $(p, w', p') \in R^k$, and $(p', a, q) \in R$. By the induction hypothesis we have $|w'| = k$. Thus $|w| = |w'| + |a| = k + 1 = i$. We can conclude that the lemma holds. □

Lemma 2 *Let A be an alphabet, N a set, and $R \subseteq N \times A \times N$. Let $p, p', p'' \in N$ and $w_1, w_2 \in A^*$. If $(p, w_1, p') \in R^i$, $(p', w_2, p'') \in R^j$ then $(p, w_1 w_2, p'') \in R^{i+j}$.*

Proof:

We prove the statement by induction on $|w_1 w_2|$.

Let $|w_2| = 0$ then $(p', \epsilon, p'') \in R^0$. Thus $p' = p''$. Then $(p, w_1, p'') \in R^i$.

Suppose it is proven for $|w_1 w_2| = k$ for some $k \geq 0$.

Assume $|w_1 w_2| = k + 1$ and $|w_2| \neq 0$. If $(p', w_2, p'') \in R^j$ then there exist $q \in N$, $w \in A^*$, and $a \in A$ such that $w_2 = wa$, $(p', w, q) \in R^{j-1}$, and $(q, a, p'') \in R$. By the induction hypothesis $(p, w_1 w, q) \in R^{i+(j-1)}$. Since $(q, a, p'') \in R$ we have $(p, w_1 w \cdot a, p'') \in R^{(i+(j-1))+1}$ by definition. Thus $(p, w_1 w_2, p'') \in R^{i+j}$.

We can conclude that the lemma holds. □

3.4 Equivalence

Let A be an alphabet and $R \subseteq A^* \times A^*$ an equivalence relation over A^*. For $w \in A^*$ the *equivalence class of* R containing w, denoted by $[w]_R$, is the set $[w]_R = \{v \in A^* | (v, w) \in R\}$ and $A^*/R = \{[w]_R | w \in A^*\}$. When there is no confusion about the relation, we will write $[w]$. R is *right-cancellative* if, for all $uv, wv \in A^*$, $uvRwv$ implies uRw. R is *left-cancellative* if, for all $vu, vw \in A^*$, $vuRvw$ implies uRw. The equivalence relation R is *cancellative* if R is both right-cancellative and left-cancellative. R is *event-preserving* if, for all $v, w \in A^*$, $(v, w) \in R$ implies $|v|_a = |w|_a$ for all $a \in A$. For an event-preserving equivalence over A^* , $|[w]|$ the length of an equivalence class containing the word $w \in A^*$ is well-defined by $|[w]| = |w|$.

Let R be event-preserving. The *translate function* $\zeta_R : A^*/R \to E_A$ is defined in the following way: $\zeta_R([w]_R) = ev(w)$ for all $[w] \in A^*/R$.
Note that ζ_R is well-defined if R is event-preserving.

Let A be an alphabet and let $S \subseteq A^* \times A^*$ be an arbitrary binary relation over A^*. The *equivalence induced by* S, denoted by $\sim_S$, is defined by $(S \cup S^{-1})^*$. Let $S_r = \{(xu, yu)| \ (x, y) \in S$ and $u \in A^*\}$ be the relation S extended to the right and $S_l = \{(ux, uy| \ (x, y) \in S$ and $u \in A^*\}$ be the relation S extended to the left. The *right congruence induced by* S is defined by $\approx_S = (S_r \cup S_r^{-1})^*$. Let $S_{lr} = \{(uxv, uyv)| \ (x, y) \in S$ and $u, v \in A^*\}$, then $\equiv_S = (S_{lr} \cup S_{lr}^{-1})^*$ is the *congruence induced by* S. $u \doteq_S v$ if and only if $(u, v) \in S_{lr}$.
Note that $\sim_S$ is an equivalence, $(S_r)^{-1} = (S^{-1})_r$, $S_{rl} = S_{lr}$, $(S_{lr})^{-1} = (S^{-1})_{lr}$, $\approx_S = \sim_{S_r}$, and $\equiv_S = \sim_{S_{lr}}$.
Note that each equivalence relation R with $S \subseteq R$ contains $(S \cup S^{-1})^*$, thus $\sim_S$ is the least equivalence relation over A^* containing S.
Note that $\sim_S$, $\approx_S$ and $\equiv_S$ are event-preserving whenever S is event-preserving.

4 Graphs

In this subsection we introduce our graph-theoretical notions directly for edge labelled graphs, as these are the graphs we are interested in rather than in unlabelled graphs.

4.1 Edge labelled graphs

An *edge labelled (directed) graph*, or *elgraph* for short, is a triple $G = (N, A, \rightarrow)$, where N is the set of nodes of G, A is the labelling alphabet of G and the binary relation $\rightarrow \subseteq N \times A \times N$ labelled with A is the set of labelled edges of G.
We usually write $p \xrightarrow{a} q$ for $(p, a, q) \in \rightarrow$ and $p \xrightarrow{u}^* q$ for $(p, u, q) \in \rightarrow^*$.

Let $G = (N, A, \rightarrow)$ be an elgraph. If $(p, u, q) \in \rightarrow^*$ for some $p, q \in N$ and $u \in A^*$ we write $p \xrightarrow{u}^* q$ and say that *there exists a path* (*labelled with* u) *from* p *to* q. Note that $p \xrightarrow{\epsilon}^* q$ holds if and only if $p = q$. G is *acyclic* if whenever both $p \xrightarrow{u}^* q$ and $q \xrightarrow{v}^* p$ hold for some $p, q \in N$ and $u, v \in A^*$, then $p = q$ and $u = v = \epsilon$. G is *deterministic* if whenever both $p \xrightarrow{a} q$ and $p \xrightarrow{a} q'$ hold for some $p, q, q' \in N$ and $a \in A$, then $q = q'$. For two vertices $p, q \in N$ of an elgraph G the set $Path_{p,q}(G)$ of all path labels in G from p to q is defined as $Path_{p,q}(G) = \{w \in A^* | p \xrightarrow{w}^* q\}$. Clearly, in a deterministic elgraph G each path label in $Path_{p,q}(G)$ corresponds with a unique path from p to q. G is *event-preserving* if whenever $p \xrightarrow{u}^* q$ and $p \xrightarrow{v}^* q$ for some $p, q \in N$ and $u, v \in A^*$, then $|u|_a = |v|_a$ for all $a \in A$.

If we know that G is event-preserving, then we can conclude that the elgraph is acyclic. This is proven in the next lemma.

Lemma 3 *Let $G = (N, A, \rightarrow)$ be an elgraph. If G is event-preserving then G is acyclic.*

Proof:

Suppose $p, q \in N$ and $u, v \in A^*$ are such that $p \xrightarrow{u}^* q$ and $q \xrightarrow{v}^* p$. Then by lemma 2 $p \xrightarrow{uv}^* p$. Since $p \xrightarrow{\epsilon}^* p$ and G is event-preserving we have $|\epsilon|_a = |uv|_a$ for all $a \in A$. Thus $u = v = \epsilon$ and $p = q$. Hence G is acyclic. □

Let $G = (N, A, \rightarrow)$ be an elgraph and $r \in N$. Then r is a *root* of G if for all $p \in N$ there exists a path from r to p.

The next lemma is very easy to see. If we have an elgraph which is acyclic and we have a root, then this root has to be unique.

Lemma 4 *Let $G = (N, A, \rightarrow)$ be an elgraph and r a root of G.*
If G is acyclic, then r is the unique root of G.

Proof:

Suppose there exists $r' \in N$ such that r' is also a root of G. Since r' is a root of G, there exists a path from r' to r. We know that r is a root thus there exists a path from r to r'. Since G is acyclic we have $r' = r$. □

An *elgraph with an initial node* is a 4-tuple $G = (N, A, \rightarrow, p_0)$, where $(N, A, \rightarrow)$ is an elgraph and $p_0 \in N$.

A *rooted elgraph*, *relgraph* for short, is an elgraph with an initial node $G = (N, A, \rightarrow, p_0)$ such that p_0 is a root of $(N, A, \rightarrow)$.

Let $G = (N, A, \rightarrow, r)$ be a relgraph. We define the set of all path labels along the paths in G, denoted by $Path(G)$, as $Path(G) = \bigcup_{p \in N} Path_{r,p}(G)$.

A *reldepgraph* is a rooted elgraph which is deterministic and event-preserving. All notation and terminology introduced for elgraphs and relgraphs will also be used for reldepgraphs.

Corollary *Let $G = (N, A, \rightarrow, r)$ be a reldepgraph. The node r is the unique root.*

Proof:

G is a reldepgraph thus we have that G is event-preserving. By lemma 3 we have that G is acyclic. Then lemma 4 tells us that the root r is unique. □

4.2 Restriction of a reldepgraph

Let $p \in N$ be a vertex of the reldepgraph $G = (N, A, \rightarrow, r)$. The *set of all vertices q before p*, denoted by $Bef(p)$, is defined by $Bef(p) = \{q \in N |$ there exists a path from q to $p\}$. The graph $G(p)$, the *restriction of reldepgraph G to the node p*, is the restriction of G to the set $Bef(p)$ and is defined by $G(p) = (Bef(p), A, \rightarrow |_{Bef(p) \times A \times Bef(p)}, r)$.

All paths in G between vertices from $Bef(p)$ are included in the graph $G(p)$ as stated in the next lemma.

Lemma 5 *Let $G = (N, A, \rightarrow, r)$ be a reldepgraph. Let $p \in N$ and $q, s \in Bef(p)$. Then $Path_{q,s}(G) = Path_{q,s}(G(p))$*

Proof:

Since $G(p)$ is a restriction of G, we have $Path_{q,s}(G(p)) \subseteq Path_{q,s}(G)$.
Now assume that $w \in Path_{q,s}(G)$. Let $q_0 = q, q_1, \ldots, q_{n-1}, q_n = s$ be nodes along the path labelled by w. Then we have $n \geq 0$ and $q_i \xrightarrow{a_{i+1}} q_{i+1}$ for $0 \leq i \leq n-1$ and $a_i \in A$ such that $w = a_1 \ldots a_n$. Since $s \in Bef(p)$, it follows that each $q_i \in Bef(p)$. Hence each (q_i, a_{i+1}, q_{i+1}) is an edge of $G(p)$. Consequently w labels a path from q to s in $G(p)$ and is an element of $Path_{q,s}(G(p))$. □

Now we can prove that the restriction of a reldepgraph to a node is again a reldepgraph.

Theorem 6 *Let $G = (N, A, \rightarrow, r)$ be a reldepgraph and $p \in N$.*
$G(p)$ is a reldepgraph.
Proof:

Since $G(p)$ is a restriction of G, it is deterministic and event-preserving. We now only have to make sure that the root r is still the root in the graph $G(p)$. We know $r \in Bef(p)$ and r is a vertex of $G(p)$. That each vertex of $G(p)$ can be reached from r follows from lemma 5.
Thus the graph $G(p)$ is a relddop-graph. □

Note that p is a *leaf* of $G(p)$: it has no out-going edges. Moreover it is the only leaf of $G(p)$. $G(p)$ has also a root, which is unique, this means that all vertices in $G(p)$ are along a path from r to p. In the graph $G(p)$ we still have the definition of $Path_{q,s}(G(p))$. However for the set $Path_{r,p}(G(p))$ we will write $Path_{max}(G(p))$.

4.3 Some properties for reldepgraphs

Let $G = (N, A, \rightarrow, r)$ be a reldepgraph. G is *co-deterministic* if $(N, A, \rightarrow)$ is co-deterministic: whenever $q \xrightarrow{a} p$ and $q' \xrightarrow{a} p$ for some $p, q, q' \in N$ and $a \in A$, then $q = q'$.

To describe the internal structure of a reldepgraph we may use the so-called diamond properties. The forward and backward diamond properties, as described in [BR95] by Biermann and Rozoy, are illustrated in figure 1 and 2, and are defined in the following way:

G has the *forward diamond property* if for all $p, p_1, p_2 \in N$ and $a, b \in A$ whenever $a \neq b$, $p \xrightarrow{a} p_1$, and $p \xrightarrow{b} p_2$ then there exists $p' \in N$ such that $p_1 \xrightarrow{b} p'$ and $p_2 \xrightarrow{a} p'$.

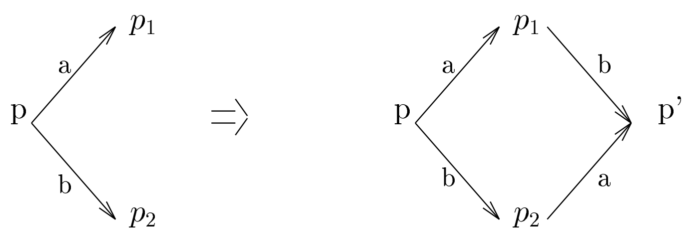


Figure 1: Forward diamond

G has the *backward diamond property* if for all $p, p_1, p_2 \in N$ and $a, b \in A$ whenever $a \neq b$, $p_1 \xrightarrow{a} p$, and $p_2 \xrightarrow{b} p$ then there exists $p' \in N$ such that $p' \xrightarrow{b} p_1$ and $p' \xrightarrow{a} p_2$.

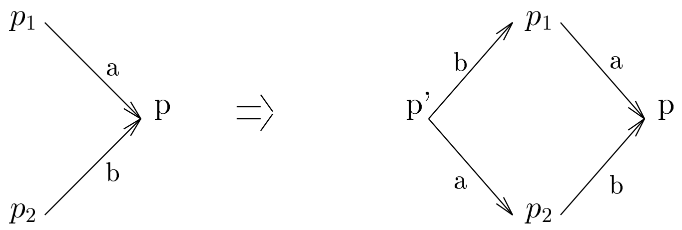


Figure 2: Backward diamond

Having defined the forward diamond property we can define the compatible forward diamond property. This means that only vertices, which are before a common vertex, have to satisfy the forward diamond property. There is a restriction that the vertex p', which closes the forward diamond property, has to be before the common vertex. This property allows us to conclude in theorem 7 that if G has the compatible forward diamond property, then all the restrictions of G to a node have the forward diamond property. The compatible forward diamond property is illustrated in figure 3.

G has the *compatible forward diamond property* if for all $p, p_1, p_2 \in N$ and for all $a, b \in A$ whenever $a \neq b$, $p \xrightarrow{a} p_1$, $p \xrightarrow{b} p_2$, and there exists $q \in N$ such that $p_1, p_2 \in Bef(q)$ then there exists $p' \in Bef(q)$ such that $p_1 \xrightarrow{b} p'$ and $p_2 \xrightarrow{a} p'$.

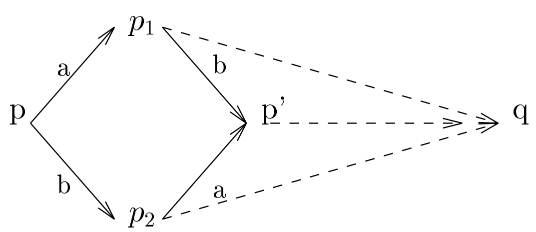


Figure 3: Compatible forward diamond

If G has the forward (backward or compatible forward, respectively) diamond property we will write: $FD(G)$ ($BD(G)$ or $CFD(G)$, respectively).

If a restriction $G(p)$ has the forward diamond property then all vertices in $Bef(p)$ satisfy the compatible forward diamond property. Thus we can conclude that if all restrictions of G have the forward diamond property then G has the compatible forward diamond property.

Theorem 7 *Let $G = (N, A, \rightarrow, r)$ be a reldepgraph. $CFD(G)$ if and only if $FD(G(p))$ for all $p \in N$.*

Proof:

Suppose $p \in N$ is such that $G(p)$ does not have the forward diamond property. Thus there exist $q, q_1, q_2 \in Bef(p)$ such that $q \xrightarrow{a} q_1$ and $q \xrightarrow{b} q_2$ for some $a, b \in A$ but there exists no $q' \in Bef(p)$ such that $q_1 \xrightarrow{b} q'$ and $q_2 \xrightarrow{a} q'$. Since $G(p)$ is a restriction of G we have the same situation in G. Thus G does not have the compatible forward diamond property.

Suppose G does not have the compatible forward diamond property. Thus there exist $q, q_1, q_2 \in N$ such that $q \xrightarrow{a} q_1$ and $q \xrightarrow{b} q_2$ for some $a, b \in A$ and there exists $p \in N$ such that $q_1, q_2 \in Bef(p)$, but there exists no $q' \in Bef(p)$ such that $q_1 \xrightarrow{b} q'$ and $q_2 \xrightarrow{a} q'$. If we look at $G(p)$, we can conclude that $G(p)$ does not have the forward diamond property, because $q_1, q_2 \in Bef(p)$ implies $q \in Bef(p)$. Thus $q \xrightarrow{a} q_1$ and $q \xrightarrow{b} q_2$ in $G(p)$ and there exists no $q' \in Bef(p)$ such that $q_1 \xrightarrow{b} q'$ and $q_2 \xrightarrow{a} q'$. □

Another way to describe the internal structure of a reldepgraph are the so-called cube and inverse cube axiom. In figure 4 the cube axiom and inverse cube axiom are visualized. For the cube axiom in part a the pre-conditions are shown and in part b the post-conditions. For the inverse cube axiom in part b the pre-conditions are shown and in part a the post-conditions. These properties are described in [DK95] by Droste and Kuske.

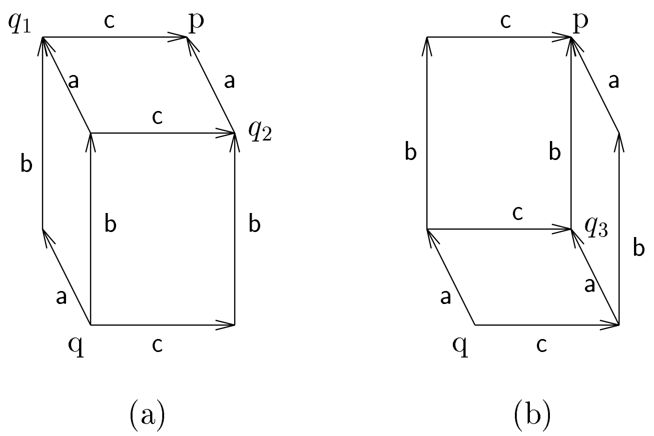


Figure 4: Cube and inverse cube axiom

G satisfies the *cube axiom* if for all $q, q_1, q_2, p \in N$ and $a, b, c \in A$ whenever $\{ab, ba\} \subseteq Path_{q,q_1}(G)$, $\{bc, cb\} \subseteq Path_{q,q_2}(G)$, and $\{bac, bca\} \subseteq Path_{q,p}(G)$, then also $\{abc, acb, cba, cab\} \subseteq Path_{q,p}(G)$ and there exists $q_3 \in N$ such that $\{ac, ca\} \subseteq Path_{q,q_3}(G)$.

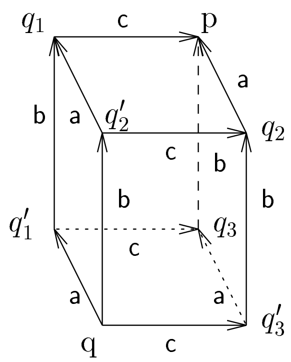
G satisfies the *inverse cube axiom* if for all $q, q_3, p \in N$ and $a, b, c \in A$ whenever $\{ac, ca\} \subseteq Path_{q,q_3}(G)$ and $\{abc, acb, cba, cab\} \subseteq Path_{q,p}(G)$, then also $\{bac, bca\} \subseteq Path_{q,p}(G)$ and there exists $q_1, q_2 \in N$ such that $\{ab, ba\} \subseteq Path_{q,q_1}(G)$ and $\{bc, cb\} \subseteq Path_{q,q_2}(G)$.

The relation between the forward diamond property and the cube axiom is stated in the next theorem.

Theorem 8 *Let $G = (N, A, \rightarrow, r)$ be a reldepgraph. If G has the forward diamond property then G satisfies the cube axiom.*

Proof:

Suppose we have $q \stackrel{a}{\rightarrow} q'_1 \stackrel{b}{\rightarrow} q_1 \stackrel{c}{\rightarrow} p$, $q \stackrel{b}{\rightarrow} q'_2 \stackrel{a}{\rightarrow} q_1$, $q \stackrel{b}{\rightarrow} q'_2 \stackrel{c}{\rightarrow} q_2$, $q \stackrel{c}{\rightarrow} q'_3 \stackrel{b}{\rightarrow} q_2 \stackrel{a}{\rightarrow} p$, for some $q, q'_1, q'_2, q'_3 \in N$. Thus we have the pre-conditions of the cube axiom. We know that G has the forward diamond diamond property. Thus there exist $q_3 \in N$ such that $q'_1 \stackrel{c}{\rightarrow} q_3$, $q'_3 \stackrel{a}{\rightarrow} q_3$, and $q_3 \stackrel{b}{\rightarrow} p$. Then G satisfies the cube axiom. $\square$



Similarly we have a relation between the backward diamond property and the inverse cube axiom.

Theorem 9 *Let $G = (N, A, \rightarrow, r)$ be a reldepgraph. If G has the backward diamond property then G satisfies the inverse cube axiom.*

Proof:

Suppose we have $q \stackrel{a}{\rightarrow} q'_1 \stackrel{b}{\rightarrow} q_1 \stackrel{c}{\rightarrow} p$, $q \stackrel{c}{\rightarrow} q'_3 \stackrel{b}{\rightarrow} q_2 \stackrel{a}{\rightarrow} p$, and $q'_1 \stackrel{c}{\rightarrow} q_3 \stackrel{b}{\rightarrow} p$ $q'_3 \stackrel{a}{\rightarrow} q_3$ for some $p, q_1, q_2, q_3, q'_1, q'_3 \in N$. Thus we have the pre-conditions of the inverse cube axiom. We know that G has the backward diamond property. Thus there exist $q'_2 \in N$ such that $q'_2 \stackrel{a}{\rightarrow} q_1$, $q'_2 \stackrel{c}{\rightarrow} q_2$, and $q \stackrel{b}{\rightarrow} q'_2$. Then G satisfies the inverse cube axiom. $\square$

4.4 Morphisms between reldepgraphs

When we have two reldepgraphs, we would like to compare these graphs. To compare two reldepgraphs we have special functions from one graph to the other graph, called morphisms. A morphism is defined in the following way.

Definition

Let $G = (N, A, \rightarrow, r)$ and $G' = (N', A, \rightarrow', r')$ be two reldepgraphs.

1. A *morphism* from G to G' is a total function $H : N \rightarrow N'$ such that $H(r) = r'$ and $p_1 \xrightarrow{a} p_2$ implies $H(p_1) \xrightarrow{a}' H(p_2)$.
2. A morphism H from G to G' is *full* if for all $q_1, q_2 \in H(N)$ and $a \in A$, whenever $q_1 \xrightarrow{a}' q_2$ then $p_1 \xrightarrow{a} p_2$ for some $p_1, p_2 \in N$ such that $H(p_1) = q_1$ and $H(p_2) = q_2$.
3. G and G' are *isomorphic* if there exists a morphism H from G to G' which is bijective and full.

If there is an injective morphism H from G into G' then G can be *embedded into* G' and we write $G \subseteq G'$. Further G *is fully mapped to* G' *by* H if H is full.

If there exists a morphism between two reldepgraphs then this morphism is unique.

Theorem 10 *Let* $G = (N, A, \rightarrow, r)$ *and* $G' = (N', A, \rightarrow', r')$ *be two reldepgraphs. If* H *and* H' *are morphisms from* G *to* G'*, then* $H = H'$*.*

Proof:

Since G is event-preserving all paths from r to a fixed $p \in N$ are labelled with words of the same length. We prove $H(p) = H'(p)$ for all $p \in N$ by induction on the length of the paths from r to p. Let $p \in N$ and let $w \in Path_{r,p}(G)$.

If $|w| = 0$, then $p = r$. Since H and H' are morphisms $H(r) = r' = H'(r)$.

Now suppose that $H(q) = H'(q)$ for all $q \in N$ for which there is a path of length $k \geq 0$ from r to q.

Assume $|w| = k + 1$. Then there exist $q \in N$, $u \in A^*$, and $b \in A$ such that $r \xrightarrow{u}^* q$, $q \xrightarrow{b} p$, and $w = ub$. Since $|u| = k$ we have $H(q) = H'(q)$ by the induction hypothesis. H and H' are morphisms thus $H(q) \xrightarrow{b}' H(p)$ and $H'(q) \xrightarrow{b}' H'(p)$. Since G' is deterministic $H(p) = H'(p)$ follows.

We can conclude that $H = H'$. $\square$

5 Partial orders and event-preserving sets

In this section we first describe our notations and conventions concerning partial orders. We use the setup of Davey and Priestley in [DP90]. The first subsection handles the general definitions used with partial orders. However we work with event-preserving sets and in the second subsection the definitions and notations for the partial order of an event-preserving set are given. Bauget and Gastin identify in [BG95] congruences by means of (modular) representations by partial orders. The necessary definitions are specified in subsection 5.3.

5.1 Partial orders

Let E be a set and let $\leq$ be a partial ordering over E. We can define an *ordering*, $<$, *of strict inequality* by $x < y$ if $x \leq y$ and $x \neq y$. If for some $x, y \in E$ both $x \not\leq y$ and $y \not\leq x$, notation $x \parallel_{\leq} y$, then x and y are *incomparable*. $\leq$ is a *linear ordering* if $x \leq y$ or $y \leq x$ for all $x, y \in S$. The set $E' \subseteq E$ is *left-closed (wrt $\leq$)* if $e \in E'$ and $e' \leq e$ implies $e' \in E'$ for all $e, e' \in E'$.

A *partially (linearly) ordered set* $P = (E, \leq)$ consists of a set E together with a partial (linear) ordering $\leq$ over E. Partially (linearly) ordered sets will be depicted by their *Hasse diagrams*. We call a partially ordered set a *poset*. We write $x \parallel_P y$ instead of $x \parallel_{\leq} y$.

Let $P = (E, \leq)$ be a poset. The *linear extension* of P, denoted by $LE(P)$, is the set $\{e_1 \ldots e_n \in E^* |\ |e_1 \ldots e_n|_e = 1$ for all $e \in E$ and for all $i, j \in \{1, \ldots, n\}$ $e_i \leq e_j$ implies $i \leq j\}$. For a poset $P = (E, \leq)$ and $x, y \in E$, we have *x is covered by y*, or *y covers x*, denoted by $x \lessdot y$, if $x < y$ and $x \leq z < y$ implies $x = z$.

Let $P = (E_P, \prec_P)$ be a finite poset. The *order independence relation* of P, denoted by I_P, is the set $\{(e_1, e_2)|\ e_1 \parallel_P e_2\}$. We can now define an equivalence relation $C_P = \{(e_1e_2, e_2e_1)|\ (e_1, e_2) \in I_P\}$.
Then $\equiv_P$ is the congruence over E_P generated by C_P.

Lemma 11 *Let $P = (E_P, \prec_P)$ be a finite poset. Let $u \in LE(P)$ and $v \in E_P^*$. If $u \equiv_P v$ then $v \in LE(P)$.*

Proof:

Suppose we have $u \in LE(P)$ and $u \dot{\equiv}_P v$. Then there exist $e_1, e_2 \in E_P$ and $x, y \in E_P^*$ such that $u = xe_1e_2y$, $v = xe_2e_1y$, and $(e_1, e_2) \in I_P$. Then clearly $v = xe_2e_1y \in LE(P)$.

Suppose we have $u \in LE(P)$ and $v, w_0, \ldots, w_n \in E_P^*$ such that $u = w_0 \dot{\equiv}_P w_1 \dot{\equiv}_P \ldots \dot{\equiv}_P w_n = v$. Then $u \equiv_P v$ and by repeatedly applying the reasoning given for $\dot{\equiv}_P$ and we have $w_0, \ldots, w_n, v \in LE(P)$. $\square$

This lemma leads to the property that any two elements of the linear extension of a poset are equivalent.

Lemma 12 *Let $P = (E_P, \prec_P)$ be a finite poset. Let $u, v \in E_P^*$. If $u, v \in LE(P)$ then $u \equiv_P v$.*

Proof:

Let $x \in E_P^*$ be the longest common prefix of u and v. Induction on $|u| - |x|$.
Let $|u| - |x| = 0$ then $u = x$ and thus $u = v$. Clearly $u \equiv_P v$ holds.
Suppose the statement is proven for $|u| - |x| = k$.
Assume $|u| - |x| = k + 1$. Let $a, b \in E_P$ and $y, z \in E_P^*$ be such that $u = xay$ and $v = xbz$. If $u, v \in LE(P)$ then $|u|_e = |v|_e = 1$ for all $e \in E_P$. Then there exist $y', y'', z', z'' \in E_P^*$ such that $u = xay'by''$ and $v = xbz'az''$. If we have this then for all $e \in alph(ay')$, $(e, b) \in I_P$. Then $u = xay'by'' \equiv_P xaby'y'' \equiv_P xbay'y''$. From lemma 11 follows $xbay'y'' \in LE(P)$. Since $|xbay'y''| - |x'| \leq k$ where x' is the longest common prefix of $xbay'y''$ and v we have by the induction hypothesis $xbay'y'' \equiv_P v$. It is now easy to see that $u \equiv_P v$. □

A *labelled poset* is a pair $LP = (P, l)$, where $P = (E, \leq)$ is a poset and l is a labelling function $l : E \rightarrow A$ for some alphabet A.
We write $LP = (E, \leq, l)$ instead of $LP = ((E, \leq), l)$.

Let $LP = (P, l) = (E, \leq, l)$ be a labelled poset. The *linear extension* of LP, denoted by $LE(LP)$, is defined by :
$LE(LP) = \{l(e_1) \ldots l(e_n) \in A^* |\ e_1, e_n \in LE(P)\}$. Thus $LE((P, l)) = l(LE(P))$ since l is a homomorphism.

Let $(P, l) = (E_P, \prec_P, l)$ be a finite labelled poset. Define the equivalence relation $C_{(P,l)} = \{(l(e_1e_2), l(e_2e_1)) | (e_1, e_2) \in I_P\}$ Then $\equiv_{(P,l)}$ is the congruence over $l(E_p)^*$ generated by $C_{(P,l)}$.

Lemma 13 *Let $P = (E_P, \prec_P)$ be a finite poset and l be a labelling function. Let $u, v \in l(E_p)^*$. If $u \in l(LE(P))$ and $u \equiv_{(P,l)} v$ then $v \in l(LE(P))$.*

Proof:

Suppose we have $u \dot{\equiv}_{(P,l)} v$. Then $(u, v) \in C_{(P,l)}$. Let $u = l(u')$ and $v = l(v')$. Then $(u', v') \in C_P$ and $u' \in LE(P)$. By lemma 11 we have $v' \in LE(P)$. Thus $v \in l(LE(P))$.

Suppose $u \equiv_{(P,l)} v$ and there exist $w_0, \ldots, w_n \in l(E_p)^*$ such that $u = w_0 \dot{\equiv}_{(P,l)} w_1 \dot{\equiv}_{(P,l)} \ldots \dot{\equiv}_{(P,l)} w_n = v$. By repeatedly applying the above reasoning for $\dot{\equiv}_{(P,l)}$ we get $u_0, \ldots, u_k, v \in l(LE(P))$. □

If we have two words which are an element of the linearizations of a labelled partial order then the two words are equivalent.

Lemma 14 *Let $P = (E_P, \prec_P)$ be a finite poset and l be a labelling function. Let $u, v \in l(E_p)^*$. If $u, v \in l(LE(P))$ then $u \equiv_{(P,l)} v$.*

Proof:

Let $u = l(u')$ and $v = l(v')$ for some $u', v' \in LE(P)$. From lemma 12 follows $u' \equiv_P v'$. Since $(C_{(P,l)})_{lr} = l((C_P)_{lr})$ we have $u \equiv_{(P,l)} v$. $\Box$

Let $LP = (E, \leq, l)$ be a labelled poset. $C \subseteq E$ is a *configuration of* LP if C is a finite left-closed subset of E. The *set of configurations of* LP is denoted by $Conf_{LP}$ and defined as $Conf_{LP} = \{C | C \text{ is a configuration of } LP\}$. The *configuration graph* of LP is the elgraph with initial node $\mathcal{C}_{LP} = (Conf_{LP}, A, \rightarrow, \emptyset)$, where $C \xrightarrow{l(e)} C'$ if $C' = C \cup \{e\}$ and $e \notin C$.

If we have a certain configuration, we know that there exists an element such that this configuration without the element is still a configuration. This leads to the fact that each configuration can be reached from a smaller configuration.

Lemma 15 *Let $LP = (E, \leq, l)$ a labelled poset. Let $C \in Conf_{LP}$. If $C \neq \emptyset$ then there exist $C' \in Conf_{LP}$ and $e \in E$ such that $C' \xrightarrow{l(e)} C$.*

Proof:

Since C is finite, there exists an $e \in C$ such that for no $e' \in C$ we have $e \leq e'$. Clearly $C \backslash \{e\}$ is left-closed and finite. Then it follows that $C \backslash \{e\} \in Conf_{LP}$ and hence $C \backslash \{e\} \xrightarrow{l(e)} C$. $\Box$

The set $Conf_{LP}$ ordered by inclusion forms a lattice, as stated in the next theorem.

Theorem 16 *Let $LP = (E, \leq, l)$ a labelled poset. Then $Conf_{LP}$ forms a lattice ordered by inclusion.*

Proof:

We have to prove that the intersection and union of two configurations is again a configuration.

Let $C_1, C_2 \in Conf_{LP}$ and $m \in E$ such that $m \in C_1 \cap C_2$. Then $m \in C_1$. Therefore if $m' \leq m$ then $m' \in C_1$ for all $m' \in E$. Also $m \in C_2$ and if $m' \leq m$ then $m' \in C_2$ for all $m' \in E$. Thus if $m' \leq m$ then $m' \in C_1 \cap C_2$ for all $m' \in E$. Then $C_1 \cap C_2$ is a configuration.

Let $C_1, C_2 \in Conf_{LP}$ and $m \in E$ such that $m \in C_1 \cup C_2$. Then $m \in C_1$ or $m \in C_2$. $m \in C_1$ implies if $m' \leq m$ then $m' \in C_1$ for all $m' \in E$. Then $m' \in C_1 \cup C_2$. Similarly $m \in C_2$. Thus $m' \leq m$ implies $m' \in C_1 \cup C_2$ for all $m' \in E$. Therefore $C_1 \cup C_2$ is a configuration. $\Box$

5.2 Partial order of an event-preserving set

Definition

Let A be an alphabet and $V \subseteq A^*$ be event-preserving. Let $w \in V$.

1. The *alphabet of events of* w is defined by $E_w = ev(w)$;
2. The *occurrence total order* of w is defined as $To(w) = (E_w, \leq_w)$ where $(a,i) \leq_w (b,j)$ if the i-th occurrence of a precedes the j-th occurrence of b in the word w;
3. The *partial order* of V, denoted by $Po(V)$, is defined as the labelled poset $Po(V) = (E_w, \leq_V, l)$, where $w \in V$ and $(a,i) \leq_V (b,j)$ if $(a,i) \leq_v (b,j)$ for all $v \in V$.

Note that $E_w = E_v$ for all $w, v \in V$.

Since the partial order of V is an intersection of the occurrence total orders of the elements of V, the linearizations of the partial order contain all elements of V.

Theorem 17 *Let A be an alphabet and $V \subseteq A^*$ an event-preserving set. Then $V \subseteq LE(Po(V))$.*

Proof:

Suppose we have a word $w \in V$ such that $w \notin LE(Po(V))$. This implies that there exist events $(a,i),(b,j) \in E_w$ such that $(a,i) \leq_V (b,j)$ and $(b,j) \leq_w (a,i)$. The partial order of V depends on all occurrence total orders of the elements of V. This means that if $(a,i) \leq_V (b,j)$ then in all the elements of V the i-th occurrence of a precedes the j-th occurrence of b. But $(b,j) \leq_w (a,i)$, a contradiction. Hence $w \in LE(Po(V))$. □

The configuration graph of the partial order of an event-preserving set is a reldepgraph.

Theorem 18 *Let A be an alphabet and $V \subseteq A^*$ an event-preserving set. The configuration graph $\mathcal{C}_{Po(V)} = (Conf_{Po(V)}, A, \rightarrow, \emptyset)$ is a reldepgraph.*

Proof:

First we show that $\emptyset$ is a root. Suppose we have $C \in Conf_{Po(V)}$. With induction on $|C|$ we prove that there exists a path from $\emptyset$ to C.

Let $|C| = 0$. Then we know that $C = \emptyset$ and $\emptyset \xrightarrow{\epsilon}^* C$ holds.

Suppose $k \geq 0$ and for each configuration with k elements there exists a path from $\emptyset$ to that configuration.

Let $|C| = k+1$. Then by lemma 15 there exist $C' \in Conf_{Po(V)}$ and $e \in E$ such that $C' \xrightarrow{l(e)} C$. Since C' has k elements the induction hypothesis implies $\emptyset \xrightarrow{w}{}^{*} C'$ for some $w \in A^*$. Thus $\emptyset \xrightarrow{wl(e)}{}^{*} C$. We can conclude that there exists a path from $\emptyset$ to C.

Next we have to show that $\mathcal{C}_{Po(V)}$ is deterministic. Suppose we have $C_1, C_2, C \in Conf_{Po(V)}$ and $e_1, e_2 \in E_V$ with $l(e_1) = l(e_2)$ such that $C \xrightarrow{l(e_1)} C_1$ and $C \xrightarrow{l(e_2)} C_2$. If $l(e_1) = l(e_2)$ then $e_1 = (a, i)$ and $e_2 = (a, j)$ for some $a \in A$ and $i, j \in \mathbb{N}$. There can be two situations. First $i = j$. Then $C_1 = C \cup \{a\} = C_2$ and thus $C_1 = C_2$. The other situation is that $i < j$. By definition $(a, i) <_V (a, j)$ thus $e_1 <_V e_2$. We have $C \xrightarrow{l(e_2)} C_2$ thus $e_1 \in C$. A contradiction, since $C \xrightarrow{l(e_1)} C_1$, Thus $\mathcal{C}_{Po(V)}$ is deterministic.

Finally we show $\mathcal{C}_{Po(V)}$ is event-preserving. Suppose we have
$C, C' \in Conf_{Po(V)}$ and $x, y \in E_V$ such that $C \xrightarrow{l(x)}{}^{*} C'$ and $C \xrightarrow{l(y)}{}^{*} C'$. By definition of $\rightarrow$ we know $alph(x) \cap C = \emptyset = alph(y) \cap C$,
$C' \backslash C = alph(x) = alph(y)$, and $|x| = |y|$. Thus $|l(x)|_a = |l(y)|_a$ for all $a \in A$.

$\mathcal{C}_{Po(V)}$ is rooted, deterministic and event-preserving, and hence $\mathcal{C}_{Po(V)}$ is a reldep-graph. $\Box$

For practical reasons we write a_i in stead of (a, i) in the configuration graph.

Example

Let $V = \{abcd, cabd, cadb\}$. The occurrence total order of the word $abcd$ is illustrated in figure 5.

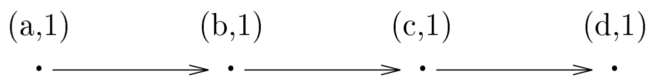


Figure 5: $To(abcd)$

After we have determined all occurrence total orders of the elements of the set V, we can construct the partial order of the set V. The partial order of V is depicted in figure 6.

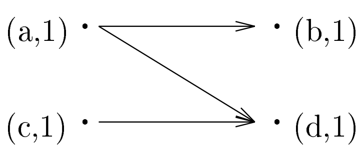


Figure 6: $Po(V)$

Now we can construct the configuration graph of the partial order of V. In this graph we write a_i instead of (a, i). In figure 7 the configuration graph is depicted.

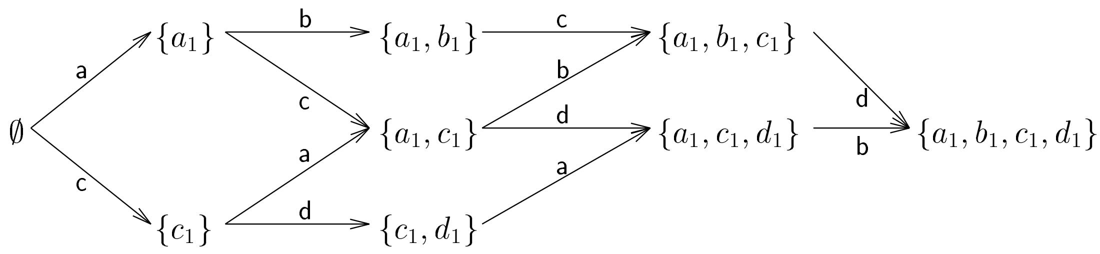


Figure 7: $\mathcal{C}_{Po(V)}$

In this example the linear extension of $Po(V)$ contains more elements then V. We have $LE(Po(V) = \{abcd, acbd, acdb, cabd, cadb, cdab\}$ and thus $V \subset LE(Po(V)$.

Let $V' = \{abcd, acbd, acdb, cabd, cadb, cdab\}$. Then $Po(V) = Po(V')$ and $V' = LE(Po(V')$. Thus there exist event-preserving sets which are equal to the linearizations of their partial order and there exist event-preserving sets which are a strict subset of the linearizations of their partial order. (See theorem 17)

If we have a path from the root to a configuration labelled with a word, then the configuration exists of all events of the word labelling the path from the root to the configuration.

Theorem 19 *Let A be an alphabet and $V \subseteq A^*$ an event-preserving set. Let $w \in A^*$ and $C \in Conf_{Po(V)}$. If $\emptyset \xrightarrow{w}^* C$ then $C = \{(a,i) | a \in A$ and $1 \leq i \leq |w|_a\} = ev(w)$.*

Proof:

Induction on $|w|$.

Let $|w| = 0$ then $w = \epsilon$ and $C = \emptyset$.

Suppose it has been proven for $0 \leq |w| \leq k$. Assume $|w| = k+1$. Then $w = w'b$ for some $w' \in A^*$ and $b \in A$ and there exists $C' \in Conf_{Po(V)}$ such that $\emptyset \xrightarrow{w'} C' \xrightarrow{b} C$. Then $|w'| = k$ and by the induction hypothesis $C' = \{(a,i) | a \in A$ and $1 \leq i \leq |w'|_a\}$. If $C' \xrightarrow{b} C$ then $C = C' \cup \{e\}$ where $l(e) = b$ and $e \notin C$. As C is left-closed, $e = (b, |w'|_b + 1)$ must hold. Thus $C = \{(a,i) | a \in A$ and $1 \leq i \leq |w|_a\}$. □

Theorem 20 *Let A be an alphabet and $V \subseteq A^*$ an event-preserving set. Then $Conf_{Po(V)} = \{ev(x) | x \in Prefix(LE(Po(V)))\}$.*

Proof:

Let x be a prefix of $w \in LE(Po(V))$. Since $ev(x) \subseteq ev(w) = ev(V)$ we only have to prove that $ev(x)$ is left-closed with respect to $\leq_V$. Then $ev(x) \in Conf_{Po(V)}$. So assume $e = (a,i) \in ev(x)$ and $e' = (b,j) \leq_V (a,i)$. Then $(b,j) \leq_w (a,i)$ and because x is a prefix of w containing the i-th occurrence of a, it also contains the j-th occurrence of b. Hence $(b, j) \in ev(x)$, and thus $ev(x)$ is left closed and an element of $Conf_{Po(V)}$.

Next we prove with induction on $|C|$ that for each $C \in Conf_{Po(V)}$ there exists $x \in Prefix(LE(Po(V)))$ such that $C = ev(x)$.
Let $|C| = 0$. Then $C = ev(\epsilon)$ and $\epsilon \in Prefix(LE(Po(V)))$.
Suppose the statement has been proved for all $C \in Conf_{Po(V)}$ with $|C| \leq k$ for some $k \geq 0$.
Assume $|C| = k+1$, thus $C \neq \emptyset$. By lemma 15 there exists $C' \in Conf_{Po(V)}$ such that $C' \stackrel{l_A(e)}{\longrightarrow} C$ for some $e \in ev(V)$ with $\{e\} = C \backslash C'$. By the induction hypothesis $C' = ev(x)$ for some $x \in Prefix(LE(Po(V)))$. Since C is left-closed $e = (l_A(e), |x|_{l_a(e)} + 1)$ and so $C = ev(xl_A(e))$. Let $l_A(e) = a$. As $x \in Prefix(LE(Po(V)))$ there exists $y \in A^*$ such that $xy \in LE(Po(V))$. Futhermore $ev(xa) \subseteq ev(V) = ev(xy)$ and so $y = y_1 a y_2$ for some $y_1, y_2 \in A^*$ with $a \notin alph(y_1)$. C is left-closed implies that $e = (a, |x|_a + 1) \parallel_{Po(V)} e'$ for all $e' \in ev(xy_1) \backslash ev(xa)$. Thus $xay_1y_2 \in LE(Po(V))$. This implies that $C = ev(xa)$ with $xa \in Prefix(LE(Po(V)))$. $\square$

Example continued

We have an event-preserving set $V = \{abcd, cabd, cadb\}$.
The set of linearizations $LE(Po(V)) = \{abcd, acbd, acdb, cabd, cadb, cdab\}$.
The set of congigurations $Conf_{Po(V)} = \{\emptyset, \{(a,1)\}, \{(c,1)\}, \{(a,1),(b,1)\}, \{(a,1),(c,1)\}, \{(c,1),(d,1)\} \{(a,1),(b,1),(c,1)\}, \{(a,1),(c,1),(d,1)\}, \{(a,1),(b,1),(c,1),(d,1)\}\}$.
It is clear that the configuration $\{(c,1),(d,1)\}$ is the set of events of a prefix of the linearization *cdab* of the partial order of V.

The configuration graph $\mathcal{C}_{Po(V)}$ has the forward diamond property and has the backward diamond property. This is stated in the following theorem.

Theorem 21 *Let A be an alphabet and $V \subseteq A^*$ an event-preserving set. Then $\mathcal{C}_{Po(V)}$ has the forward and backward diamond property.*

Proof:

First we prove that $\mathcal{C}_{Po(V)}$ has the forward diamond property. Suppose we have $C, C_1, C_2 \in Conf_{Po(V)}$ and $e_1, e_2 \in E_A$ such that $C \cup \{e_1\} = C_1$ and $C \cup \{e_2\} = C_2$. Then $C \stackrel{l_A(e_1)}{\longrightarrow} C_1$ and $C \stackrel{l_A(e_2)}{\longrightarrow} C_2$. Since $C_1 \cup C_2 \in Conf_{Po(V)}$ and $C_1 \cup C_2 = C \cup \{e_1, e_2\}$, we have $C_1 \stackrel{l_A(e_2)}{\longrightarrow} C'$ and $C_2 \stackrel{l_A(e_1)}{\longrightarrow} C'$, where $C' = C \cup \{e_1, e_2\}$. Next we prove that $\mathcal{C}_{Po(V)}$ has the backward diamond property. Suppose we have $C, C_1, C_2 \in Conf_{Po(V)}$ and $e_1, e_2 \in E_A$ such that $C_1 \cup \{e_1\} = C$ and $C_2 \cup \{e_2\} = C$. Then $C_1 \stackrel{l_A(e_1)}{\longrightarrow} C$ and $C_2 \stackrel{l_A(e_2)}{\longrightarrow} C$. Since $C_1 \cap C_2 \in Conf_{Po(V)}$ and $C_1 \cap C_2 = C \cap \{e_1, e_2\}$, we have $C' \stackrel{l_A(e_2)}{\longrightarrow} C_1$ and $C' \stackrel{l_A(e_1)}{\longrightarrow} C_2$, where $C' = C \cap \{e_1, e_2\}$. $\square$

In fact the configuration graph $\mathcal{C}_{Po(V)}$ is a distributive lattice, see [DP90].

5.3 Represented by partial orders

Bauget and Gastin identify in [BG95] congruences by means of (modular) representations by partial orders. The definitions are specified in the following way:

Definition

Let A be an alphabet and R an equivalence relation over A.

1. R can be *represented by a partial order* if R is event-preserving and $[w]_R = LE(Po([w]_R))$ for all $w \in A^*$;
2. R can be *modularly represented by a partial order* if it can be represented by a partial order and if for all $u, v \in A^*$
   (a) $\leq_{[uv]_R} \cap E_u \times E_u = \leq_{[u]_R}$,
   (b) $\leq_{[uv]_R} \cap t_u(E_v) \times t_u(E_v) = t_u(\leq_{[v]_R})$, and
   (c) $\leq_{[uv]_R} \cap t_u(E_v) \times E_u = \emptyset$.

Note that R is event-preserving, thus $Po([w]) = (E_w, \leq_{[w]}, l)$ is defined for $w \in A^*$.

When we have an equivalence which can be represented by partial orders and the concatenation of two partial orders exist only of the edges of the two partial orders and some edges between the events of the first partial order and the events of the second partial order, then we say that the equivalence can be modularly represented by partial orders.

The first condition of the definition, $\leq_{[uv]_R} \cap E_u \times E_u = \leq_{[u]_R}$, is the requirement that the concatenation of the partial orders contains the first partial order. The second condition is the requirement that the concatenation contains the second partial order. The condition $\leq_{[uv]_R} \cap t_u(E_v) \times E_u = \emptyset$ makes sure that there are no edges added which are from events of the second partial order to events of the first partial order. This condition is always satisfied as proven in the next lemma.

Lemma 22 *Let A be an alphabet and R be an equivalence relation such that R is event-preserving and R can be represented by partial orders.*
Then $\leq_{[uv]} \cap t_u(E_v) \times E_u = \emptyset$ for all $u, v \in A^*$.

Proof:

Suppose we have $u, v \in A^*$ such that $\leq_{[uv]} \cap t_u(E_v) \times E_u \neq \emptyset$. Then there exist $(a, i) \in t_u(E_v)$ and $(b, j) \in E_u$ such that $(a, i) \leq_{[uv]} (b, j)$. Then for all $w \in LE(Po([uv]))$ the i-th occurrence of a occurs before the j-th occurrence of b in w. But $(a, i) \in t_u(E_v)$ and $(b, j) \in E_u$. Thus when we consider the word uv we have $(b, j) \leq_{uv} (a, i)$. Contradiction. □

To illustrate that there exist equivalences which can not be represented by partial orders, the next example is given.

Example

Let $A = \{a, b, c\}$ and let R be the equivalence generated by $\{(aab, baa)\}$. Let $w = aaab$. Then $[w]_R = \{aaab, abaa\}$. The partial order of $[w]_R$ is depicted in figure 8.

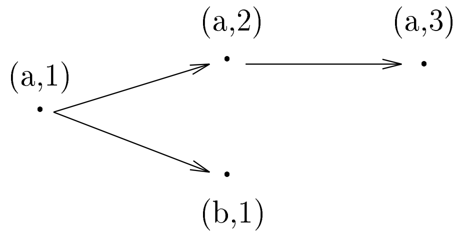


Figure 8: $Po([w]_R)$

The linearizations of the partial order contain the word $aaba$, which is not in $[w]_R$. Thus $[w] \subset LE(Po([w]_R))$.
R is event-preserving but $[w]_R \neq LE(Po([w]_R))$, thus R can not be represented by partial orders.

The next example contains an equivalence which can be represented by partial orders.

Example

Let $A = \{a, b, c, d\}$ and R is the congruence generated by $\{(ab, ba), (ac, ca)\}$. Let $v = acbac$ and $w = dab$. The partial orders of $[v]_R$ and $[w]_R$ are depicted in figure 9.

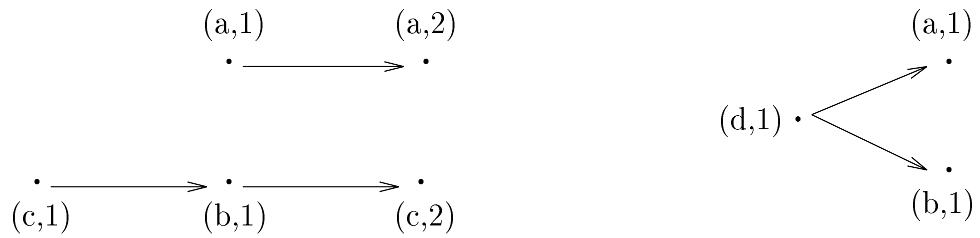


Figure 9: $Po([v]_R)$ and $Po([w]_R)$

As we will prove later in theorem 28 $[w]_R = LE(Po([w]_R))$ for all $p \in A^*/R$, thus R can be represented by partial orders. Let $u = acbacdab$, then $u = vw$. The partial order of $[u]_R$ is depicted in figure 10.

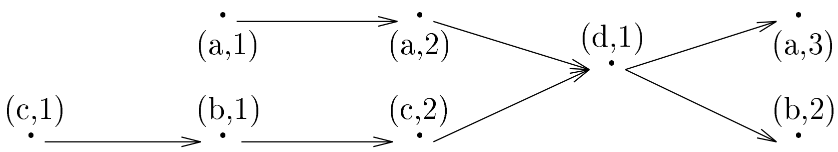


Figure 10: $Po([u]_R)$

When we evaluate the partial order of $u = vw$ then we can conclude that the partial order of $[u]_R$ is a disjoint union of the two partial orders of $[v]_R$ and $[w]_R$ and we have added some edges from $Po([v]_R)$ to $Po([w]_R)$. The equivalence relation R can be modularly represented by partial orders.

In the last example the concatenation of two partial orders is the partial order containing the two partial orders and some edges from the first to the second one. However there exist congruences which can be represented by partial orders and the concatenation of the partial orders is not defined as the union of the partial orders and adding some edges between the first and the second partial order. In this case the equivalence relation can not be modularly represented by partial orders. In the next example this is shown.

Example

Let $A = \{p, c\}$ and R is the congruence induced by $\{(ppc, pcp)\}$. Let $v = p$ and $w = cp$. The partial orders are depicted in figure 11.

Figure 11: $Po([v]_R)$ and $Po([w]_R)$

If we examine all equivalence classes we can conclude that $[w]_R = LE(Po([w]_R))$ for all $[w]_R \in A^*/R$, thus R can be represented by partial orders. Let $u = pcp$, then $[u]_R = \{pcp, ppc\}$. The partial order is depicted in figure 12.

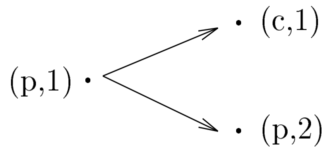


Figure 12: $Po([u]_R)$

It is clear that the partial order of $[u]_R$ does not include the partial order of $[w]_R$. Thus $\leq_{[pcp]} \cap t_p(E_{cp}) \times t_p(E_{cp}) \neq t_p(\leq_{[cp]})$. And therefore R can not be modularly represented by partial orders.

6 Mazurkiewicz traces

Let A be an alphabet and I a symmetric irreflexive binary relation (over A), I is an independence relation indicating that whenever a and b are independent then the sequential observations ab and ba are to be considered as equivalent observations of independently occurring a and b. This is the basis of the theory as developed by *Mazurkiewicz*, [M87].

6.1 Mazurkiewicz trace equivalence

Let A be an alphabet. A binary relation $I \subseteq A \times A$ is called an *independence relation over* A if it is symmetric and irreflexive. For an independence relation I over A we denote by C_I the *commutation relation* induced by I, which is defined by $C_I = \{(ab, ba) | (a, b) \in I\}$. Note that C_I is symmetric and whenever $(u, v) \in C_I$ then $|u| = |v| = 2$, $alph(u) = alph(v)$, and $u \neq v$.

Definition

Let A be an alphabet and I an independence relation over A. Let $x, y \in A^*$.

1. $x \doteq_I y$ if there exist $u, v \in A^*$ and $(a, b) \in I$ such that $x = uabv$ and $y = ubav$.
2. The *Mazurkiewicz trace equivalence* $\equiv_I$ is defined by $\equiv_I = (\doteq_I)^*$.
3. $\langle x \rangle_I = \{z \in A^* | \ z \equiv_I x\}$ the equivalence class of x, is the *Mtrace* (over I) containing x.

It is easy to see that the Mazurkiewicz trace equivalence $\equiv_I$ so defined is in fact a congruence over A^*. The congruence $\equiv_I$ is the least congruence containing C_I, the commutation relation induced by I.

Theorem 23 *Let A be an alphabet and I an independence relation over A. Then $\equiv_I = \equiv_{C_I}$.*

Proof:

From the definition we can conclude that $\doteq_I = (C_I)_{lr}$ and since $\doteq_I$ is symmetric it follows that $\equiv_I = (\doteq_I)^* = (\doteq_I \cup \doteq_I^{-1})^* = \equiv_{C_I}$. □

<u>Example</u>

Suppose we have $A = \{a, b, c\}$ and $I = \{(b, c), (c, b)\}$. If we only consider words with length less then 4, we have the following Mtraces:
$\langle \epsilon \rangle_I$, $\langle a \rangle_I$, $\langle b \rangle_I$, $\langle c \rangle_I$,
$\langle aa \rangle_I$, $\langle bb \rangle_I$, $\langle cc \rangle_I$, $\langle ab \rangle_I$, $\langle ac \rangle_I$, $\langle ba \rangle_I$, $\langle ca \rangle_I$, $\langle bc \rangle_I = \langle cb \rangle_I$,

$\langle aaa\rangle_I$, $\langle aab\rangle_I$, $\langle aac\rangle_I$, $\langle aba\rangle_I$, $\langle aca\rangle_I$, $\langle baa\rangle_I$, $\langle caa\rangle_I$,
$\langle bbb\rangle_I$, $\langle bba\rangle_I$, $\langle bab\rangle_I$, $\langle abb\rangle_I$,
$\langle ccc\rangle_I$, $\langle cca\rangle_I$, $\langle cac\rangle_I$, $\langle acc\rangle_I$, $\langle bac\rangle_I$, $\langle cab\rangle_I$,
$\langle abc\rangle_I = \langle acb\rangle_I$, $\langle bca\rangle_I = \langle cba\rangle_I$, $\langle bbc\rangle_I = \langle bcb\rangle_I = \langle cbb\rangle_I$, and
$\langle bcc\rangle_I = \langle bcb\rangle_I = \langle ccb\rangle_I$.
Notice that there are 5 equivalence classes which have two or more elements.

$\equiv_I$ is event-preserving and so $x \doteq_I y$ implies $ev(x) = ev(y)$. In lemma 24 this property is strengthened.

Lemma 24 *Let A be an alphabet and I an independence relation over A. Let $x, y \in A^*$. Then $x \equiv_I y$ if and only if $ev(x) = ev(y)$ and for all distinct $(a,i),(b,j) \in ev(x)$, $(a,i) \leq_x (b,j)$ and $(b,j) \leq_y (a,i)$ together imply $(a,b) \in I$.*

Proof:

First assume that $x \equiv_I y$ and let $x_0, \ldots, x_n \in A^*$ be such that $x_0 = x$ and $x_n = y$ and $x_{i-1} \doteq_I x_i$ for all $1 \leq i \leq n$.
If $n = 0$ then $x = y$ and the statement holds.
If $n = 1$ then $x \doteq_I y$. Thus there exist $u, v \in A^*$ and $(c,d) \in I$ such that $x = ucdv$ and $y = udcv$. Clearly $ev(x) = ev(y)$ and the only distinct pair $(a,i),(b,j)$ with $(a,i) \leq_x (b,j)$ and $(b,j) \leq_y (a,i)$ are $(a,i) = (c, |uc|_c)$ and $(b,j) = (d, |ud|_d)$. So the statement holds.
Now let $n \geq 2$ and assume that the conclusion holds for all $x_i \equiv_I x_{i+j}$ with $0 \leq i \leq i+j \leq n$ and $0 \leq j \leq n-1$. Since $x_{n-1} \doteq_I x_n$ we have $ev(x) = ev(x_{n-1}) = ev(y)$. Assume that distinct $(a,i),(b,j) \in ev(x)$ are such that $(a,i) \leq_x (b,j)$ and $(b,j) \leq_y (a,i)$. If $(b,j) \leq_{x_{n-1}} (a,i)$ then $(a,b) \in I$ by the induction hypothesis applied to $x_0 \equiv_I x_{n-1}$. If $(a,i) \leq_{x_{n-1}} (b,j)$ then also $(a,b) \in I$ by the induction hypothesis applied to $x_{n-1} \equiv_I x_n$.
Thus the statement holds.

Next, let $x, y \in A^*$ be such that $ev(x) = ev(y)$ and for all distinct $(a,i),(b,j) \in ev(x) : (a,i) \leq_x (b,j)$ and $(b,j) \leq_y (a,i)$ implies $(a,b) \in I$.
Let u be the longest common prefix of x and y. If $u = x = y$ then $x \equiv_I y$ by definition.
Assume that $x = uav$ and $y = ubw$ for some $v, w \in A^*$ and $a, b \in A$ with $a \neq b$. Since $ev(x) = ev(y)$ it follows that there exist $v_1, v_2 \in A^*$ such that $v = v_1 b v_2$ with $b \notin alph(v_1)$. Thus $x = uav_1bv_2$ and $y = ubw$ which implies that for all $c \in alph(av_1)$ we have $(b,c) \in I$. Hence $x = uav_1bv_2 \equiv_I uabv_1v_2 \equiv_I ubav_1v_2$. Since we have not disturbed the ordering of the events in av_1v_2 and in w and since the common prefix of $ubav_1v_2$ and $y = ubw$ is at least one symbol longer than u we can repeatedly apply a similar reasoning until we have reached y as the common prefix. This proves $x \equiv_I ubav_1v_2 \equiv_I y$ and thus the statement holds.

We can now conclude that the lemma holds. □

Having defined what it means for two observations to be equivalent we can prove that the defined congruence $\equiv_I$ is cancellative. To prove this we only need the earlier proven lemma 24.

Corollary *Let A be an alphabet and I an independence relation over A. Then $\equiv_I$ is cancellative.*
Proof:
Let $u, v, x, y \in A^*$ be such that $uxv \equiv_I uyv$. We have to prove that $x \equiv_I y$. From lemma 24 we know that $ev(uxv) = ev(uyv)$. It is easy to see that this implies $ev(x) = ev(y)$. Now consider two arbitrary but distinct events $(a, i), (b, j) \in ev(x)$ such that $(a, i) \leq_x (b, j)$ and $(b, j) \leq_y (a, i)$. Then $(a, i + |u|_a) \leq_{uxv} (b, j + |u|_b)$ and $(b, j + |u|_b) \leq_{uyv} (a, i + |u|_a)$ and thus by lemma 24 again $(a, b) \in I$. As (a, i) and (b, j) were arbitrarily chosen we can apply the if part of lemma 24 to conclude that $x \equiv_I y$. $\square$

From lemma 24 we conclude that each Mtrace is an event-preserving set. Thus each Mtrace $p \in A^*/\equiv_I$ defines a partial order $Po(p) = (ev(p), \leq_p, l_A)$, which is the labelled intersection of the occurrence total orders of the words in p.

Example continued

We had $A = \{a, b, c\}$ and $I = \{(b, c), (c, b)\}$. We can look at the partial order of each Mtrace. In this example we show the partial orders of some Mtraces. First we construct the Hasse diagram of $Po(\langle aba \rangle_I)$ depicted in figure 13).

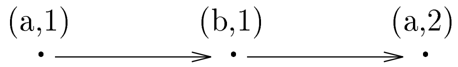


Figure 13: $Po(\langle aba \rangle_I)$

The equivalence class $\langle aba \rangle_I$ has only one element. The equivalence class $\langle abc \rangle_I$ has however 2 elements. To illustrate the differences in the partial order of these Mtraces the Hasse diagram of $Po(\langle abc \rangle_I)$ is drawn in figure 14.

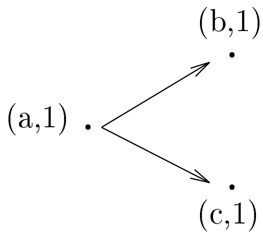


Figure 14: $Po(\langle abc \rangle_I)$

There are also equivalence classes with 3 elements. The last figure 15 has the Hasse diagram of the Mtrace $\langle bbc \rangle_I$, an equivalence class with 3 elements.

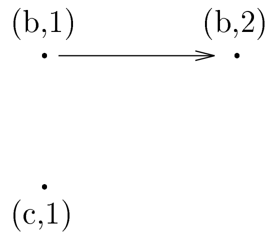


Figure 15: $Po(\langle bbc \rangle_I)$

As a corollary to lemma 24 we obtain the following relationship between the incomparable events in an Mtrace and its underlining independence relation.

Lemma 25 *Let A be an alphabet and I be an independence relation over A. Let p be a Mtrace and $(a, i), (b, j) \in ev(p)$.*
If $(a, i) \parallel_{Po(p)} (b, j)$ then $(a, b) \in I$.

Proof:

Suppose $(a, i), (b, j) \in ev(p)$ are such that $(a, i) \parallel_{Po(p)} (b, j)$. Then there exist $u, v \in p$ such that $(a, i) \leq_u (b, j)$ and $(b, j) \leq_v (a, i)$. Since $u, v \in p$ we have $u \equiv_I v$. Applying lemma 24 we can conclude that $(a, b) \in I$. □

The opposite of the lemma, $(a, b) \in I$ implies $(a, i) \parallel_{Po(p)} (b, j)$ for some i, j is however not true. This will be illustrated in the following example.

<u>Example continued</u>

Let $A = \{a, b, c\}$ be an alphabet and let $I = \{(b, c), (c, b)\}$.
For the Mtrace $p = \langle cab \rangle_I = \{cab\}$ the Hasse diagram of $Po(p)$ is pictured in figure 16.

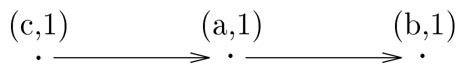


Figure 16: $Po(p)$

Thus $(b, c) \in I$ but $(b, 1) \parallel_{Po(p)} (c, 1)$ does not holds.

The next lemma shows that Mtraces are event-preserving sets which coincide with the linearizations of their labelled posets.

Lemma 26 *Let A be an alphabet and I be an independence relation. Let $u \in A^*$. Then $\langle u \rangle_I = LE(Po(\langle u \rangle_I))$.*

Proof:

As $\langle u \rangle_I \subseteq LE(Po(\langle u \rangle_I))$ always holds, by theorem 17, we only have to prove the converse inclusion. So let $x \in LE(Po(\langle u \rangle_I))$. Then $ev(x) = ev(u)$.

Assume that there are distinct $(a,i),(b,j) \in ev(x)$ such that $(a,i) \leq_x (b,j)$ and $(b,j) \leq_u (a,i)$. Then $(a,i) \parallel_{Po(\langle u \rangle_I)} (b,j)$ and hence by lemma 25 $(a,b) \in I$. Then by lemma 24 $x \equiv_I u$ and the inclusion is proven. Thus the statement holds.

This result can also be directly concluded from lemma 13. We know that $u \in LE(Po\langle u \rangle_I)$ and if $w \in \langle u \rangle_I$ then $u \equiv_{Po(\langle u \rangle_I)} w$. □

Thus when we have two observations which are linearizations of the same partial order of a Mtrace we can by lemma 26 conclude that these observations are in the same Mtrace and therefore are equivalent observations.

Lemma 27 *Let A be an alphabet and I be an independence relation over A. Let $u, v \in A^*$. $u \equiv_I v$ if and only if $Po(\langle u \rangle_I) = Po(\langle v \rangle_I)$.*

Proof:

The only-if direction is trivial, so we only have to prove that if $Po(\langle u \rangle_I) = Po(\langle v \rangle_I)$ then $u \equiv_I v$. By lemma 26 we know $LE(Po(\langle u \rangle_I)) = \langle u \rangle_I$ and $LE(Po(\langle v \rangle_I)) = \langle v \rangle_I$. Thus $Po(\langle u \rangle_I) = Po(\langle v \rangle_I)$ implies $\langle u \rangle_I = \langle v \rangle_I$. □

Note that $\equiv_I$ can be represented by partial orders since $\langle u \rangle_I = LE(Po(\langle u \rangle_I))$. Moreover Po is injective.

The property $\equiv_I$ can be represented by partial orders, from lemma 26, can be strengthened to the property $\equiv_I$ can be modularly represented by partial orders.

Theorem 28 *Let A be an alphabet and I an independence relation over A. $\equiv_I$ can be modularly represented by a partial order.*

Proof:

That $\equiv_I$ can be represented by partial orders follows from lemma 26. Then we have to prove the two properties. First we have $\leq_{\langle uv \rangle_I} \cap E_u \times E_u = \leq_{\langle u \rangle_I}$.

Let $w \in A^*$ such that $E_w = E_u$ and $wy \in \langle uv \rangle_I$ for some $x \in A^*$. Proof with induction on $|u| - |x|$, where x is the longest common prefix of u and w. Let $|u| - |x| = 0$, then $u = w$ and $w \in \langle u \rangle_I$.
Suppose it has been proven for $0 \leq |u| - |x| \leq k$. Assume $|u| - |x| = k+1$.
Let $w = xay_1$ and $u = xby_2$ for some $a, b \in A$ and $y_1, y_2 \in A^*$.
Note that $a \neq b$. We have $(a, |x|_a + 1) \parallel_{Po(\langle uv \rangle_I)} (b, |x|_b + 1)$ since $wy, uv \in \langle uv \rangle_I$. From lemma 25 follows $(a,b) \in I$. Thus there exist $w', u', y_3, y_3' \in A^*$ such that $u' = xaby_3 \in \langle u \rangle_I$ and $w' = xaby_3' \in \langle w \rangle_I$. The longest common prefix of u' and w' is xab and by the induction hypothesis we have $xaby_3' \in \langle u \rangle_I$. Therefore $\langle w \rangle_I = \langle u \rangle_I$.

Second we have to prove that $\leq_{\langle uv \rangle_I} \cap\, t_u(E_v) \times t_u(E_v) = t_u(\leq_{\langle v \rangle_I})$.
Let $w \in A^*$ such that $E_w = E_v$ and $yw \in \langle uv \rangle_I$ for some $x \in A^*$. Proof with induction on $|v| - |x|$, where x is the longest common suffix of u and w.
Let $|v| - |x| = 0$, then $v = w$ and $w \in \langle v \rangle_I$.
Suppose it has been proven for $0 \leq |v| - |x| \leq k$.

Assume $|v| - |x| = k + 1$. Let $w = y_1 a x$ and $v = y_2 b x$ for some $a, b \in A$ and $y_1, y_2 \in A^*$. Note that $a \neq b$. Since $yw, uv \in \langle uv \rangle_I$ we have $(a, |uv|_a - |x|_a) \,\|_{Po(\langle uv \rangle_I)}\, (b, |uv|_b - |x|_b)$. From lemma 25 follows $(a, b) \in I$. Thus there exist $w', u', y_3, y_3' \in A^*$ such that $w' = y_3' bax \in \langle w \rangle_I$ and $u' = y_3 bax \in \langle v \rangle_I$. The longest common suffix of w' and u' is bax and by the induction hypothesis we have $y_3' bax \in \langle v \rangle_I$. Therefore $\langle w \rangle_I = \langle v \rangle_I$.

We can conclude that the theorem holds. □

6.2 Prefix graphs

Definition

Let A be an alphabet and I be an independence relation over A. Let $x, y \in A^*$.

1. The *concatenation of two Mtraces* $\langle x \rangle_I$ and $\langle y \rangle_I$ is denoted by $\langle x \rangle_I \cdot \langle y \rangle_I$ and $\langle x \rangle_I \cdot \langle y \rangle_I = \langle xy \rangle_I$;
2. The *trace monoid* over A and I denoted by $M(A, I)$ is the quotient monoid $A^* / \equiv_I$, with concatenation $\cdot$ and unit $\langle \epsilon \rangle_I$;
3. The *prefix ordering* $\preceq_I$ on $M(A, I)$ is defined in the following way: $\langle x \rangle_I \preceq_I \langle y \rangle_I$ if there exists $w \in A^*$ such that $\langle x \rangle_I \cdot \langle w \rangle_I = \langle y \rangle_I$.

The operation $\cdot$ is well-defined. If $x' \in \langle x \rangle_I$ and $y' \in \langle y \rangle_I$ then, since $\equiv_I$ is a congruence, $xy \equiv_I x'y \equiv_I x'y'$ and thus $\langle xy \rangle_I = \langle x'y' \rangle_I$.
Note that $\cdot$ is associative and so $\langle x \rangle_I \cdot (\langle y \rangle_I \cdot \langle z \rangle_I) = \langle xyz \rangle_I = (\langle x \rangle_I \cdot \langle y \rangle_I) \cdot \langle z \rangle_I$.
We will usually omit $\cdot$ and simply write $\langle x \rangle_I \langle y \rangle_I$ rather than $\langle x \rangle_I \cdot \langle y \rangle_I$.
Since $\cdot$ is well-defined, the ordering $\preceq_I$ is also well-defined.

Note that $\langle \epsilon \rangle_I \preceq_I p$ for all $p \in M(A, I)$ and hence $\langle \epsilon \rangle_I$ is the least element of the poset $(M(A, I), \preceq_I)$.

<u>Example continued</u>

When we have $A = \{a, b, c\}$ and $I = \{(b, c), (c, b)\}$, as before, we can construct the Hasse diagram, figure 17, for the prefix-ordering. Because the set $M(A, I)$ is very large we depicted only the restriction of the set $M(A, I)$ to all words with length less then 4.

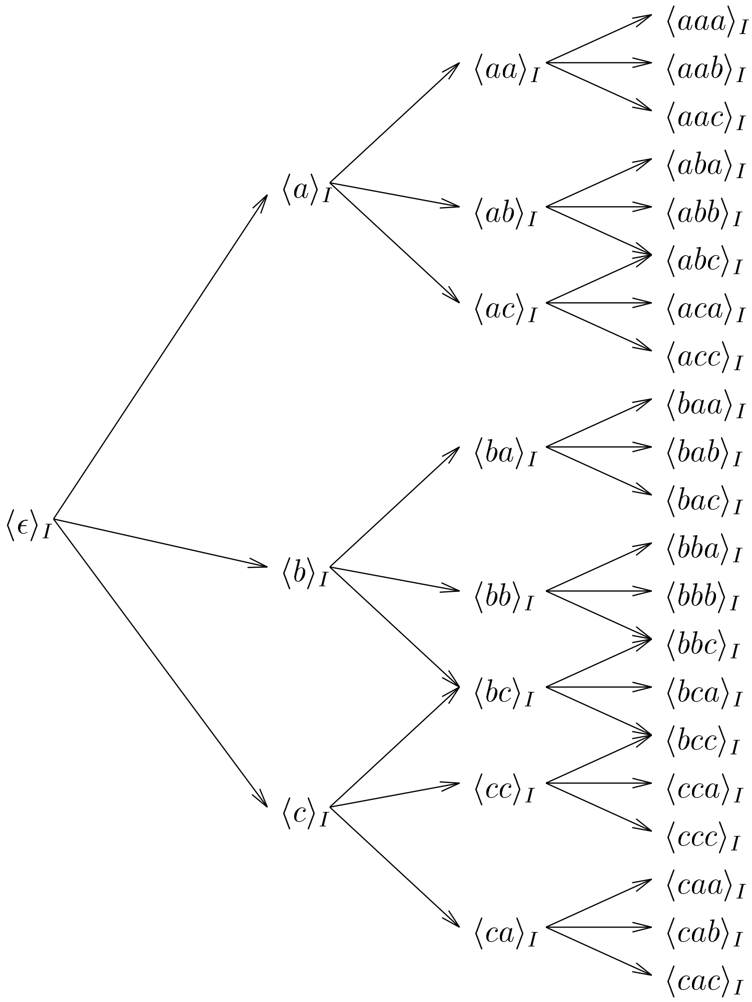


Figure 17: $\preceq_I$

With this prefix ordering we can define a graph with Mtraces for nodes and the prefix ordering defining the edges.

Definition

Let A be an alphabet and I be an independence relation over A.
Define the *prefix graph* of $M(A, I)$ as the elgraph with initial node
$G(A, I) = (M(A, I), A, \rightarrow_{\equiv_I}, \langle\epsilon\rangle_I)$, where
$\langle x\rangle_I \xrightarrow{a}_{\equiv_I} \langle y\rangle_I$ if $\langle x\rangle_I\langle a\rangle_I = \langle y\rangle_I$.

The next lemma shows that the relation $\rightarrow_{\equiv_I}$ has a strong connection with the ordering $\preceq$. This connection is in fact so strong that we can conclude that $p \prec\!\!\cdot_I\, q$ for two vertices of $G(A, I)$ implies $p \xrightarrow{a}_{\equiv_I} q$ for some $a \in A$ and vice versa.

Lemma 29 *Let A be an alphabet and I an independence relation over A. Let $p, q \in M(A, I)$. $p \prec\!\cdot_I q$ if and only if $p \xrightarrow{a}_{\equiv_I} q$ for some $a \in A$.*

Proof:

If $p \prec\!\cdot_I q$ then there exists $w \in A^*$ such that $p\langle w \rangle = q$. From the definition of $\prec\!\cdot_I$ follows that $w \neq \epsilon$ and $|w| \leq 1$. Then we know that there exists $a \in A$ such that $p \cdot \langle a \rangle_I = q$. By the definition of $\rightarrow_{\equiv_I}$ we have $p \xrightarrow{a}_{\equiv_I} q$.

If $p \xrightarrow{a}_{\equiv_I} q$ then $p \preceq_I q$ and $p \neq q$. Suppose $p \preceq_I z \prec_I q$ then there exists $v, w \in A^*$ such that $q = z\langle w \rangle_I = p\langle v \rangle_I \langle w \rangle_I = p\langle a \rangle_I$. Then there can be two situations. First $v = a$ and $w = \epsilon$. But then $z = q$ in contrast with $z \prec q$. The second situation is $v = \epsilon$ and $w = a$. Then $p = z$ and thus by definition $p \prec\!\cdot_I q$.

Thus the statement holds. □

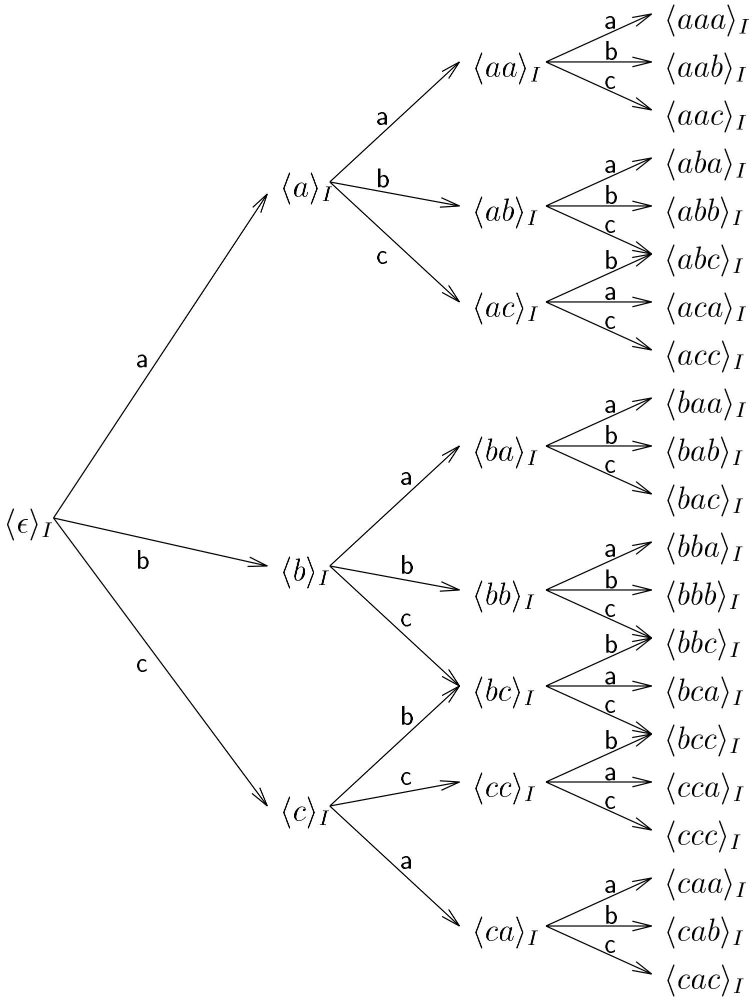


Figure 18: $G(A, I)$

Example continued

The prefix graph is depicted in figure 18. The only difference between figure 17 and figure 18 is that the edges are labelled. This resemblence is specified in lemma 29.

The defined prefix graph is a reldepgraph with root $\langle \epsilon \rangle_I$.

Theorem 30 *Let A be an alphabet and I be an independence relation over A. $G(A, I)$ is a reldepgraph.*

Proof:

As observed before, $\langle \epsilon \rangle_I$ is the least element of the poset $(M(A, I), \preceq)$. Since by repeated application of lemma 29, $p \preceq q$ if and only if $p \xrightarrow{w}^*_{\equiv_I} q$ for some $w \in A^*$, it now follows that $\langle \epsilon \rangle_I$ is a root of $G(A, I)$.

We have that $G(A, I)$ is deterministic since if $p \xrightarrow{a}_{\equiv_I} q$ and $p \xrightarrow{a}_{\equiv_I} q'$ for some $p, q, q' \in M(A, I)$ and $a \in A$ then $p\langle a \rangle_I = q$ and $p\langle a \rangle_I = q'$. Therefore $q = q'$.

Finally that $G(A, I)$ is event-preserving can be seen as follows. Suppose we have $p \xrightarrow{u}_{\equiv_I} q$ and $p \xrightarrow{v}_{\equiv_I} q$ for some $p, q \in M(A, I)$ and $u, v \in A^*$. Then $q = p\langle v \rangle_I$ and $q = p\langle u \rangle_I$. Since $\equiv_I$ is cancellative we know $\langle v \rangle_I = \langle u \rangle_I$. Since Mtraces are event preserving, $|u|_a = |v|_a$ for all $a \in A$. Hence $G(A, I)$ is event-preserving.

Thus we can conclude that $G(A, I)$ is a reldep-graph. □

6.3 Some properties of prefix graphs

Let A be an alphabet and I an independence relation over A.

If we investigate the prefix graph we can conclude that the prefix graph has certain properties. In this section these properties are proven. First we already know that the prefx graph is deterministic. But if we consider a vertex and we look at the labels of the edges which are directed to this vertex then we can conclude that the prefix graph is also co-deterministic. The explanation and thus the proof is very simple. The prefix graph is co-deterministic since $\equiv_I$ is cancellative. Thus if there are two edges labelled with a to a vertex, then the two vertices from which the edges are directed have to be the same congruence class and are therefore the same.

Theorem 31 *Let A be an alphabet and I be an independence relation over A. Then $G(A, I)$ is co-deterministic.*

Proof:

Suppose there are $\langle v \rangle_I, \langle w \rangle_I, p \in M(A, I)$ and $a \in A$ such that $\langle v \rangle_I \xrightarrow{a}_{\equiv_I} p$ and $\langle w \rangle_I \xrightarrow{a}_{\equiv_I} p$. Then we know $p = \langle va \rangle_I$ and $p = \langle wa \rangle_I$. Since $\equiv_I$ is cancellative we have $v \equiv w$. Thus the theorem holds. □

The next theorem shows that the prefix graph has the compatible forward diamond property. This property holds since we have an independence relation which has no requirements on the future or past of the executions. Then if from a vertex p there are edges labelled with a to q_1 and with b to q_2 and there exists a vertex p' such that q_1 and q_2 are before p', the events a and b have to be concurrent and there exist a vertex q such that the requirements for the compatible forward diamond property are satisfied.

Theorem 32 *Let A be an alphabet and I be an independence relation over A. Then $G(A, I)$ has the compatible forward diamond property.*

Proof:

Assume we have the nodes $\langle u \rangle_I, p \in M(A, I)$ and $a, b \in A$ with $a \neq b$ such that $\langle ua \rangle_I, \langle ub \rangle_I \in M(A, I)$ and $\langle uav \rangle_I = \langle ubw \rangle_I = p$ for some $v, w \in A^*$. From the if-proof of lemma 24 we know $uav_1bv_2 \equiv_I uabv_1v_2 \equiv_I ubav_1v_2$ for some $v_1, v_2 \in A^*$ with $b \notin alph(v_1)$. The Mazurkiewicz trace equivalence is cancellative thus if $uabv_1v_2 \equiv_I ubav_1v_2$ then $uab \equiv_I uba$ and $\langle uab \rangle_I \xrightarrow{v_1v_2}_{\equiv_I} p$. We can therefore conclude that $G(A, I)$ has the compatible forward diamond property. □

Example continued

In the prefix graph depicted before in figure 18 is easy to see that the prefix graph does not always have the forward diamond property.

The prefix graph has the backward diamond property. If there exists a vertex q to which two edges are directed labelled with a and b then we know that these events a and b are concurrent by the given independence relation. Thus then the requirements of the backward diamond property are satisfied.

Theorem 33 *Let A be an alphabet and I be an independence relation over A. Then $G(A, I)$ has the backward diamond property.*

Proof:

Suppose we have $p, \langle v \rangle_I, \langle w \rangle_I \in M(A, I)$ and $a, b \in A$ such that $\langle v \rangle_I \xrightarrow{a}_{\equiv_I} p$ and $\langle w \rangle_I \xrightarrow{b}_{\equiv_I} p$ where $a \neq b$. Thus $va \equiv_I wb$. Since $\equiv_I$ is event-preserving there exist $v_1, v_2 \in A^*$ such that $va = v_1bv_2a$ with $b \notin alph(v_2)$. By lemma 24 we know that $(b, c) \in I$ for all $c \in alph(v_2a)$.
Hence $va = v_1bv_2a \equiv_I v_1v_2ba \equiv_I v_1v_2ab \equiv_I wb$. Since $\equiv_I$ is cancellative we have $\langle v_1v_2 \rangle_I \xrightarrow{b}_{\equiv_I} \langle v \rangle_I$ and $\langle v_1v_2 \rangle_I \xrightarrow{a}_{\equiv_I} \langle w \rangle_I$. Hence $G(A, I)$ has the backward diamond property. □

The result of the next lemma is used in the proof of theorem 35.

Lemma 34 *Let A be an alphabet and I be an independence relation over A. Let $p, q \in M(A, I)$ and $a, b \in A$.*
If $\{ab, ba\} \subseteq Path_{p,q}(G(A, I))$ then $(a, b) \in I$.

Proof:

$\{ab, ba\} \subseteq Path_{p,q}(G(A, I)$ implies $p \xrightarrow{ab}{}^{*}_{\equiv_I} q$ and $p \xrightarrow{ba}{}^{*}_{\equiv_I} q$.
Thus $p\langle ab\rangle_I = q$ and $p\langle ba\rangle_I = q$. Suppose $u \in p$ for some $u \in A^*$, then we know that $uab \equiv_I uba$ and by lemma 24 $(a, b) \in I$. □

Even though the prefix graph does not have the forward diamond property, it satisfies the cube axiom. Because whenever a and b may occur independently of one another after a certain history then a and b can occur independently of one another after each other history of events.

Theorem 35 *Let A be an alphabet and I be an independence relation over A. Then $G(A, I)$ satisfies the cube axiom.*

Proof:

Suppose we have $p, p_1, p_2, q \in M(A, I)$ and $a, b, c \in A$ such that
$\{ab, ba\} \subseteq Path_{p,p_1}(G(A, I))$, $\{bc, cb\} \subseteq Path_{p,p_2}(G(A, I))$,
and $\{bac, bca\} \subseteq Path_{p,q}(G(A, I))$. By lemma 34 we know $(a, b), (b, c) \in I$.
If $\{bac, bca\} \subseteq Path_{p,q}(G(A, I))$ then $\{ac, ca\} \subseteq Path_{p\langle b\rangle_I,q}(G(A, I))$ and thus by lemma 34 $(a, c) \in I$.

If we know that $(a, c) \in I$ then $p\langle ac\rangle_I = p\langle ca\rangle_I$.

Therefore $\{ac, ca\} \subseteq Path_{p,p_3}(G(A, I))$ with $p_3 = p\langle ac\rangle_I$.

Since $(a, b), (a, c), (b, c) \in I$ we have $\langle abc\rangle_I = \{abc, acb, cba, cab\} \cup \{bac, bca\}$. Then $\{abc, acb, cba, cab\} \subseteq Path_{p,q}(G(A, I))$. □

It is easy to see that the prefix graph satisfies the inverse cube axiom as stated in the next theorem.

Theorem 36 *Let A be an alphabet and I be an independence relation over A. Then $G(A, I)$ satisfies the inverse cube axiom.*

Proof:

From theorem 9 and theorem 33 follows that $G(A, I)$ satisfies the inverse cube axiom. □

6.4 The relation between $G(A, I)(p)$ and $\mathcal{C}_{Po(p)}$

Let A be an alphabet and let I be an independence relation over A.
Let $p \in M(A, I)$. Define the set of prefixes of p, denoted by $Pref(p)$, as the set $Pref(p) = \{q \in M(A, I)|$ there exists $w \in A^*$ such that $q\langle w\rangle_I = p\}$.

The set $Pref(p)$ is the set consisting of those vertices of $M(A, I)$ which are before p in $G(A, I)$.

Lemma 37 *Let A be an alphabet and let I be an independence relation over A. Let $p \in M(A, I)$. Then $Pref(p) = Bef(p)$.*

Proof:

$Pref(p) = \{q \in M(A,I) |$ there exists $w \in A^*$ such that $q\langle w\rangle_I = p\}$.
For all $w \in A^*$ $q\langle w\rangle_I = p$ if and only if $q \xrightarrow{w}{}^*_{\equiv_I} p$.

Hence $Pref(p) = \{q \in M(A,I) |$ there exists a path from q to $p \in G(A,I)\}$.
Thus $Pref(p) = Bef(p)$. □

Thus $G(p) = (Bef(p), A, \longrightarrow_{\equiv_I} |_{Bef(p)\times A\times Bef(p)}, \langle\epsilon\rangle_I)$ can now be described as $G(p) = (Pref(p), A, \longrightarrow_{\equiv_I} |_{Pref(p)\times A\times Pref(p)}, \langle\epsilon\rangle_I)$ which is a reldepgraph.

Example continued

Let $A = \{a, b, c\}$ and $I = \{(b,c), (c,b)\}$, as before. In figure 19, 20, and 21 $G(A,I)(\langle aba\rangle_I)$, $G(A,I)(\langle abc\rangle_I)$, and $G(A,I)(\langle bbc\rangle_I)$ are drawn.

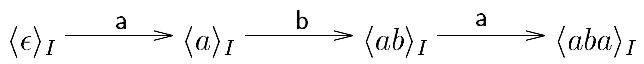


Figure 19: $G(A,I)(\langle aba\rangle_I)$

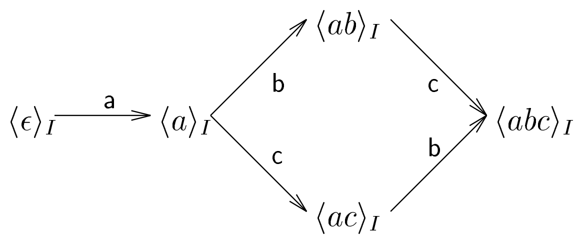


Figure 20: $G(A,I)(\langle abc\rangle_I)$

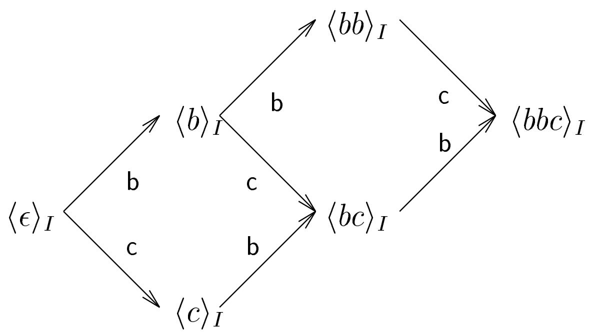


Figure 21: $G(A,I)(\langle bbc\rangle_I)$

All the reldepgraphs in the example have the forward diamond property. This property always holds for restrictions of $G(A,I)$ and is stated in the next theorem.

Theorem 38 *Let A be an alphabet and let I be an independence relation over A. Let $p \in M(A,I)$. Then $G(A,I)(p)$ has the forward diamond property.*

Proof:

We know that $G(A, I)$ has the compatible forward diamond property from theorem 32. From theorem 7 follows that $G(A, I)(p)$ has the forward diamond property. □

When we construct the configuration graph $\mathcal{C}_{Po(p)}$ for some $p \in M(A, I)$ we get a reldepgraph. The prefix graph $G(A, I)(p)$ is also a reldepgraph and we can compare these graphs by the translate function $\zeta_{\equiv_I}$, defined in section 3.4, restricted to the set $Pref(p)$. We denote this restriction by ζ_p.

Note that ζ_p is a total function mapping traces $\langle u \rangle_I \in Pref(p)$ to their set of events $ev(u) \subseteq E_A$. Since $\langle u \rangle_I \in Pref(p)$ implies that u is a prefix of some $v \in p$ we have by theorem 20 $\zeta_p(\langle u \rangle_I) = ev(u) \in Conf_{Po(p)}$.

Example continued

Let $A = \{a, b, c\}$ and $I = \{(b, c), (c, b)\}$, as before. We examine the restrictions of $G(A, I)$ to the Mtraces $\langle aba \rangle_I$, $\langle abc \rangle_I$, and $\langle bbc \rangle_I$. The posets of the Mtraces $\langle aba \rangle_I$, $\langle abc \rangle_I$, and $\langle bbc \rangle_I$ are already visualised in figures 13 to 15. The configuration graphs of the partial orders of these Mtraces are drawn in figure 22, 23, and 24.

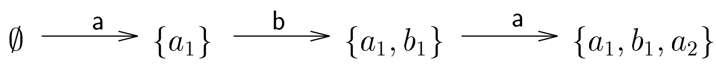


Figure 22: $\mathcal{C}_{Po(\langle aba \rangle_I)}$

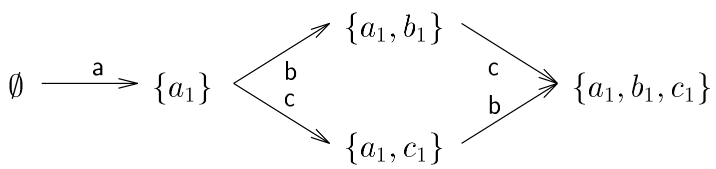


Figure 23: $\mathcal{C}_{Po(\langle abc \rangle_I)}$

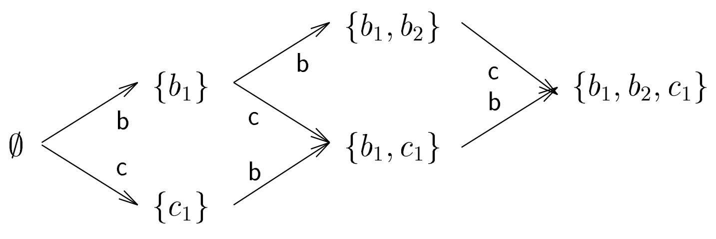


Figure 24: $\mathcal{C}_{Po(\langle bbc \rangle_I)}$

The total function $\zeta_p : Pref(p) \to Conf_{Po(p)}$ is the morphism from $G(A, I)(p)$ to $\mathcal{C}_{Po(p)}$ as proven in the next theorem.

Theorem 39 *Let A be an alphabet and I be an independence relation over A. Let $p \in M(A,I)$. ζ_p is the morphism from $G(A,I)(p)$ to $\mathcal{C}_{Po(p)}$.*

Proof:

We have to show that if $q \xrightarrow{a}_{\equiv_I} q'$ for some $q, q' \in M(A,I)$ and $a \in A$ then $\zeta_p(q) \xrightarrow{a} \zeta_p(q')$. We have $\zeta_p(\langle \epsilon \rangle_I) = \emptyset$ by the definition of ζ_p.

Suppose we have $q, q' \in M(A,I)$ and $a \in A$ such that $q \xrightarrow{a}_{\equiv_I} q'$. Then $q' = q\langle a \rangle_I$ and thus $ev(q') = ev(q) \cup \{(a, |q|_a + 1)\}$. Hence $\zeta_p(q) \xrightarrow{a} \zeta_p(q')$.

By theorem 10, ζ_p is the morphism from $G(A,I)(p)$ to $\mathcal{C}_{Po(p)}$. □

Finally we have the following theorem which shows that the restriction of a prefix graph to a Mtrace p is isomorphic with the configuration graph of the partial order of p.

Theorem 40 *Let A be an alphabet and I be an independence relation over A. Let $p \in M(A,I)$. Then $G(A,I)(p)$ is isomorphic with $\mathcal{C}_{Po(p)}$.*

Proof:

From lemma 39 we already know that ζ_p is the morphism from $G(A,I)(p)$ to $\mathcal{C}_{Po(p)}$. We have to show that ζ_p is bijective and full.

First we show that ζ_p is injective. Suppose we have $C \in Conf_{Po(p)}$ such that $C = \zeta_p(\langle v \rangle_I) = \zeta_p(\langle w \rangle_I)$ for some $w, v \in A^*$. Thus $ev(w) = ev(v)$. Suppose there are distinct $(a,i), (b,j) \in ev(v)$ such that $(a,i) \leq_{\langle v \rangle_I} (b,j)$ and $(b,j) \leq_{\langle w \rangle_I} (a,i)$. Since $\langle w \rangle_I, \langle v \rangle_I \in Pref(p)$ we have $(a,i) \parallel_{Po(p)} (b,j)$. From lemma 25 follows that $(a,b) \in I$. We can now use lemma 24 and conclude that $\langle w \rangle_I = \langle v \rangle_I$.

Next we prove that ζ_p is surjective.

Suppose $C \in Conf_{Po(p)}$. We will prove by induction on $|C|$ that there exists $q \in M(A,I)$ such that $C = \zeta_p(q)$.

Let $|C| = 0$. Then $C = \emptyset$ and by the definition of ζ_p we have $\emptyset = \zeta_p(\langle \epsilon \rangle_I)$.

Suppose it has been proven for all configurations $C \in Conf_{Po(p)}$, where $0 \leq |C| \leq k$.

Assume $|C| = k+1$, then there exist $C' \in Conf_{Po(t)}$ and $e \in E_A$ such that $C' \cup \{e\} = C$. Then $|C'| = k$ and by the induction hypothesis there exists $\langle u' \rangle_I \in M(A,I)$ such that $C' = \zeta_p(\langle u' \rangle_I)$. Let $\langle u \rangle_I = \langle u' \rangle_I \langle l_A(e) \rangle_I$, then

$\zeta_p(\langle u \rangle_I) = ev(u') \cup \{(l_A(e), |u'|_{l_A(e)} + 1)\}$

$= \zeta_p(\langle u' \rangle_I \cup \{(l_A(e), |u'|_{l_A(e)} + 1)\}$

$= C' \cup \{(l_A(e), |u'|_{l_A(e)} + 1)\}$. Thus $\zeta_p(\langle u \rangle_I) = C$.

At last we prove that ζ_p is full. Assume $C' = \zeta_p(\langle u \rangle_I)$ and $C' \xrightarrow{a} C$. Then $a = l_A(e)$, $e \notin C'$ and $C = C' \cup \{e\}$ for some $e \in E_A$.

$\zeta_p(\langle ua \rangle_I) = ev(ua) = ev(u) \cup \{(a, |u|_a + 1)\} = C' \cup \{(a, |u|_a + 1)\}$. From lemma 19 follows $\zeta_p(\langle ua \rangle_I) = C$.

Having proved that ζ_p is bijective and full, we know that $G(A,I)(p)$ is isomorphic with $\mathcal{C}_{Po(p)}$. □

6.5 Conclusions

The trace theory developed by Mazurkiewicz is very special since the Mazurkiewicz trace equivalence can be modularly represented by partial orders, the prefix graph has the compatible forward and backward diamond property and the configuration graph of a partial order of a vertex is isomorph to the restriction of the prefix graph to this vertex.

The property *Mazurkiewicz trace equivalences can be a modularly represented by a partial order* always holds when *the prefix graph restricted to a vertex and the configuration graph of the partial order of the vertex are isomorphic.* This because we know that $p = Path_{max}(G(A, I)(p)) = Path_{max}(\mathcal{C}_{Po(p)}) = LE(Po(p))$. Thus the most important result of this section is theorem 40.

Now we can examine in the next section the generalizations of the Mazurkiewicz trace theory.

7 Generalization of traces

In this section we consider two different generalizations of Mtraces. In the first subsection the generalization described by Biermann and Rozoy in [BR95] is handled. The requirements are a dependency on the past, which results in a congruence on the right, and an event-preserving equivalence.

The other generalization, by Bauget and Gastin in [BG95], requires an event-preserving equivalence but now the dependency on the past is more restricted, which results in a congruence. This all is in the second subsection, section 7.2.

7.1 Congruence on the right

Biermann and Rozoy describe in [BR95] a theory in which possible permutations of events depend only on their past. This leads to a relation which contains equivalent executions, represented by words. This relation is not a binary relation over A, but a binary relation over A^*. As in the Mtrace theory it is required that the relation is event-preserving. In other words the resulting *c*ongruence on the *r*ight has to *p*reserve the *o*ccurrences of the letters. This crop trace equivalence is defined in subsubsection 7.1.1. Because we have a congruence on the right we have not a trace monoid, but in subsubsection 7.1.3 follows the definition of quasi-concatenation and quasi-prefixes. In the next subsubsection the definition of the quasi-prefix graph follows. After the restriction of the quasi-prefix graph we compare two different ways to get the information about the concurrency of events in a crop trace in subsubsection 7.1.4 and subsubsection 7.1.5.

7.1.1 Crop trace equivalence

Definition

Let A be an alphabet and $R \subseteq A^* \times A^*$ an event-preserving relation. Let $x, y \in A^*$.

1. The *crop trace equivalence* induced by R is $\approx_R$, the right-congruence induced by R.
2. $[x\rangle_R = \{z \in A^* |\ z \approx_R x\}$ the equivalence class containing x, is the *crop trace* containing x.

Note that crop trace equivalences are event-preserving. Moreover we write $[x\rangle$ if R is clear.

7.1.2 Quasi-concatenation and quasi-prefixes

Let A be an alphabet and R an event-preserving relation over A^*. In the last section we defined the crop trace equivalence $\approx_R$. If we investigate $A^*/\approx_R$ we find that $x \approx_R x'$ does not necessary imply that $[yx\rangle_R = [yx'\rangle_R$ for all $y \in A^*$. Implying that concatenation of crop traces is not well-defined by $[y\rangle_R[x\rangle_R = [yx\rangle_R$.

Example

Suppose we have $A = \{a, b, c\}$ and $R = \{(bc, cb)\}$.
Let $p = [bc\rangle = \{bc, cb\}$, $x = bc$, $x' = cb$, and $q = [a\rangle$.
Then $x, x' \in p$ and $q[x\rangle \neq q[x'\rangle$ since $abc \not\approx_R acb$.
But $[x\rangle q = [x'\rangle q$ since $bca \approx_R cba$. The last equation is always valid since$\approx_R$ is a congruence on the right.

Thus we have no induced concatenation in $A^*/\approx_R$. But since $\approx_R$ is a congruence on the right, we can define a *quasi-concatenation*, denoted by $\Diamond$. This quasi-concatenation is defined in the following way: $[x\rangle_R \Diamond v = [xv\rangle_R$ for all $x, v \in A^*$. Now the operation $\Diamond$ is well-defined. We will write $[x\rangle_R \Diamond v \Diamond w$ instead of $([x\rangle_R \Diamond v)\Diamond w$.

Lemma 41 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$ and $u, v \in A^*$.*
Then $p\Diamond u \Diamond v = p \Diamond uv$.

Proof:

Suppose $p = [x\rangle_R$ for some word $x \in A^*$. Then $p\Diamond u\Diamond v = [x\rangle_R \Diamond u \Diamond v$.
We have $[x\rangle_R\Diamond u\Diamond v = ([x\rangle_R \Diamond u)\Diamond v = [xu\rangle_R \Diamond v = [xuv\rangle_R = [x\rangle_R \Diamond uv$.
Therefore $p\Diamond u \Diamond v = p\Diamond uv$ holds. $\Box$

Now we have defined a quasi-concatenation, we can also define a set with prefixes and an ordering between the crop traces.

Definition

Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$.

1. The *set of quasi-prefixes of* p, denoted by $Pre(p)$, is defined by
   $Pre(p) = \{q \in A^*/\approx_R \mid \exists w \in A^* : q\Diamond w = p\}$;
2. The *quasi-prefix ordering* denoted by $\preceq_R$, is defined in the following way $[x\rangle_R \preceq_R [y\rangle_R$ if there exists $w \in A^*$ such that $[x\rangle_R \Diamond w = [y\rangle_R$.

Note that $[\epsilon\rangle_R \preceq p$ for all $p \in A^*/\approx_R$ and hence $[\epsilon\rangle_R$ is the least element of the poset $(A^*/\approx_R, \preceq_R)$.

The defined quasi-prefix ordering is like the prefix ordering a partial order, which is stated in the following lemma.

Lemma 42 *Let A be an alphabet and R an event-preserving relation over A^*. The ordering $\preceq_R$ is a partial order.*

Proof:

First we prove that the ordering $\preceq_R$ is reflexive.
Suppose $p \in A^*/\approx_R$ then we have $p \preceq_R p$ since $p\Diamond\epsilon = p$.

Futher we have to show that $\preceq_R$ is anti-symmetric.
Suppose we have $p, q \in A^*/\approx_R$ such that $p \preceq_R q$ and $q \preceq_R p$.
Then there exist $u, v \in A^*$ such that $p\Diamond u = q$ and $q\Diamond v = p$.
Hence $u = v = \epsilon$ and $p = q$.

The last thing we have to prove is the transitivity of $\preceq_R$.
Assume we have $p_1 \preceq_R p_2$ and $p_2 \preceq_R p_3$ for some $p_1, p_2, p_3 \in A^*/\approx_R$.
Then there exist $u, v \in A^*$ such that $p_1\Diamond u = p_2$ and $p_2\Diamond v = p_3$.
Hence $p_1\Diamond(uv) = p_3$ and $p_1 \preceq_R p_3$. □

7.1.3 Quasi-prefix graphs

Let A be an alphabet and R an event-preserving relation over A^*. Using $\preceq_R$ we can define a graph with the crop traces from $A^*/\approx_R$ as nodes and the quasi-prefix ordering between these crop traces as edges.

Definition

Let A be an alphabet and R an event-preserving relation over A^*. Define the *quasi-prefix graph* of $A^*/\approx_R$ as the elgraph with initial node $G(A, R_r) = (A^*/\approx_R, A, \rightarrow_{\approx_R}, [\epsilon\rangle_R)$, where $[u\rangle_R \xrightarrow{a}_{\approx_R} [v\rangle_R$ for some $u, v \in A^*$ and $a \in A$ if $[u\rangle_R\Diamond a = [v\rangle_R$.

The quasi-prefix ordering has a strong relation with the function $\rightarrow_{\approx_R}$.

Lemma 43 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p, q \in A^*/\approx_R$. Then $p \prec\!\cdot_R q$ if and only if $q \xrightarrow{a}_{\approx_R} q$ for some $a \in A$.*

Proof:

If $p \prec\!\cdot_R q$ then there exists $w \in A^*$ such that $p\Diamond w = q$. From the definition of $\prec\!\cdot_R$ follows that $w \neq \epsilon$ and $|w| \leq 1$. Then we know that there exists $a \in A$ such that $p\Diamond a = q$. From the definition of $\rightarrow_{\approx_R}$ follows that $p \xrightarrow{a}_{\approx_R} q$.

If $p \xrightarrow{a}_{\approx_R} q$ then $p \preceq q$ and $p \neq q$. Suppose $p \preceq z \prec q$ for some $z \in A^*/\approx_R$ then there exist $v, w \in A^*$ such that $q = z\Diamond w = p\Diamond v\Diamond w = p\Diamond a$. Then there can be two situations. First $v = a$ and $w = \epsilon$. But then $z = q$ and $z \prec q$, a contradiction. Then the second situation. Suppose $v = \epsilon$ and $w = a$. Then $p = z$ and $p \prec\!\cdot q$ follows. □

We can prove the next theorem which states that the quasi-prefix graph is a reldepgraph with root $[\epsilon\rangle_R$.

Theorem 44 *Let A be an alphabet and R an event-preserving relation over A^*. $G(A, R_r)$ is a reldepgraph.*

Proof:

As observed before, $[\epsilon\rangle_R$ is the least element of the poset $(A^*/\approx_R, \preceq_R)$. Since by repeated application of lemma 43, $p \preceq q$ if and only if $p \xrightarrow{w}^*_{\approx_R} q$ for some $w \in A^*$. It now follows that $[\epsilon\rangle_R$ is a root of $G(A, R_r)$.

We also have to show that the graph $G(A, R_r)$ is deterministic. Suppose we have $p_1, p_2, p_3 \in A^*/\approx_R$ such that $p_1 \xrightarrow{a}_{\approx_R} p_2$ and $p_1 \xrightarrow{a}_{\approx_R} p_3$ for some $a \in A$. Then $p_2 = p_1 \Diamond a = p_3$.

If $p \xrightarrow{u}^*_{\approx_R} q$ and $p \xrightarrow{v}^*_{\approx_R} q$ then $q = p \Diamond u = p \Diamond v$. Since $\approx_R$ is event-preserving we have $|u|_a = |v|_a$ for all $a \in A$, hence $G(A, R_r)$ is event-preserving.

Since $G(A, R_r)$ is rooted, deterministic, and event-preserving, the elgraph $G(A, R_r)$ with initial node is a reldepgraph. □

7.1.4 The relation between $G(A, R_r)(p)$ and $\mathcal{C}_{Po(p)}$

We now investigate the restriction of the quasi-prefix graph. First we prove that the set of quasi-prefixes of a crop trace coincides with the set of all the crop traces before this crop trace.

Lemma 45 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$. Then $Pre(p) = Bef(p)$.*

Proof:

Assume we have $y \in A^*/\approx_R$ such that $y \in Pre(p)$. Then we know there exists $u \in A^*$ such that $y \Diamond u = p$. Then $y \xrightarrow{u}_{\approx_R} p$ and thus there exists a path labelled with u from y to p. Therefore $y \in Bef(p)$.

Assume we have $y \in A^*/\approx_R$ such that $y \in Bef(p)$. Then we know there exists a path from y to p. Suppose this path is labelled with u, where $u \in A^*$. Then $y \xrightarrow{u}^*_{\approx_R} p$ and thus $y \Diamond u = p$. Therefore $y \in Pre(p)$.

We can now conclude that the sets $Bef(p)$ and $Pre(p)$ are equal. □

Note that $G(A, R_r)(p) = (Bef(p), A, \rightarrow_{\approx_R}|_{Bef(p) \times A \times Bef(p)}, [\epsilon\rangle_R)$ can also be written as $G(A, R_r)(p) = (Pre(p), A, \rightarrow_{\approx_R}|_{Pre(p) \times A \times Pre(p)}, [\epsilon\rangle_R)$.

Next we show that all words labelling the paths leading from the root to the leaf together form this crop trace.

Theorem 46 *Let A be an alphabet and R an event-preserving relation over A^*. let $[p\rangle \in A^*/\approx_R$. Then $Path_{max}(G(A, R_r)([p\rangle)) = [p\rangle$.*

Proof:

For all $w \in A^*$, if $[\epsilon\rangle_R \xrightarrow{w}_{\approx_R} [p\rangle$ then by repeatedly applying lemma 43 $[p\rangle = [\epsilon\rangle_R \Diamond w = [w\rangle_R$. Thus if $w \in Path_{max}(G(A, R_r)([p\rangle))$ then $w \in [p\rangle$.

By induction on $|p|$ we prove that if $w \in [p\rangle$ then $w \in Path_{maz}(G(A, R_r)([p\rangle))$. Let $|p| = 0$, then $p = \epsilon$ and $w = \epsilon$. Then $\epsilon \in Path_{maz}(G(A, R_r)([p\rangle))$.
Suppose it has been proven for $0 \leq |p| \leq k$.
Assume $|p| = k+1$ and $w \in [p\rangle$. Then $|w| = k+1$, thus there exist $w' \in A^*$ and $a \in A$ such that $w = w'a$. Suppose $w' \in [p'\rangle$. $|p'| = k$ and by the induction hypothesis we have $w' \in Path_{max}(G(A, R_r)([p'\rangle))$. Since $w' \in [p'\rangle$ and $w'a \in [p\rangle$ we have $[p'\rangle \xrightarrow{a}_{\approx_R} [p\rangle$. Therefore $w \in Path_{max}(G(A, R_r)([p\rangle))$. $\Box$

We know that the crop trace equivalence induced by R is event-preserving, thus we can construct a partial order of crop traces, as defined in subsection 5.2. Then by theorem 17 we have the following corollar.

Corollary *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$. Then $p \subseteq LE(Po(p))$.*

This is the first main difference with the Mazurkiewicz trace theory. If p is a Mtrace then $p = LE(Po(p))$, lemma 26. In the next example we show that there are crop trace equivalences such that $p \subset LE(Po(p))$.

Example

Let $A = \{a, b, c\}$ and $R = \{(abc, cba)\}$. Suppose $p = [abc\rangle_R$. The partial order of p is depicted in figure 25.

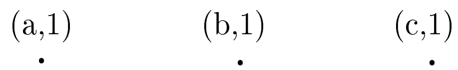


Figure 25: $Po([abc\langle_R)$

Then $LE(Po(p)) = \{abc, acb, bac, bca, cab, cba\}$ and $p = \{abc, cba\}$.

From the partial order of a crop trace, $Po(p)$, we construct the configuration graph $\mathcal{C}_{Po(p)}$. The two graphs, $G(A, R_r)(p)$ and $\mathcal{C}_{Po(p)}$, can be compared using the translate function ζ_p, which is $\zeta_{\equiv_R}$, defined in section 3.4, restricted to $Bef(p)$.

Theorem 47 *Let A be an alphabet and R be an event-preserving relation over A^*. Let $p \in A^*/\equiv_R$. ζ_p is the morphism from $G(A, R_r)(p)$ to $\mathcal{C}_{Po(p)}$.*

Proof:

First we show that ζ_p is a total function. Suppose we have $[u\rangle_R \in Pre(p)$. Then there exist $v, w \in A^*$ such that $uw = v \in p$. By theorem 20 we have $\zeta_p([u\rangle_R) = ev(u) \in Conf_{Po(p)}$. Thus for all $q \in Pre(p)$ we have $\zeta_p(q) = ev(q) \in Conf_{Po(p)}$. Hence ζ_p is a total function from $Pre(p)$ to $Conf_{Po(p)}$.

Now we have to prove that ζ_p is a morphism. We have $\zeta_p([\epsilon\rangle_R) = \emptyset$ by the definition of ζ_p.

Suppose we have $q, q' \in A^*/\equiv_R$ and $a \in A$ such that $q \xrightarrow{a}_{\equiv_R} q'$. Then $q' = q\Diamond a$ and thus $ev(q') = ev(q) \cup \{(a, |q|_a + 1)\}$. Hence $\zeta_p(q) \xrightarrow{a} \zeta_p(q')$.

By theorem 10, ζ_p is the morphism from $G(A, R_r)(p)$ to $\mathcal{C}_{Po(p)}$. □

From the last corollary we know that $Po(p)$ has the property that p is contained in the set of linearizations. In contrast to the case of Mtraces however this inclusion is in general strict. Biermann and Rozoy examined the situation in which the crop trace p equals the linearizations of the labelled poset $Po(p)$, from [BR95].

Theorem 48 *[BR95] Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$. If $p = LE(Po(p))$ then ζ_p is surjective and full.*

Proof:

Since $p = Le(Po(p)$, theorem 20 implies that $Conf_{Po(p)} = \{ev(x) | x \in Prefix(p)\} = \{\zeta_p([u\rangle_R) | u \in Pre(p)\}$. Hence ζ_p is surjective.

Assume $C \xrightarrow{a} C'$ for some $C, C' \in Conf_{Po(p)}$. Then $C = ev([u\rangle_R)$ and $C' = ev([ua\rangle_R)$ for some $[u\rangle_R \in Pre(p)$. By definition $[u\rangle_R \xrightarrow{a}_{\approx_R} [ua\rangle_R$.

Now we have proven that ζ_p is surjective and full if $p = LE(Po(p))$. □

In the first example we show that the theorem holds and cannot be strengthened.

Example

Let $A = \{a, b, c\}$ and $R = \{(ac, ca), (bc, cb), (bac, bca), (abc, acb), (cab, cba)\}$. Let $p = [abc\rangle_R = \{abc, acb, cab, cba, bca, bac\}$. It is easy to see that there exists no ordering between the events of p. The set of linearizations of the partial order of p is equal to p. The quasi-prefix graph restricted to p and the configuration graph of the partial order of p are depicted in figure 26.

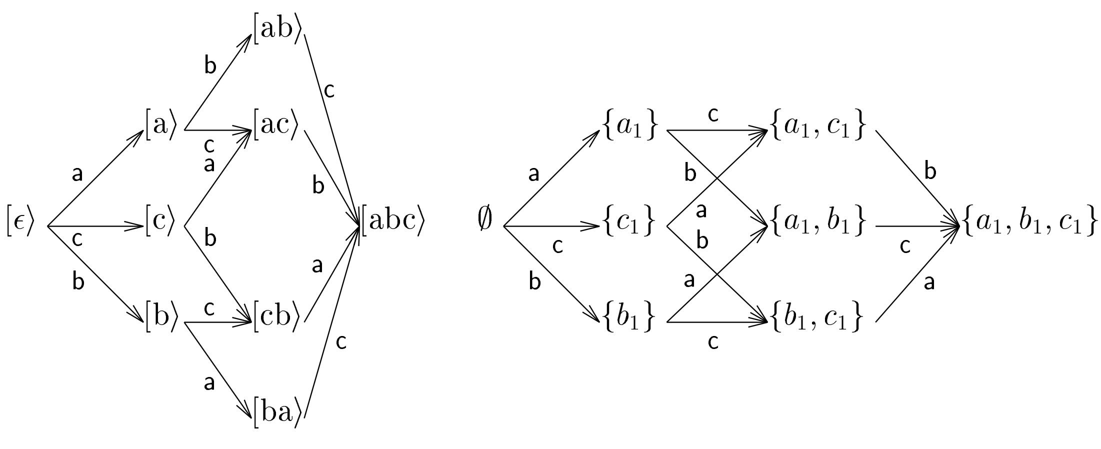


Figure 26: $G(A, R_r)(p)$ and $\mathcal{C}_{Po(p)}$

When we investigate these graphs we can conclude that ζ_p is surjective, full, and not injective.

The converse of the theorem does not hold. The next example illustrates this.

Example

Let $A = \{a, b\}$ and $R = \{(abab, baba)\}$. Let $p = [abab\rangle_R = \{abab, baba\}$. The quasi-prefix graph restricted to the crop trace p and the Hasse diagram of the partial order of p are depicted in figure 27 and 28.

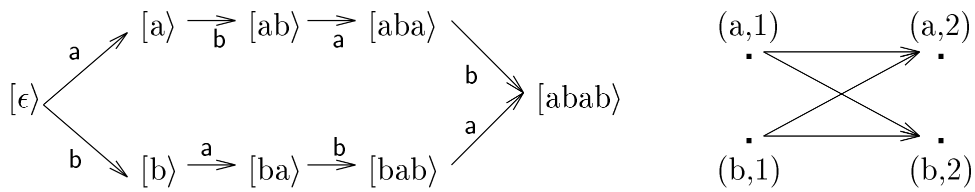


Figure 27: $G(A, R_r)(p)$ and $Po(p)$

The configuration graph of the partial order of p is depicted in figure 28. The function ζ_p is surjective and full, but p is a strict subset of the set of linearizations of the partial order of p.

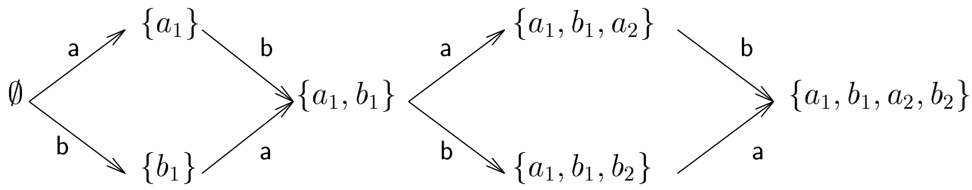


Figure 28: $\mathcal{C}_{Po(p)}$

Next we consider the case that ζ_p is injective. Then the conclusion $G(A, R_r)(p)$ is co-deterministic is easy.

Theorem 49 *[BR95] Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$. If ζ_p is injective then $G(A, R_r)(p)$ is co-deterministic.*

Proof:

Suppose ζ_p is injective and we have $[w_1\rangle_R \xrightarrow{a}_{\approx_R} [w\rangle_R$ and $[w_2\rangle_R \xrightarrow{a}_{\approx_R} [w\rangle_R$ for some $w, w_1, w_2 \in A^*$ and $a \in A$.

Then $\zeta_p([w_1\rangle_R) = \zeta_p([w_2\rangle_R) = \zeta_p([w\rangle_R)\backslash\{(a, |w|_a)\}$. ζ_p is injective thus $[w_1\rangle_R = [w_2\rangle_R$. Therefore $G(A, R_r)(p)$ is co-deterministic. □

First an example is given to illustrate the theorem.

Example

Let $A = \{a, b, c\}$ and $R = \{abc, cab\}$. Let $p = [abc\rangle = \{abc, cab\}$. The quasi-prefix graph, the Hasse diagram of the partial order of p and the configuration graph of the partial order of p are depicted in figure 29.

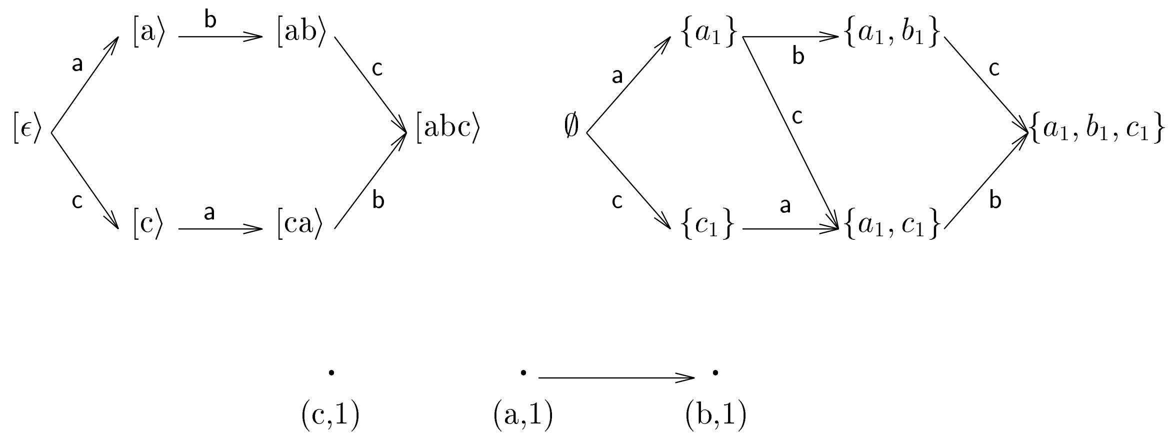


Figure 29: $G(A, R_r)(p)$, $Po(p)$ and $\mathcal{C}_{Po(p)}$

Then $p = \{abc, cab\} \subset LE(Po(p)) = \{abc, cab, acb\}$, ζ_p is injective (and surjective). The quasi-prefix graph has not the forward nor the backward diamond property and $G(A, R_r)(p)$ is co-deterministic.

The next example illustrates the fact that the converse of the theorem does not hold.

Example

Let $A = \{a, b\}$ and $R = \{(abab, baba)\}$. $G(A, R_r)([abab\rangle)$, $Po([abab\rangle)$ are depicted in figure 27 and $\mathcal{C}_{Po([abab\rangle)}$ in figure 28. We have $G(A, R_r)([abab\rangle)$ is co-deterministic, but $\zeta_{[abab\rangle}$ is not injective.

Next we prove that if $G(A, R_r)(p)$ has the forward or backward diamond property, then the function ζ_p is injective and full. Theorem 13 from [BR95]. The two situations, $G(A, R_r)(p)$ has the forward diamond property and $G(A, R_r)(p)$ has the backward diamond property, are divided into two cases. First we consider the case $G(A, R_r)(p)$ has the forward diamond property. Before we can prove in theorem 55 that $G(A, R_r)(p)$ having the forward diamond property implies that ζ_p is injective and full, we prove some lemmas which are needed in the proof of the theorem.

In the first lemma we prove that, if $G(A, R_r)(p)$ has the forward diamond property and if we know that there exists a path from vertex x to y, which contains the letter a, and there exists a path labelled with a from x to a vertex z then there exists a path from z to y. Further we know that if the path from x via z to y is labelled with aw' and the path from x to y is labelled with w then aw' is a permutation of w. In figure 30 this situation and the situation in the proof of this lemma is visualized.

Lemma 50 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$. If $FD(G(A, R_r)(p))$ then*

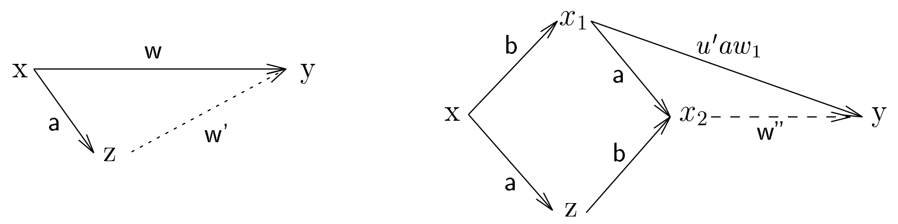


Figure 30: Visualization of lemma 50

for all $x, y, z \in Pre(p)$, $w \in A^*$, *and* $a \in A$
if $x \xrightarrow{w}^*_{\approx_R} y$, $x \xrightarrow{a}_{\approx_R} z$, *and* $|w|_a \geq 1$ *then there exists* $w' \in A^*$ *such that* $z \xrightarrow{w'}^*_{\approx_R} y$ *and* aw' *is a permutation of* w.

Proof:

Induction on $|u|$, where u is the least prefix of w such that $|u|_a = 1$.
Let $|u| = 1$. Then $w = aw''$ for some $w'' \in A^*$. $G(A, R_r)(p)$ is deterministic thus $z \xrightarrow{w''}^*_{\approx_R} y$. Suppose the statement has been proven for all paths labelled with w such that the length of the least prefix of w containing a is $\leq k$. Assume $|u| = k + 1$. Then there exist $w_1, v \in A^*$ such that $w = vaw_1$. Suppose the first letter of v is b, $v = bu'$, and $x \xrightarrow{b}_{\approx_R} x_1$ for some $x_1 \in Pre(p)$, $b \in A$, and $u' \in A^*$.
$FD(G(A, R_r)(p))$ thus we know that $x_1 \xrightarrow{a}_{\approx_R} x_2$ and $z \xrightarrow{b}_{\approx_R} x_2$ for some $x_2 \in Pre(p)$. Then we have $x_1 \xrightarrow{u'aw_1}^*_{\approx_R} y$, $x_1 \xrightarrow{a}_{\approx_R} x_2$, and $1 \leq |u'aw_1|_a \leq k$.
Now by the induction hypothesis, there exists $w'' \in A^*$ such that $x_2 \xrightarrow{w''}^*_{\approx_R} y$ and aw'' is a permutation of $u'aw_1$.
Thus $z \xrightarrow{bw''}^*_{\approx_R} y$ and abw'' is a permutation of w.
We can conclude that the lemma holds. □

Lemma 50 can be generalized to the following lemma. In figure 31 the situation and the situation during the proof is visualized.

Lemma 51 *Let* A *be an alphabet and* R *an event-preserving relation over* A^*. *Let* $p \in A^*/\approx_R$. *If* $FD(G(A, R_r)(p))$ *then*
for all $x, y, z \in Pre(p)$ *and* $w, v \in A^*$
if $x \xrightarrow{w}^*_{\approx_R} y$, $x \xrightarrow{v}^*_{\approx_R} z$, *and* $|w|_a \geq |v|_a$ *for all* $a \in A$ *then there exists* $v' \in A^*$ *such that* $z \xrightarrow{v'}^*_{\approx_R} y$ *and* vv' *is a permutation of* w.

Proof:

Induction on $|v|$.
Let $|v| = 0$ then $v = \epsilon$. Thus $z \xrightarrow{w}^*_{\approx_R} y$ and w is a permutation of w.
Suppose it has been proven for $0 \leq |v| \leq k$.
Assume $|v| = k + 1$. Then $v = av'$ for some $a \in A$ and $v' \in A^*$. Suppose

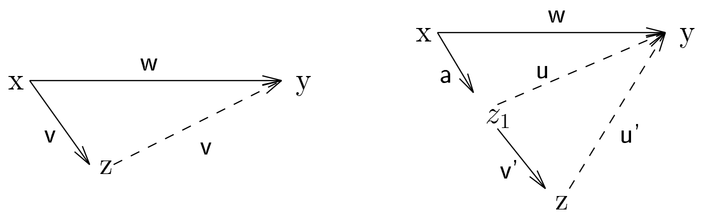


Figure 31: Visualization of lemma 51

$x \xrightarrow{a}_{\approx_R} z_1$. By lemma 50 there exists $u \in A^*$ such that $z_1 \xrightarrow{u}^{*}_{\approx_R} y$ and au is a permutation of w. We have $z_1 \xrightarrow{u}^{*}_{\approx_R} y$, $z_1 \xrightarrow{v'}_{\approx_R} z$, and $|u|_a \geq |v'|_a$ for all $a \in A$. Then $|v'| = k$ and by the induction hypothesis we have there exists $u' \in A^*$ such that $z \xrightarrow{u'}_{\approx_R} y$ and $v'u'$ is a permutation of u. Thus $z \xrightarrow{u'}_{\approx_R} y$ and $av'u' = vu'$ is a permutation of w. $\Box$

The last lemma, which is needed for the proof of theorem 55, states that, if $G(A, R_r)(p)$ has the forward diamond property and there are two distinct paths labelled with w' and w'' from a vertex to two other vertices and w' is a permutation of w'' then these latter vertices are one and the same. Again a visualization of this lemma is given in the next figure.

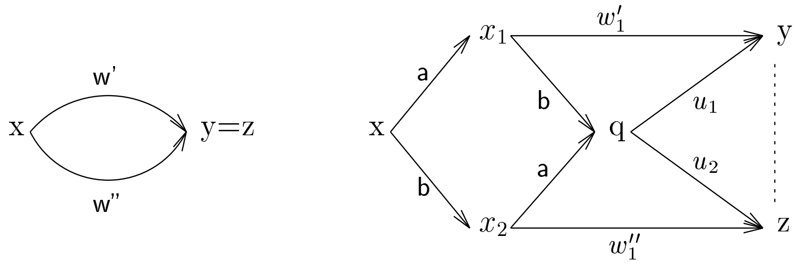


Figure 32: Visualization of lemma 52

Lemma 52 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$. If $FD(G(A, R_r)(p))$ then for all $x, y, z \in Pre(t)$ and $w', w'' \in A^*$ if $x \xrightarrow{w'}^{*}_{\approx_R} y$, $x \xrightarrow{w''}^{*}_{\approx_R} z$, and $|w'|_a = |w''|_a$ for all $a \in A$ then $y = z$.*

Proof:

Induction on $|w'|$.

Let $|w'| = 1$ then $w' = w'' = a$ for some $a \in A$. $G(A, R_r)(p)$ is deterministic thus $y = z$. Suppose the statement has been proven for $|w'| \leq k$. Assume $|w'| = k+1$. Then there exist $w'_1, w''_1 \in A^*$ such that $w' = aw'_1$ and $w'' = bw''_1$. If $a = b$ we are done, so assume $a \neq b$. Let $x \xrightarrow{a}_{\approx_R} x_1$ and $x \xrightarrow{b}_{\approx_R} x_2$ for some $x_1, x_2 \in Pre(p)$. We have $FD(G(A, R_r)(p))$ thus there exists

$q \in Pre(p)$ such that $x_1 \xrightarrow{b}_{\approx_R} q$ and $x_2 \xrightarrow{a}_{\approx_R} q$. We know $x \xrightarrow{a}_{\approx_R} x_1$, $x_1 \xrightarrow{w_1'}{}^{*}_{\approx_R} y$, $x_1 \xrightarrow{b}_{\approx_R} q$, and $|w_1'|_b \geq 1$.

By lemma 50 there exists $u_1 \in A^*$ such that $q \xrightarrow{u_1}{}^{*}_{\approx_R} y$ and bu_1 is a permutation of w_1'. Similarly there exists $u_2 \in A^*$ such that $q \xrightarrow{u_2}{}^{*}_{\approx_R} z$ and au_2 is a permutation of w_1''.

We know $q \xrightarrow{u_1}{}^{*}_{\approx_R} y$, $q \xrightarrow{u_2}{}^{*}_{\approx_R} z$, and $|u_1|_a = |u_2|_a$ for all $a \in A$. Then $|u_1| = |u_2| = k - 1$ and by the induction hypothesis we have $y = z$.

We can conclude that the lemma holds. □

Finally we can prove that there is a connection between the forward diamond property of $G(A, R_r)(p)$and the function ζ_p.

Theorem 53 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$.*
If $FD(G(A, R_r)(p))$ then ζ_p is injective.

Proof:

Suppose $[w_1\rangle, [w_2\rangle \in Pre(p)$ and $\zeta_p([w_1\rangle_R) = \zeta_p([w_2\rangle_R) = C$ for some $C \in Conf_{Po(p)}$. Then $ev(w_1) = ev(w_2)$. We have $[\epsilon\rangle_R \xrightarrow{w_1}{}^{*}_{\approx_R} [w_1\rangle_R$, $[\epsilon\rangle_R \xrightarrow{w_2}{}^{*}_{\approx_R} [w_2\rangle_R$, and $|w_1|_a = |w_2|_a$ for all $a \in A$. By lemma 52 we have $[w_1\rangle_R = [w_2\rangle_R$. □

Theorem 54 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$.*
If $FD(G(A, R_r)(p))$ then ζ_p is full.

Proof:

Suppose we have $[w_1\rangle_R, [w_2\rangle_R \in Pre(p)$ and $C_1, C_2 \in Conf_{Po(p)}$ such that $\zeta_p([w_1\rangle_R = C_1$ and $\zeta_p([w_2\rangle_R = C_2$. From theorem 53 follows that $[w_1\rangle_R$ and $[w_2\rangle_R$ are uniquely determined. Assume $C_1 \xrightarrow{a} C_2$, then we have to prove that $[w_1\rangle_R \xrightarrow{a}_{\approx_R} [w_2\rangle_R$.

From theorem 46 follows $[\epsilon\rangle_R \xrightarrow{w_1}{}^{*}_{\approx_R} [w_1\rangle_R$ and $[\epsilon\rangle_R \xrightarrow{w_2}{}^{*}_{\approx_R} [w_2\rangle_R$. Since $\zeta_p([w_1\rangle_R = ev(w_1) = C_1$ and $\zeta_p([w_2\rangle_R = ev(w_2) = C_2$ we have $ev(w_2) = ev(w_1) \cup \{(a, |w_1|_a + 1)\}$. From lemma 51 follows $[w_1\rangle_R \xrightarrow{a}_{\approx_R} [w_2\rangle_R$. □

This all leads to the following theorem.

Theorem 55 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$. If $FD(G(A, R_r)(p))$ then ζ_p is injective and full.*

Example

Let $A = \{a, b, c\}$ and let $R = \{(ab, ba), (acb, abc), (bca, bac)\}$. Let $p = [abcd\rangle_R = \{abc, acb, bac, bca\}$. The quasi-prefix graph restricted to p and the Hasse diagram of the partial order of p are depicted in figure 33.

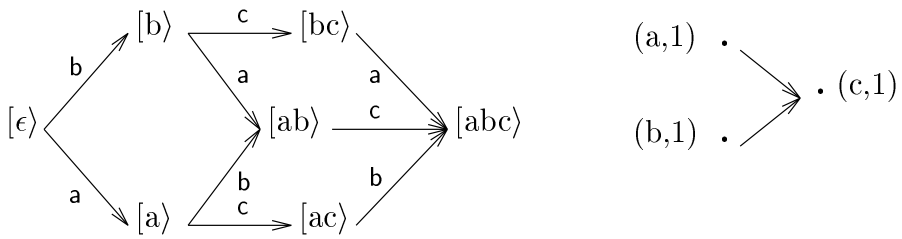


Figure 33: $G(A, R_r)(p)$ and $Po(p)$

In figure 34 the configuration graph of the partial order is given. We know that $G(A, R_r)(p)$ has the forward diamond property, thus the translate function ζ_p is injective and full. Moreover the function is surjective.

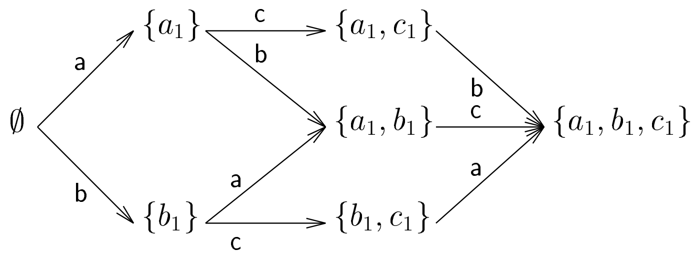


Figure 34: $\mathcal{C}_{Po(p)}$

Now we concentrate on the other situation, $G(A, R_r)(p)$ has the backward diamond property. With this property we also need that $G(A, R_r)(p)$ is co-deterministic or else we can not draw any conclusions. In this situation we also need three lemmas for the proof of theorem 61.

In the first lemma we also have a path labelled with w from a vertex y to x and a path labelled with v from z to x. If we know that $ev(v) \subseteq ev(w)$ then there exists a path labelled with v' from y to z and $v'v$ is a permutation of w. In figure 35 this situation is visualized.

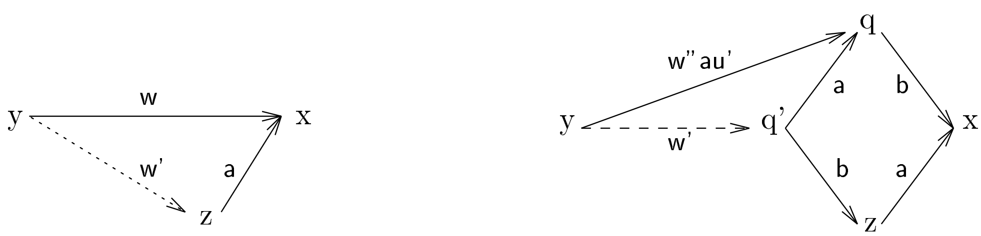


Figure 35: Visualization of lemma 56

Lemma 56 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$. $BD(G(A, R_r)(p))$ and co-deterministic.*

For all $x, y, z \in Pre(p)$, $w \in A$, and $a \in A$
$y \xrightarrow{w}{}^{*}_{\approx_R} x$, $z \xrightarrow{a}_{\approx_R} x$, *and* $|w|_a \geq 1$ *implies there exists* $w' \in A^*$ *such that* $y \xrightarrow{w'}{}^{*}_{\approx_R} z$ *and* $w'a$ *is a permutation of* w.

Proof:

Induction on $|u|$, where u is the least suffix of w such that $|u|_a = 1$.
Let $|u| = 1$, then $w = w''u = w''a$ for some $w'' \in A^*$. $G_{\sim}(t)$ is co-deterministic thus $y \xrightarrow{w''}{}^{*}_{\approx_R} z$ and $w''a$ is a permutation of w.
Suppose the statement has been proven for all paths with $u \leq k$.
Assume $|u| = k + 1$. Then there exists $w'' \in A^*$ such that $w = w''u$, also there exist $b \in A$ and $u' \in A^*$ such that $u = au'b$ and $x \xrightarrow{w''au'}{}^{*}_{\approx_R} q$ for some $q \in Pre(p)$. Thus $q \xrightarrow{b}_{\approx_R} x$ and $z \xrightarrow{a}_{\approx_R} x$. $G(A, R_r)(p)$ has the backward diamond property thus there exists $q' \in Pre(p)$ such that $q' \xrightarrow{a}_{\approx_R} q$, $q \xrightarrow{a}_{\approx_R} x$, $q' \xrightarrow{b}_{\approx_R} z$, and $z \xrightarrow{a}_{\approx_R} x$. Then $|w''au'| = k$ and by the induction hypothesis there exists $w' \in A^*$ such that $y \xrightarrow{w'}{}^{*}_{\approx_R} q'$ and $w'a$ is a permutation of w''. Then there exists $w' \in A^*$ such that $x \xrightarrow{w'b}{}^{*}_{\approx_R} z$ and $w'ba$ is a permutation of w.

We can conclude that the lemma holds. □

Lemma 56 can be generalized like lemma 50, which leads to the next lemma.

Lemma 57 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$. $BD(G(A, R_r)(p))$ and co-deterministic.*
For all $x, y, z \in Pre(p)$ and $w, v \in A$
$y \xrightarrow{w}{}^{*}_{\approx_R} x$, $z \xrightarrow{v}{}^{*}_{\approx_R} x$, *and* $|w|_a \geq |v|_a$ *for all* $a \in A$ *implies there exists* $v' \in A^*$ *such that* $y \xrightarrow{v'}{}^{*}_{\approx_R} z$ *and* $v'v$ *is a permutation of* w.

Proof:

Induction on $|v|$.
Let $|v| = 0$ then $v = \epsilon$ and $y \xrightarrow{w}{}^{*}_{\approx_R} z$, where w is a permutation of w.
Suppose it has been proven for $0 \leq |v| \leq k$.
Assume $|v| = k + 1$. Then there exist $u \in A^*$ and $a \in A$ such that $v = v'a$. Suppose $z \xrightarrow{v'}{}^{*}_{\approx_R} z_1$ for some $z_1 \in Pre(p)$. Since $|w|_b \geq |v|_b$ for all $b \in A$ we have $|w|_a \geq 1$. By lemma 56 there exists $u \in A^*$ such that $y \xrightarrow{u}{}^{*}_{\approx_R} z_1$ and ua is a permutation of w. Then $z \xrightarrow{v'}{}^{*}_{\approx_R} z_1$, $y \xrightarrow{u}{}^{*}_{\approx_R} z_1$, and $|u| = k$. By the induction hypothesis we have there exists $y' \in A^*$ such that $y \xrightarrow{u'}_{\approx_R} z$ and $u'v'$ is a permutation of u. Then $y \xrightarrow{u'}{}^{*}_{\approx_R} z$ and $u'v'a = u'v$ is a permutation of w. □

The last lemma has as result that if there are two distinct paths from two other vertices y and z to a vertex x and these path labels are permutations of each other then these latter vertices y and z are one and the same. This only holds if $G(A, R_r)(p)$ has the backward diamond property and if $G(A, R_r)(p)$ is co-deterministic.

Lemma 58 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$. If $BD(G(A, R_r)(p))$ and $G(A, R_r)(p)$ is co-deterministic then for all $x, y, z \in Pre(p)$ and $w', w'' \in A^*$*
$y \xrightarrow{w'}{}^*_{\approx_R} x$, $z \xrightarrow{w''}{}^*_{\approx_R} x$ and $|w'|_a = |w''|_a$ for all $a \in A$ implies $y = z$.

Proof:

Induction on $|w'|$.
Let $|w'| = 1$, then $w' = w'' = a$ for some $a \in A$. $G(A, R_r)(p)$ is co-deterministic thus $y = z$. Suppose the statement has been proven for all the situations in which the paths are labelled with a word of length $\leq k$.
Assume $|w'| = k + 1$. There exist $w_1', w_1'' \in A^*$ such that $w' = w_1'a$ and $w'' = w_1''b$. $G(A, R_r)(p)$ has the backward diamond property thus there exists $q \in Pre(p)$ such that $q \xrightarrow{b}_{\approx_R} q'$, $q' \xrightarrow{a}_{\approx_R} x$, $q \xrightarrow{a}_{\approx_R} q''$, and $q'' \xrightarrow{b}_{\approx_R} x$.
We have $y \xrightarrow{w_1'}{}^*_{\approx_R} q'$, $q \xrightarrow{b}_{\approx_R} q'$, and $|w_1'|_b \geq 1$ thus applying lemma 56 there exists $w_2' \in A^*$ such that $y \xrightarrow{w_2'}{}^*_{\approx_R} q$ and $w_2'b$ is a permutation of w_1'.
Similarly there exists $w_2'' \in A^*$ such that $z \xrightarrow{w_2''}{}^*_{\approx_R} q$ and $w_2''b$ is a permutation of w_1''. Thus $y \xrightarrow{w_2'}{}^*_{\approx_R} q$, $z \xrightarrow{w_2''}{}^*_{\approx_R} q$, and $|w_2'|_a = |w_2''|_a$ for all $a \in A$.
Then $|w_2'| = |w_2''| = k - 1$ and by the induction hypothesis $y = z$.

We can conclude that the lemma holds. □

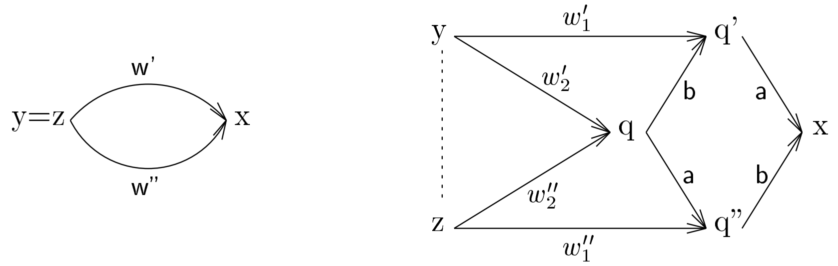


Figure 36: Visualization of lemma 56

Finally we can prove that if $G(A, R_r)(p)$ has the backward diamond property and is co-deterministic, then ζ_p is injective and full. We have divided theorem 61 into theorem 59 and 60. Thus first we prove that ζ_p is injective.

Theorem 59 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$.*
$G(A, R_r)(p)$ has the backward diamond property and is co-deterministic implies ζ_p is injective.

Proof:

Suppose we have $q, q' \in Pre(p)$ such that $\zeta_p(q) = \zeta_p(q')$. If $\zeta_p(q) = \zeta_p(q')$ then $|q|_a = |q'|_a$ for all $a \in A$, thus $ev(q) = ev(q')$. Because $q, q' \in Pre(p)$ there exist paths in $G(A, R_r)(p)$ labelled with $w', w'' \in A^*$ from q and q' to p. Thus $q \xrightarrow{w'}_{\approx_R} p$ and $q' \xrightarrow{w''}_{\approx_R} p$.
Also $[\epsilon\rangle_R \xrightarrow{v'}_{\approx_R} q$ and $[\epsilon\rangle_R \xrightarrow{v''}_{\approx_R} q'$ for some $v' \in q$ and $v'' \in q'$. Then $[\epsilon\rangle_R \xrightarrow{v'w'}_{\approx_R} p$ and $[\epsilon\rangle_R \xrightarrow{v''w''}_{\approx_R} p$. $\approx_R$ is event-preserving thus $|v'w'|_a = |v''w''|_a$ for all $a \in A$. We had $ev(q) = ev(q')$ thus $|w'|_a = |w''|_a$ for all $a \in A$ From lemma 58 follows $q = q'$. □

And second we prove that ζ_p is full if the quasi-prefix graph restricted to a crop trace has the backward diamond property and is co-deterministic.

Theorem 60 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$.
$G(A, R_r)(p)$ has the backward diamond property and is co-deterministic implies ζ_p is full.*

Proof:

Suppose we have $[w_1\rangle_R, [w_2\rangle_R \in Pre(p)$ and $C_1, C_2 \in Conf_{Po(p)}$ such that $\zeta_p([w_1\rangle_R) = C_1$ and $\zeta_p([w_2\rangle_R) = C_2$. From theorem 59 follows that $[w_1\rangle_R$ and $[w_2\rangle_R$ are uniquely determined. Assume $C_1 \xrightarrow{a} C_2$, then we have to prove that $[w_1\rangle_R \xrightarrow{a}_{\approx_R} [w_2\rangle_R$. Let $\emptyset \xrightarrow{w}{}^* C_1$ for some $w \in A^*$. Then $C_1 = ev(w)$ by lemma 19 and $C_2 = ev(w) \cup \{(a, |w|_a + 1)\}$. Since $[w_1\rangle_R, [w_2\rangle_R \in Pre(p)$ we have $[w_1\rangle_R \xrightarrow{u_1}{}^*_{\approx_R} p$ and $[w_2\rangle_R \xrightarrow{u_2}{}^*_{\approx_R} p$ for some $u_1, u_2 \in A^*$. Since $\zeta_p([w_1\rangle_R) = ev(w_1) = C_1$ and $\zeta_p([w_2\rangle_R) = ev(w_2) = C_2$ we have $ev(w_1) = ev(w)$ and $ev(w_2) = ev(w_1) \cup \{(a, |w|_a + 1)\}$. We know $[\epsilon\rangle_R \xrightarrow{w_1}_{\approx_R} [w_1\rangle_R \xrightarrow{u_1}_{\approx_R} p$ and $[\epsilon\rangle_R \xrightarrow{w_2}_{\approx_R} [w_2\rangle_R \xrightarrow{u_2}_{\approx_R} p$. Then $|w_1u_1|_b = |w_2u_2|_b$ for all $b \in A$. Thus $|u_1|_a = |u_2|_a - 1$. By lemma 57 there exists $w' \in A^*$ such that $[w_1\rangle_R \xrightarrow{w'}{}^*_{\approx_R} [w_2\rangle_R$ and $w'u_2$ is a permutation of u_1. Therefore $[w_1\rangle_R \xrightarrow{a}_{\approx_R} [w_2\rangle_R$. □

Finally we conclude the following theorem holds.

Theorem 61 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$.
$G(A, R_r)(p)$ has the backward diamond property and is co-deterministic implies ζ_p is injective and full.*

<u>Example</u>

Let $A = \{a, b, c, d\}$ and $R = \{(abc, acb), (acd, adc), (cbd, cdb)\}$.
Let $p = [abcd\rangle$. The quasi-prefix graph restricted to p is together with the partial order of p depicted in figure 37.

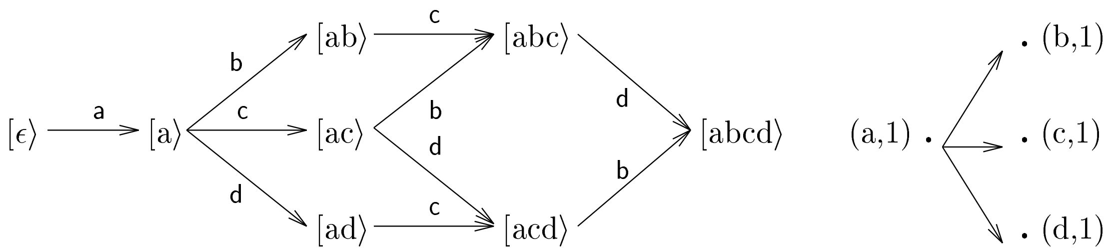


Figure 37: $G(A, R_r)(p)$ and $Po(p)$

The configuration graph of the partial order of p is depicted in figure 38. The quasi-prefix graph restricted to p has the backward diamond property and is co-deterministic. The function ζ_p is injective and full.

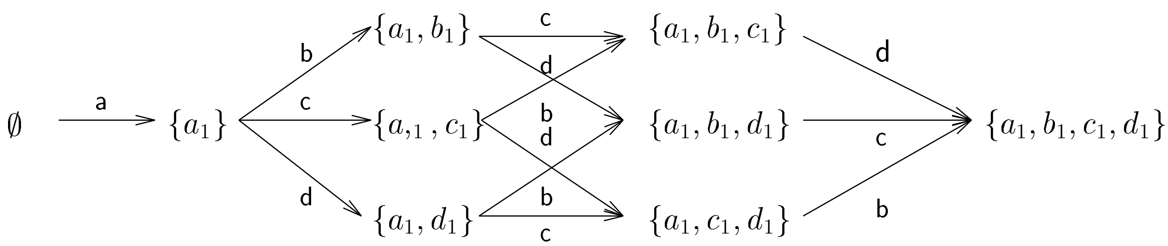


Figure 38: $\mathcal{C}_{Po(p)}$

<u>Example</u>

Let $R = \{(ab, ba), (bc, cb), (abc, acb), (acb, bac), (bac, bca), (bca, cba)\}$ and $A = \{a, b, c\}$. Let $p = [abc\rangle = \{abc, acb, bac, bca, cba\}$.

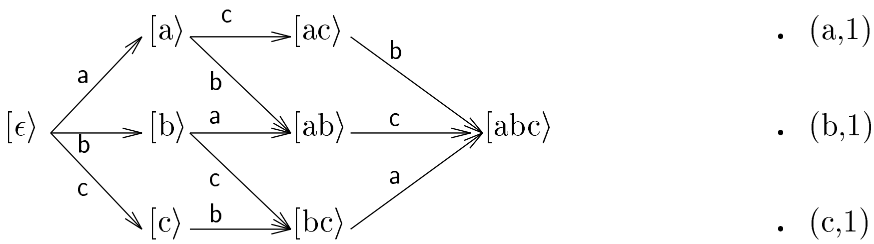


Figure 39: $G(A, R_r)(p)$ and $Po(p)$

The quasi-prefix graph restricted to p is together with the Hasse diagram of the partial order of p depicted in figure 39.

The configuration graph of the partial order of p is depicted in figure 40. $G(A, R_r)(p)$ has not the backward diamond property and is co-deterministic. The translate function ζ_p is surjective and injective, but not full.

When we have the results of theorem 55 and 61 we can conclude that the following theorem, stated in [BR95] holds.

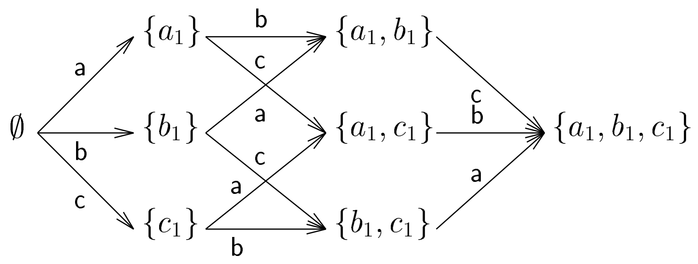


Figure 40: $\mathcal{C}_{Po(p)}$

Theorem 62 *[BR95] Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$.*
If $FD(G(A, R_r)(p))$ or $BD(G(A, R_r)(p))$ and $G(A, R_r)(p)$ is co-deterministic then ζ_p is injective and full.

Moreover as Biermann and Rozoy prove in [BR95] $G(A, R_r)(p)$ is isomorphic with $\mathcal{C}_{Po(p)}$ if and only if $G(A, R_r)(p)$ has both the forward and backward diamond property.

Theorem 63 *[BR95] Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$.*
$G(A, R_r)(p)$ and $\mathcal{C}_{Po(p)}$ are isomorphic if and only if $G(A, R_r)(p)$ has the forward and backward diamond property.

If $G(A, R_r)(p)$ and $\mathcal{C}_{Po(p)}$ are isomorphic then theorem 21 implies that $G(A, R_r)(p)$ has the forward and backward diamond property. If $G(A, R_r)(p)$ has the forward and backward diamond property then $G(A, R_r)(p)$ is a distributive lattice and isomorphic with $\mathcal{C}_{Po(p)}$.

Example

Let $A = \{a, b, c\}$ and $R = \{(ab, ba), (bac, bca)\}$. Let $p = [abc\rangle$. Then $p = \{abc, bac, bca\}$. In figure 41 $G(A, R_r)(p)$, the partial order of p and the configuration graph is depicted.

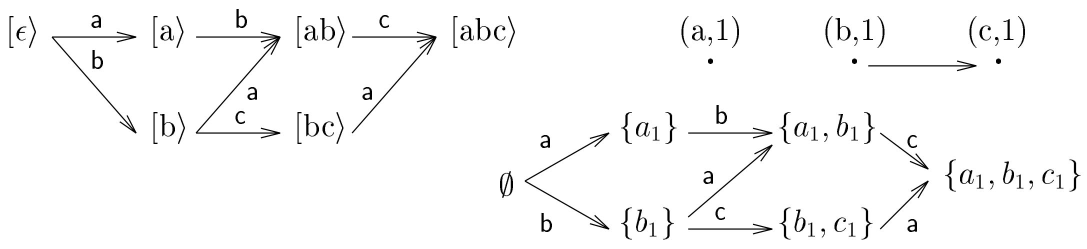


Figure 41: $G(A, R)(p)$, $Po(p)$ and $\mathcal{C}_{Po(p)}$

It is easy to see that the two reldepgraphs are isomorphic and that $G(A, R_r)(p)$ has the forward and backward diamond property.

7.1.5 From $\mathcal{C}_{Po(p)}$ to $G(A, R_r)(p)$

Let A be an alphabet and R an event-preserving relation over A^*. In section 7.1.4 we have proven that the total function ζ_p is the morphism from $G(A, R_r)(p)$ to $\mathcal{C}_{Po(p)}$. However we can also look at a function ψ_p on $Conf_{Po(p)}$ in order to establish a morphism from $\mathcal{C}_{Po(p)}$ to $G(A, R_r)(p)$ is such morphism exists.

Definition

Let A be an alphabet and R an event-preserving relation over A^*. The function ψ_p is defined for all $C \in Conf_{Po(p)}$ in the following way:
$\psi_p(C) = \{w \in A^* | (\emptyset, w, C) \in \rightarrow^*\}$.

The function ψ_p is a total function, But $\psi_p(C)$ is not always an element of $Pre(p)$.

Example

Let $A = \{a, b, c, d\}$ and $R = \{(bdc, bcd), (abc, acb), (cbd, cdb), (acd, adc), (dbc, dcb)\}$.
Let $p = [abcd\rangle_R = \{abdc, abcd, acbd, acdb, adcb, adbc\}$.

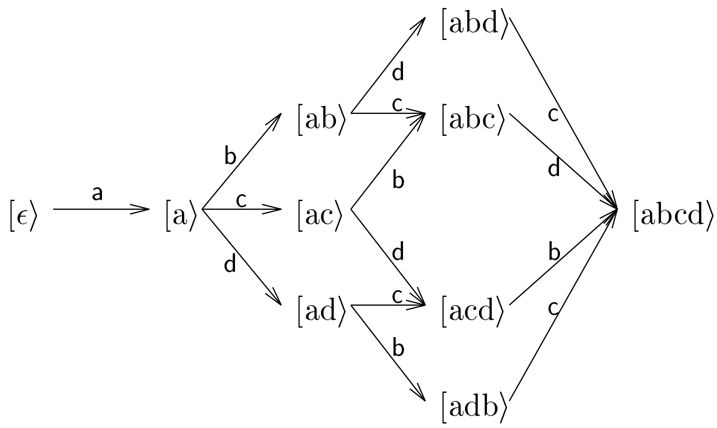


Figure 42: $G(A, R_r)([abcd\rangle_R)$

First we draw the quasi-prefixgraph. $G(A, R_r)(p)$ is depicted in figure 42. The labelled poset $Po(p)$ is depicted in figure 43.

When we have the labelled poset $Po(p)$ we can construct the graph of configurations. This graph is depicted in figure 44.

When we evaluate $\mathcal{C}_{Po(p)}$, $G(A, R_r)(p)$, and ψ_p we see that if $C = \{(a, 1), (b, 1), (d, 1)\}$ then $\psi_p(C) = \{abd, adb\}$. Since $abd \not\approx_R adb$, $\psi_p(C) \notin Pre(p)$.

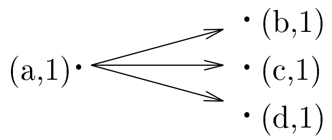


Figure 43: $Po(p)$

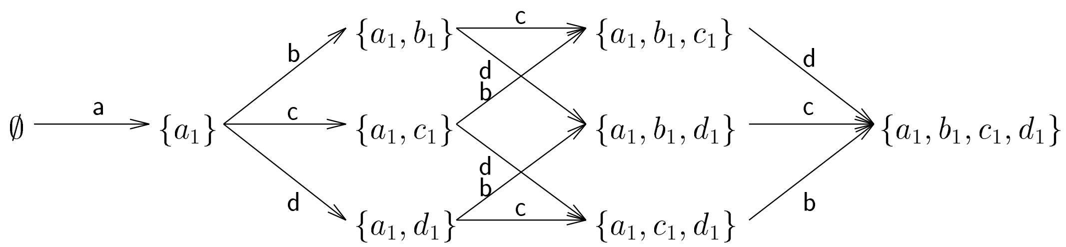


Figure 44: $\mathcal{C}_{Po(p)}$

Now we know that the function ψ_p is not always a morphism from $\mathcal{C}_{Po(p)}$ to $G(A,R_r)(p)$.If there exists a path labelled with w from the root to a configuration C then the set $\psi_p(C)$ contains the set $[w\rangle$.

Lemma 64 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$, $C \in Conf(Po(p))$, and $w \in A^*$. If $\emptyset \overset{w}{\rightarrow}^* C$ then $[w\rangle_R \subseteq \psi_p(C)$.*

Proof:

Induction on $|C|$.

Let $|C| = 0$, then $C = \emptyset$. Thus $\emptyset \overset{\epsilon}{\rightarrow}^* C$ and $[\epsilon\rangle_R = \psi_p(\emptyset)$.

Suppose it has been proven for all configurations C', where $0 \leq |C'| \leq k$.

Assume $|C| = k+1$ and $\emptyset \overset{w}{\rightarrow}^* C$. Then there exist $e \in E_A$, $C' \in Conf_{Po(p)}$, and $w' \in A^*$ such that $C' \overset{l_A(e)}{\rightarrow} C$, $\emptyset \overset{w'}{\rightarrow}^* C'$and $w = w' l_A(e)$. $|C'| = k$ and by the induction hypothesis we have $[w'\rangle_R \subseteq \psi_p(C')$. $C' \cup \{e\} = C$, thus $\psi_p(C) = \psi_p(C' \cup \{e\})$.

Then $[w'\rangle_R \Diamond l_A(e) \subseteq \psi_p(C') \Diamond l_A(e)$= $\psi_p(C' \cup \{e\}) = \psi_p(C)$. $\square$

However the function ψ_p is a morphism whenever for all paths labelled with w from the root to a configuration C $\psi_p(C) = [w\rangle_R$.

Theorem 65 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$. If $\emptyset \overset{w}{\rightarrow}^* C$ implies $\psi_p(C) = [w\rangle_R$ for all $C \in Conf_{Po(p)}$ and $w \in A^*$ then ψ_p is a morphism.*

Proof:

We have $\psi_p(\emptyset) = [\epsilon\rangle_R$. Then we have to prove that $C_1 \overset{a}{\rightarrow} C_2$ implies $\psi_p(C_1) \overset{a}{\rightarrow}_{\approx_R} \psi_p(C_2)$. Suppose we have $C_1, C_2 \in Conf_{Po(p)}$ and $e \in E_A$ such that $C_1 \overset{l_A(e)}{\rightarrow} C_2$. Suppose $\emptyset \overset{w}{\rightarrow} C_1$ for some $w \in A^*$. Then $\psi_p(C_1) = [w\rangle_R$.

$C_1 \cup \{e\} = C_2$, thus $\psi_p(C_2) = \psi_p(C_1 \cup \{e\}) = \psi_p(C_1)\Diamond l_A(e)$. Therefore $\psi_p(C_1) \xrightarrow{a}_{\approx_R} \psi_p(C_2)$. $\square$

It is therefore easy to see that if we have a morphism between the configuration graph of an partial order of a vertex and the quasi-prefix graph of this vertex then ψ_p is injective.

Theorem 66 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$. If ψ_p is the morphism from $\mathcal{C}_{Po(p)}$ to $G(A, R_r)(p)$ then ψ_p is injective.*

Proof:

Suppose $[w\rangle_R = \psi(C_1) = \psi(C_2)$ for some $w \in A^*$ and $C_1, C_2 \in Conf_{Po(p)}$. By definition we have $[w\rangle_R = \psi(C_1) = \{x \in A^* | \ (\emptyset, x, C_1) \in \rightarrow^*\} = \psi(C_2) = \{x \in A^* | \ (\emptyset, x, C_2) \in \rightarrow^*\}$.

Thus if $y \in [w\rangle_R$ then $\emptyset \xrightarrow{y}^* C_1$ and also $\emptyset \xrightarrow{y}^* C_2$. But $\mathcal{C}_{Po(p)}(p)$ is deterministic thus $C_1 = C_2$. $\square$

If $\mathcal{C}_{Po(p)}$ is embedded into $G(A, R_r)(p)$ then $\mathcal{C}_{Po(p)}$ and $G(A, R_r)(p)$ are isomorphic. This theorem is from [BR95].

Theorem 67 *[BR95] Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$. If $\mathcal{C}_{Po(p)} \subseteq G(A, R_r)(p)$ then $\mathcal{C}_{Po(p)}$ and $G(A, R_r)(p)$ are isomorphic.*

Proof:

We know that ψ_p from $Conf_{Po(p)}$ to $Pre(p)$ is injective since $\mathcal{C}_{Po(p)} \subseteq G_r(A, I)(p)$. Thus all we have to prove is that ψ_p is surjective and full.

Let $|[w\rangle_R|$ be the length of all the words in the crop trace. First we prove with induction on $|[w\rangle_R|$, $w \in A^*$, that ψ from $Conf_{Po(p)}$ to $Pre(p)$ is surjective.

Let $|[w\rangle_R| = 0$, then $w = \epsilon$ and $\psi(\emptyset) = [\epsilon\rangle_R$ holds.

Suppose ψ is surjective for all crop traces with words with length less or equal to k.

Assume $|[w\rangle_R| = k + 1$. There exist $a \in A$ and $w' \in A^*$ such that $w = w'a$. Because a appears behind w' the occurrence of a after w' is allowed by $Po(p)$. Then $|w'| = k$ and by the induction hypothesis there exist a configuration $C \in Conf_{Po(p)}$ such that $\psi(C) = [w'\rangle_R$. Then there exists a configuration C' such that $C' = C \cup \{(a, |w'a|_a)\}$ and $\psi(C') = [w\rangle_R$. Thus ψ is surjective.

All we have to prove is that ψ is full. Suppose $[w\rangle_R \xrightarrow{a}^*_{\approx_R} [w\rangle_R \Diamond a$ for some $w \in A^*$ and $a \in A$. Then $[w\rangle_R \in Pre(p)$ thus there exists $C \in Conf_{Po(p)}$ such that $w \in \psi(C)$. Also $[w\rangle_R \Diamond a \in Pre(p)$ thus there exists $C' \in Conf_{Po(p)}$ such that $wa \in \psi_p(C')$. For all $b \in A \backslash \{a\}$ holds $|w|_b = |wa|_b$ and $|w|_a + 1 = |wa|_a$. Therefore $C \cup \{(a, |wa|_a)\} = C'$. By the definition of $\rightarrow$ we have $(C, a, C') \in \rightarrow^*$. Thus ψ is full.

Thus $G(A, R_r)(p)$ and $\mathcal{C}_{Po(p)}$ are isomorphic. $\square$

If we join the theorems 66 and 67 we get the next theorem.

Theorem 68 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in A^*/\approx_R$. If ψ_p is a morphism from $\mathcal{C}_{Po(p)}$ to $G(A, R_r)$ then $\mathcal{C}_{Po(p)}$ and $G(A, R_r)$ are isomorphic.*

In the crop trace theory we do not always have a morphism from the configuration graph of the partial order of a crop trace to the quasi-prefix graph restricted to the crop trace. However for Mtraces the function ψ_p is always a morphism. Moreover then the function ψ_p is always injective, surjective and full, since the configuration graph and the prefix graph are always isomorphic.

7.1.6 Represented by partial orders

A crop trace equivalence cannot always be represented by partial orders. In the next example a crop trace equivalence will be given which cannot be represented by partial orders.

Example

Let $A = \{a, b\}$ and $R = \{(aab, baa), (bba, abb)$. Let $p = [aabb\rangle_R$. Then $p = \{aabb, baab, bbaa, abba\}$. The quasi-prefix graph and the Hasse diagram of the partial order of p are depicted in figure 45.

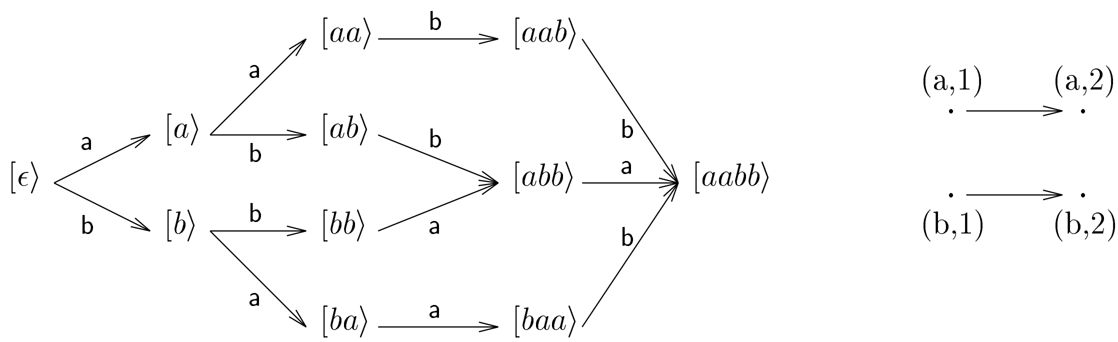


Figure 45: $G(A, R_r)(p)$ and $Po(p)$

The set of linearizations of $Po(p)$ contains also the words $abab$ and $baba$. These words are not in p and thus R cannot be represented by partial orders.

7.1.7 Conclusions

First we consider the relation between Mtraces and crop traces. Let A be an alphabet and I an independence relation over A. Then $\equiv_I$ is a congruence, thus also a right-congruence, and the quasi-concatenation $\Diamond$ is well-defined for $A^*/\equiv_I$.

Since $p[v\rangle_I = p\Diamond v$ for all $p \in A^*/\equiv_I$ and $v \in A^*$ we have $G(A, I) = G(A, (C_I)_r)$. This implies that results from last subsubsections also hold for the prefix graphs of Mtraces. Thus we can conclude that the quasi-prefix graph restricted to a Mtrace is isomorph to the configuration graph of the partial order of the Mtrace. This also follows from section 6.

Crop traces lack some properties of Mtraces; the equivalence is not represented by partial orders, the quasi-prefix graph has not the (compatible) forward or backward diamond property nor satisfies the (inverse) cube axiom. All these properties were satisfied by the Mtraces.

We can only conclude that if the quasi-prefix graph of a crop trace has the forward or backward diamond property and is co-deterministic then the morphism from the quasi-prefix graph to the configuration graph is injective and surjective. An interesting result is that if there exists a morphism from the configuration graph to the quasi-prefix graph then these graphs are isomorphic, which is a result of the fact that there always exists a morphism from the quasi-prefix graph to the configuration graph.

7.2 Congruence

In this subsection we handle the generalization of Bauget and Gastin in [BG95]. Bauget and Gastin describe a theory in which possible permutations of events depend only on a part of their past. The relation, which defines equivalent executions, is again event-preserving. Since the permutations of the events only depend on a part of their past, two equivalent observations will still be equivalent when they are being observed after the same or equivalent past executions. In other words we consider *c*ongruences which *p*reserve the *o*ccurrences of the letters.

This cop trace equivalence is defined in section 7.2.1. We will define the cop trace monoid and the prefix graph in the next subsubsection. Like with the crop trace theory we can look at the morphism between the prefix graph and the graph of configurations, see subsubsection 7.2.3.

7.2.1 Cop trace equivalence

Definition

Let A be an alphabet and R an event-preserving relation over A^*.

1. The *cop trace equivalence* induced by R is defined by $\equiv_R$, the congruence induced by R;
2. $\langle x\rangle_R = \{z \in A^* | \ z \equiv_R x\}$ the equivalence class containing x is the *cop trace* containing x.

Note that cop trace equivalences are event-preserving. Moreover we write $\langle x\rangle$ if R is clear.

7.2.2 The prefix graph and the configuration graph

Let A be an alphabet and R an event-preserving relation over A^*. $\equiv_R$ is a congruence thus we have a concatenation $\cdot$ defined as in subsection 6.2 by $\langle x\rangle \cdot \langle y\rangle = \langle xy\rangle$. The concatenation is well-defined. Because if $x \equiv_R x'$ and $y \equiv_R y'$ then $xy \equiv_R x'y \equiv_R x'y'$.

With the concatenation $\cdot$ we have a trace monoid as in subsection 6.2, $M(A, R)$ is the quotient monoid $A^*/\equiv_R$, with concatenation $\cdot$ and unit $\langle\epsilon\rangle_R$. Thus we have a prefix ordering $\preceq_R$ and a prefix graph $G(A, R) = (M(A, R), A, \rightarrow_{\equiv_R}, \langle\epsilon\rangle_R)$ similar to Mtraces. Moreover we have the following properties.

Theorem 69 *Let A be an alphabet and R an event-preserving relation over A^*. For all $p, q \in M(A, R)$ $p \prec\!\!\cdot_R q$ if and only if $p \xrightarrow{a}_{\equiv_R} q$ for some $a \in A$.*

Theorem 70 *Let A be an alphabet and R an event-preserving relation over A^*. Then $G(A,R)$ is a reldepgraph.*

Example

Let $A = \{a, b, c\}$ and $R = \{(ac, ca), (bc, cb), (cab, cba)\}$. Let $p = \langle abc \rangle_R$. The prefix graph of the cop trace p is depicted in figure 46.

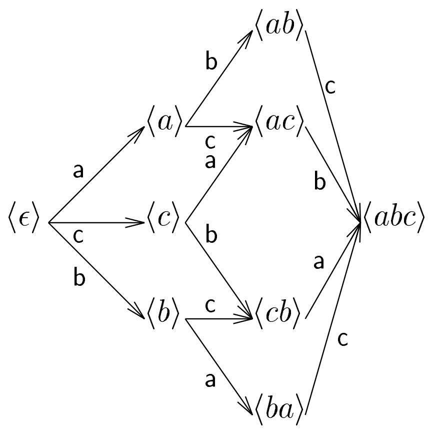


Figure 46: $G(A,R)(p)$

The prefix graph has not the forward diamond property, is not co-deterministic and does not satisfy the cube nor inverse cube axiom. Thus it is special when the prefix graph has all these properties.

This example shows that there exist cop trace equivalences such that the prefix graph has not the forward diamond property and such that the prefix graph is not co-deterministic.

Example continued

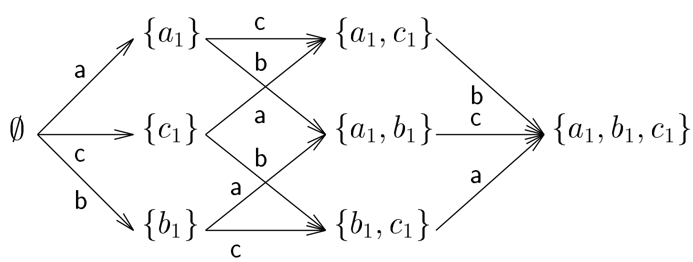


Figure 47: $\mathcal{C}_{Po(p)}$

The configuration graph of $Po(p)$ is depicted in figure 47. It is clear that the prefix graph and the configuration graph are not isomorphic.

Example

Let $A = \{a, b, c\}$ and $R = \{(ac, ca), (abc, acb), (cab, cba)\}$. Let $p = \langle abc \rangle_R$. The prefix graph restricted to p is depicted in figure 48.

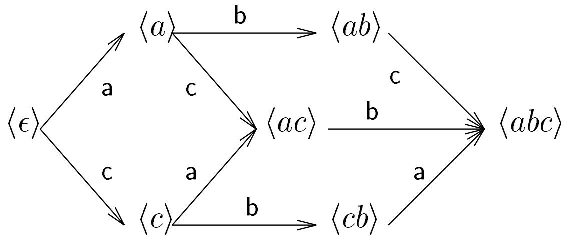


Figure 48: $G(A, R)(p)$

In the last example an event-preserving relation is given such that the prefix graph restricted to a cop trace does not have the backward diamond property. Thus after these examples we can conclude that the prefix graph restricted to a cop trace does not have special properties like with Mtraces.

7.2.3 From $G(A, S)(p)$ to $\mathcal{C}_{Po(p)}$

Clearly, each cop trace equivalence $\equiv_R$ is a crop trace equivalence $\approx_S$, with $S = R_l$, since $R_{lr} = (R_l)_r$. Thus we can use the crop trace theory and the quasi-concatenation, defined as $\langle x \rangle_R \Diamond a = [x\rangle_S \Diamond a = [xa\rangle_S = \langle xa \rangle_R$. Since the equivalence classes coincide, $\Diamond a$ defines $\xrightarrow{a}_{\equiv_R}$. We define the quasi-prefix graph $G(A, R_{lr}) = G(A, S_r) = (A^*/ \approx_S, A, \rightarrow_{\approx_S}, [\epsilon\rangle_S)$. By theorem 44 this is a reldep-graph.

Example continued

The quasi-prefix graph of $[abc\rangle_R$ is depicted in figure 49. The difference between the crop traces and cop traces is very clear.

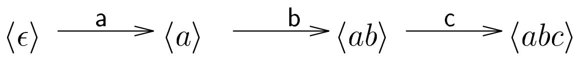


Figure 49: $G(A, R_r)(p)$

The translate function restricted to the set $Pre(p)$, denoted by ζ_p as defined in section 7.1.4, is the morphism from $G(A, R)(p)$ to $\mathcal{C}_{Po(p)}$. Now from theorem 48 follows:

Theorem 71 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in M(A, R)$. If $p = LE(Po(p))$ then ζ_p is surjective and full.*

From theorem 49 follows:

Theorem 72 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in M(A,R)$. If ζ_p is injective then $G(A,S)(p)$ is co-deterministic.*

From theorem 62 follows:

Theorem 73 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in M(A,R)$.*
If $FD(G(A,R)(p))$ or $BD(G(A,R)(p))$ and $G(A,R)(p)$ is co-deterministic then $G(A,R)(p) \subseteq \mathcal{C}_{Po(p)}$ and ζ_p is full.

From theorem 63 follows:

Theorem 74 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in M(A,R)$.*
$G(A,R)(p)$ and $\mathcal{C}_{Po(p)}$ are isomorphic if and only if $G(A,R)(p)$ has the forward and backward diamond property.

We also have the function ψ_p from the the configuration graph of a vertex to the quasi-prefix graph restricted to the vertex. Since the cop trace equivalence is a crop trace equivalence we can directly use the result from section 7.1.5.

From theorem 68:

Theorem 75 *Let A be an alphabet and R an event-preserving relation over A^*. Let $p \in M(A,R)$. If ψ is a morphism from $\mathcal{C}_{Po(p)}$ to $G(A,R)(p)$ then $\mathcal{C}_{Po(p)}$ and $G(A,R)(p)$ are isomorphic.*

7.2.4 Difference between crop and cop traces

In the example of the subsubsections 7.2.2 and 7.2.3 the difference between $[\rangle_R$ and $\langle\rangle_R$ is shown. Since there exists a difference between right-congruences and congruences there exists a difference between crop and cop traces.

Example

Suppose we have the event-preserving relation $R = \{(ab, ba)\}$. Let $p = [abc\rangle_R$ and $p' = \langle abc\rangle_R$. Then $p = \{abc, bac\}$ and $p' = \{abc, bac\}$. However if we investigate the cop trace $\langle abcab\rangle_R$ and the crop trace $[abcab\rangle_R$ then the cop trace has besides the elements of the crop trace also as the elements $abcba$ and $bacba$. Notice that we can not construct an event-preserving relation R' such that $\approx_R = \equiv_{R'}$.

This example illustrates the fact that there exist right-congruences induced by event-preserving relations which cannot be described by congruences induced by event-preserving relations. Thus we can conclude that the crop trace theory is more general than crop trace theory.

7.2.5 Conclusions

Cop traces are different from the defined Mtraces. This follows from theorem 74 which states that the prefix graph restricted to a vertex and the configuration graph of the partial order of the vertex are only isomorphic if the prefix graph restricted to a vertex has the forward and backward diamond property. In the first example a prefix graph restricted to a vertex this is shown, which has not the forward and not the backward diamond property. Like with crop trace theory the cop trace theory has no properties; the equivalence is not represented by partial orders, the prefix graph has not the (compatible) forward or backward diamond property nor satisfies the (inverse) cube axiom.

In subsubsection 7.2.4 there is shown that the crop trace theory is more general than the cop trace theory.

8 Restrictions

In this section we consider congruences and right-congruences generated by event-preserving relations satisfying certain restrictions. First we study the so-called local traces from [H94]. Next we will study congruences which have a restricted event-preserving relation. In subsection 8.2 we require that the event-preserving relation is a context commutation relation, which leads to the theory of cc traces. Bauget and Gastin describe in [BG95] congruences with a left-context commutation relation, leading to lcc traces and rcc traces in subsection 8.3. Biermann and Rozoy describe in [BR95] not only the crop trace theory but also the 1-context trace theory. This theory can be generalized to the k-context trace theory, which is formulated in subsection 8.4.

8.1 Subset permutations

Local traces are a generalization of Mtraces. They were introduced in [H94] as a trace based semantics for non-safe Petri nets. Here we follow the set up of [KR95]. Local traces are aimed at capturing concurrency between sets of events occurring after a certain history. Thus concurrency is decribed by means of a relation between words (histories) and sets of letters.

8.1.1 Local trace equivalence

Let A be an alphabet. A *local independence relation* over A is a relation $L \subseteq A^* \times P_f(A)$.

Definition

Let A be an alphabet and L a local independence relation over A. Let $x, y \in A^*$.

1. $x \dot{\approx}_L y$ if there exist $u, v, w, z \in A^*$ and $(u, S) \in L$ such that $x = uvz$, $y = uwz$ and $|v|_a = |w|_a \leq 1$ for all $a \in A$, and $alph(v) \subseteq S$.
2. The *local trace equivalence* $\approx_L$ is defined by $\approx_L = (\dot{\approx}_L)^*$.
3. $[x\rangle_L = \{z \in A^* | z \approx_L x\}$ the equivalence class of x, is the *local trace* containing x.

Note that we write $[x\rangle$ if L is clear from the context.

8.1.2 Difference between crop and local traces

It is easy to see that the so defined local trace equivalence is a right-congruence. The congruence $\approx_L$ is the least right-congruence containing R_L, the relation defined as follows. For a local independence relation L over A
$R_L = \{(uv, uw) |$ there exist $(u, S) \in L$ and $S' \subseteq S$ such that $|v|_a = |w|_a = 1$ for all $a \in S'$ and $|v|_a = |w|_a = 0$ for all $a \in A \backslash S'\}$.

Theorem 76 *Let A be an alphabet and L a local independence relation over A. Then $\approx_L = \approx_{R_L}$.*

Proof:

Easy to see is $\dot{\approx}_L = (R_L)_r$, as defined in section 3.4. First we prove that $\dot{\approx}_L$ is symmetric. Suppose we have $x, y \in A^*$ such that $x \dot{\approx}_L y$. Then there exist $u, v, w, z \in A^*$ and $(u, S) \in L$ such that $x = uvz$, $y = uwz$ and $|v|_a = |w|_a \leq 1$ for all $a \in A$, and $alph(v) \subseteq S$. But then also $uwz \dot{\approx}_L uwx$ and thus $y \dot{\approx}_L x$. Thus $\dot{\approx}_L \cup (\dot{\approx}_L)^{-1} = \dot{\approx}_L$ and then $\approx_L = (\dot{\approx}_L)^* = (\dot{\approx}_L \cup (\dot{\approx}_L)^{-1})^* = ((R_L)_r \cup (R_L)_r^{-1}) = \approx_{R_L}$. □

Note that the relation R_L is an event-preserving relation, thus the congruence on the right generated by R_L is a crop trace equivalence. Thus we have the quasi-concatenation, $\Diamond$, defined as $[x\rangle_L \Diamond a = [xa\rangle_L$ and the quasi-prefix graph $G(A, L_r) = (A^* / \approx_L, A, \rightarrow_{\approx_L}, [\epsilon\rangle_L)$, which is a reldepgraph. We have the quasi-prefix ordering $\preceq_L$, defined as $[x\rangle_L \preceq_L [y\rangle_L$ if there exists $w \in A^*$ such that $[x\rangle_L \Diamond w = [y\rangle_L$. If $[x\rangle_L \prec\!\cdot_L [y\rangle_L$ for some $a \in A$ then $[x\rangle_L \xrightarrow{a}_{\approx_L} [y\rangle_L$.

The local traces are however more restricted than the crop traces. This will be illustrated in the next example.

Example

Let $A = \{a, b, c\}$ and the event-preserving relation $R = \{(abc, cba)\}$. Let $p = [abc\rangle_R$. Then $p = \{abc, cba\}$ and $[a\rangle_R$ and $[c\rangle_R$ are prefixes of $[abc\rangle_R$. Suppose $L = \{(\epsilon, \{a, c\})$. Then $[abc\rangle_L = \{abc\}$. But we want $cba \in [abc\rangle_L$. Suppose $L' = \{(\epsilon, \{a, b, c\})\}$. Then $[abc\rangle_{L'} = \{abc, acb, bac, bca, cab, cba\}$. Thus there exists no local independence relation L such that $[abc\rangle_L = [abc\rangle_R$.

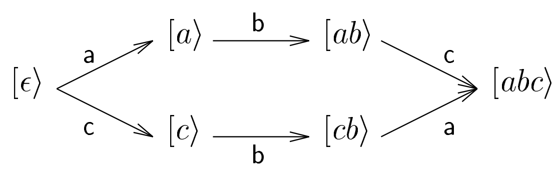


Figure 50: $G(A, R_r)([abc\rangle_R)$

8.1.3 The graph of local traces

In [H94] Kleijn and Rozoy introduce a graph of local traces induced by the local independence relation L.

Definition

Let A be an alphabet and L a local independence relation.
Let $X \subseteq A^*$, $S \in P_f(A)$, $p \in A^*/\approx_L$.

1. The operation $\triangle : (A^*/\approx_L \times P_f(A)) \rightarrow P_f(A^*)$ is defined in the following way: $p \triangle S = \{uv \in A^* | u \in p \text{ and } v \in Lin(S)\}$;
2. The *behaviour of* L is the set $Beh(L) = \{w \in A^* | \exists (w', S) \in L : w \in [w'\rangle \triangle S\}$.
3. $[X\rangle$ is the *set of local traces* $t = [w\rangle_L$ such that $w \in X$;
4. The *language of local traces associated with* L is $Lan(L) = [Prefix(Beh(L))\rangle$;
5. The *graph of local traces (associated with* L*)* is the elgraph with initial node $G_L = (Lan(L), A, \rightarrow_L, [\epsilon\rangle_L)$, where $t \xrightarrow{a}_L t'$ if $t \Diamond a = t'$ and $t, t' \in Lan(L)$.

The defined graph of local traces is a reldep graph, as shown in the next theorem.

Theorem 77 *Let A be an alphabet and L a local independence relation. G_L is a reldep-graph.*

Proof:

First we show that $[\epsilon\rangle_L$ is a root of G_L.
Suppose we have $p \in Lan(L)$. With induction on the length of p we prove that there exists a path from $[\epsilon\rangle_L$ to p:
Let $|p| = 0$, then $p = [\epsilon\rangle_L$ and $[\epsilon\rangle_L \xrightarrow{\epsilon}{}^*_L p$ holds.
Suppose there exists a path from $[\epsilon\rangle_L$ to all $p \in Lan(L)$ where $|p| \leq k$.
Assume $|p| = k + 1$. Then there exist $p' \in Lan(L)$ and $a \in A$ such that $p = p' \Diamond a$ (because of the function $Prefix$). Then we have $p' \xrightarrow{a}_L p$ and $|p'| = k$. By the induction hypothesis we have that there exists a path from $[\epsilon\rangle_L$ to p'. Thus there exists a path from $[\epsilon\rangle_L$ to p.
We can conclude that $[\epsilon\rangle_L$ is a root of G_L.

We have that G_L is deterministic.
Suppose $p \xrightarrow{a}_L q$ and $p \xrightarrow{a}_L q'$ for some $p, q, q' \in Lan(L)$ and $a \in A$ then $p \Diamond a = q$ and $p \Diamond a = q'$. Thus $q = q'$.

At last we prove that G_L is event-preserving.
Suppose we have the traces $[u\rangle_L, [u'\rangle_L \in Lan(L)$ and we know that v, w are paths in the graph G_L from $[u\rangle_L$ to $[u'\rangle_L$.
Then $[u\rangle_L \Diamond v = [u\rangle_L \Diamond w = [u'\rangle_L$ and thus $|u'|_a = |uv|_a = |uw|_a$ for all $a \in A$.
Thus we can conclude $|v|_a = |w|_a$ for all $a \in A$.

Having proved that G_L is rooted, deterministic and event-preserving, we can conclude that G_L is a reldepgraph. □

Since a local trace equivalence is also a crop trace equivalence we can construct besides the graph of local traces the quasi-prefix graph with the same local independence relation. In the next example a quasi-prefix graph and a graph of local traces is generated for a local independence relation.

Example

In this example we show that there is a difference between the two graphs, which are both reldep-graphs. First we define a local independence relation L_1:
$L_1 = \{(b, \{a, b\}), (ab, \{a, c\})\}$

When we have the local independence relation L_1 we can define the language of local traces associated with the local independence relation L_1.
$Beh(L_1) = \{bab, bba, abca, abac\}$,
$Pre(Beh(L_1)) = \{\epsilon, a, b, ab, ba, bb, aba, abc, bab, bba, abca, abac\}$, and
$Lan(L_1) = \{[\epsilon\rangle_{L_1}, [a\rangle_{L_1}, [b\rangle_{L_1}, [ab\rangle_{L_1}, [ba\rangle_{L_1}, [bb\rangle_{L_1}, [aba\rangle_{L_1}, [abc\rangle_{L_1}, [abac\rangle_{L_1}\}$.

Now we can construct the graph of local traces associated with the local independence relation L_1, in figure 51.

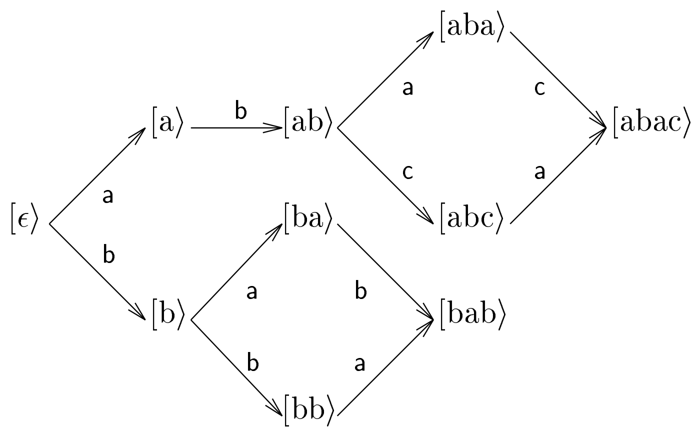


Figure 51: G_{L_1}

The graph in figure 51 is the graph of local traces associated with the local independence relation L_1. For the local independence relation L_1 we also have the local trace equivalence $\approx_{L_1}$. This trace equivalence is also a crop trace equivalence and therefore we can construct the quasi-prefix graph, denoted by $G(A,(R_{L_1})_r)$. Because the quasi-prefix graph is infinite only a part of $G(A,(R_{L_1})_r)$ is shown in figure 52. The graph of local traces, denoted by G_{L_1}, is embedded into $G(A,(R_{L_1})_r)$ and this is shown by the thick lines in figure 52.

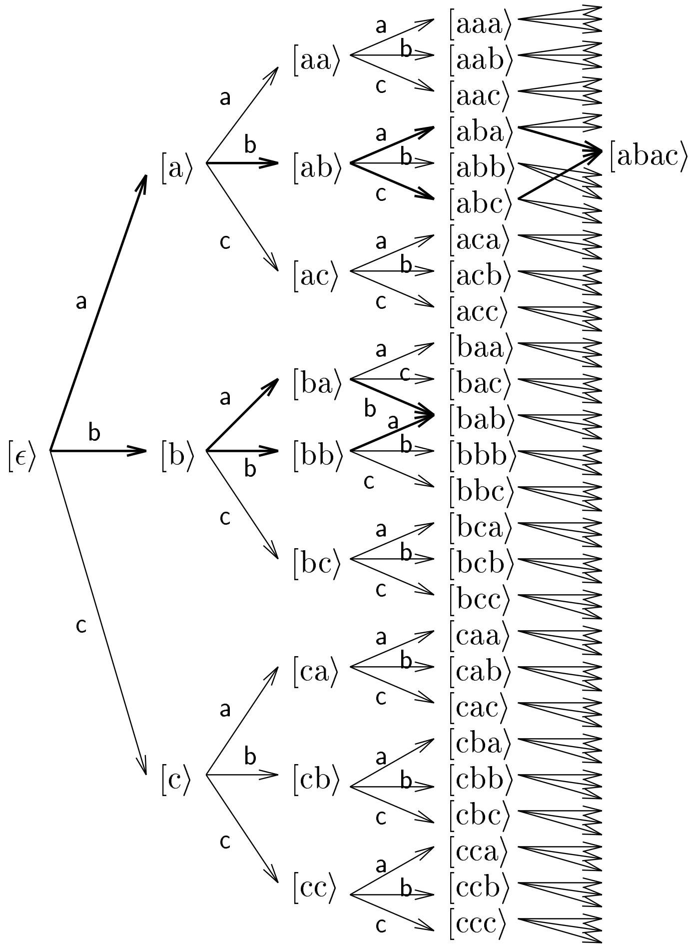


Figure 52: $G(A,(R_{L_1})_r)$

The graph of local traces is always embedded into the quasi-prefix graph. The difference between these graphs is that the parts of the quasi-prefix graph with no concurrency are left out of the graph of local traces. From the construction of the language of local traces associated with L we know that $Lan(L)$ is a subset of $A^*/\approx_L$. Now we can prove that G_L is the restriction of $G_r(A,(R_L)_r)$ to the set $Lan(L)$. So far we only knew the restriction of a reldepgraph to a node. But the restriction of a reldepgraph to a set of nodes means that only the nodes $Lan(L)$ are used as vertices and only the edges which are between nodes of $Lan(L)$ are used.

Theorem 78 *Let A be an alphabet and L a local independence relation. G_L is the restriction of $G(A,(R_L)_r)$ to the set $Lan(L)$.*

Proof:

We know G_L and $G(A,(R_L)_r)$ are each reldepgraphs with root $[\epsilon\rangle_L$. Clearly $Lan(L) \subseteq A^*/\approx_L$. By the definition of $\rightarrow_{\approx_L}$ and $\rightarrow_l$ we have $\rightarrow_{\approx_L}= \{(t,a,t')|t \in A^*/\approx_L \text{ and } t' = t\Diamond a \text{ and } a \in A\}$ and $\rightarrow_L= \{(t,a,t')|t,t' \in Lan(L) \text{ and } t' = t\Diamond a \text{ and } a \in A\}$.
Thus $\rightarrow_L=\rightarrow_{\approx_L}|_{Lan(L)\times A\times Lan(L)}$ and G_L is the the quasi-prefix graph restricted to the set $Lan(L)$. $\Box$

From now on we examine the quasi-prefix graph, since the construction of the graph of local traces is specific for local traces.

8.1.4 Properties of local traces

In this section we show the properties of local traces. First the last example is continued.

Example continued

Let $p = [abac\rangle_{L_1}$. In figure 53 the quasi-prefix graph restricted to $[abac\rangle_{L_1}$ is shown. It is clear that $G(A,(R_{L_1})_r)(p)$ has the forward and backward diamond property and $G(A,(R_{L_1})_r)(p)$ is co-deterministic.

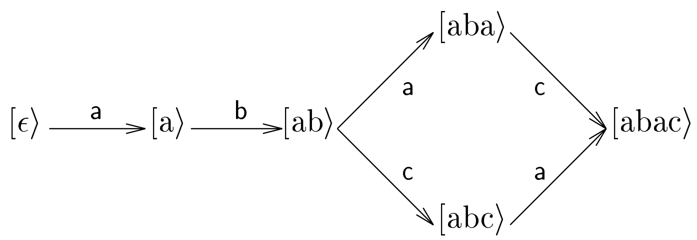


Figure 53: $G(A,(R_{L_1})_r)(p)$

However the quasi-prefix graph restricted to a local trace does not always have the forward diamond property, nor the backward diamond property, nor is always co-deterministic.

In the next example a local independence relation and a local trace are given such that the quasi-prefix graph restricted to the local trace does not have the backward diamond property.

Example

Let $A = \{a,b,c\}$ and $L_2 = \{(\epsilon,\{a,c\}),(c,\{a,b\}),(a,\{b,c\})\}$.
Let $p = [abc\rangle_{L_2} = \{abc, acb, cab, cba\}$. The quasi-prefix graph restricted to p is depicted in figure 54.

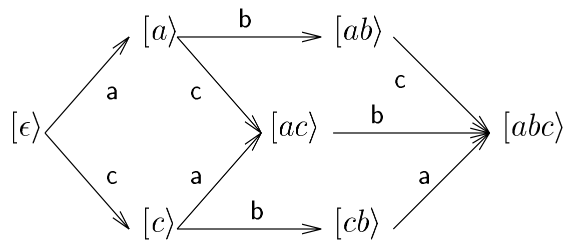


Figure 54: $G(A,(R_{L_2})_r)(p)$

The next example a local independence relation and a local trace are given such that the quasi-prefix graph restricted to the local trace is not co-deterministic and does not have the forward diamond property.

Example

Let $A = \{a, b, c\}$ and $L_3 = \{(\epsilon, \{a, c\}), (\epsilon, \{b, c\}), (b, \{a, c\}), (a, \{b, c\}), (c, \{b, a\})\}$. Let $p = [abc\rangle_{L_3}$. The quasi-prefix graph restricted to p is depicted in figure 55.

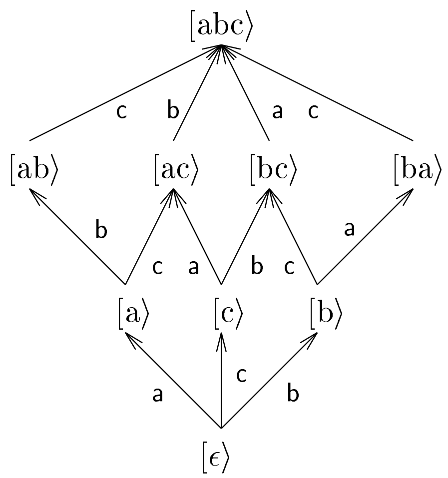


Figure 55: $G(A,(R_{L_3})_r)(p)$

Thus we have the local trace theory which does not have the properties: the quasi-prefix graph has the (compatible) forward diamond property, the quasi-prefix graph has the backward diamond property, and the quasi-prefix graph is co-deterministic. This in contrast with the Mtraces.

The translate function restricted to the set $Pre(p)$, denoted by ζ_p as defined in section 7.1.4, is the morphism from $G(A, R)(p)$ to $\mathcal{C}_{Po(p)}$. The local trace equivalence is a crop trace equivalence thus we can use the results from section 7.1.4. We can directly conclude that the following theorems hold.

From theorem 48:

Theorem 79 *Let A be an alphabet and L a local independence relation. Let $p \in A^*/\approx_L$. If $p = LE(Po(p))$ then ζ_p is surjective and full.*

From theorem 49:

Theorem 80 *Let A be an alphabet and L a local independence relation. Let $p \in A^*/\approx_L$. If ζ_p is injective then $G(A,(R_L)_r)(p)$ is co-deterministic.*

From theorem 62:

Theorem 81 *Let A be an alphabet and L a local independence relation. Let $p \in A^*/\approx_L$. If $G(A,(R_L)_r)(p)$ has the forward diamond property or $G(A,(R_L)_r)(p)$ has the backward diamond property and is co-deterministic then ζ_p is injective and full.*

From theorem 63:

Theorem 82 *Let A be an alphabet and L a local independence relation. Let $p \in A^*/\approx_L$. $G(A,(R_L)_r)(p)$ and $\mathcal{C}_{Po(p)}$ are isomorphic if and only if $(G(A,(R_L)_r)(p)$ has the forward and backward diamond property.*

We also have the function ψ_p from the the configuration graph of a vertex to the quasi-prefix graph restricted to the vertex. Since the local trace equivalence is a crop trace equivalence we can directly use the result from section 7.1.5.

From theorem 68:

Theorem 83 *Let A be an alphabet and L a local independence relation. Let $p \in A^*/\approx_L$. If ψ is a morphism from $\mathcal{C}_{Po(p)}$ to $G(A,(R_L)_r)(p)$ then $\mathcal{C}_{Po(p)}$ and $G(A,(R_L)_r)(p)$ are isomorphic.*

8.1.5 Diagonals in the quasi-prefix graph

Let A be an alphabet and L a local independence relation.

When we have a quasi-prefix graph restricted to a local trace p, and there exist $q, q_1, q_2, q' \in Pre(p)$ such that $q \xrightarrow{a}_{\approx_L} q_1$, $q \xrightarrow{b}_{\approx_L} q_2$, $q_1 \xrightarrow{b}_{\approx_L} q'$, and $q_2 \xrightarrow{a}_{\approx_L} q'$ for some $a, b \in A$, then a and b do not have to be concurrent after q. In the next example, from [BR95], this will be shown.

Example

Let $A = \{a,b,c\}$ and $L = \{(\epsilon,\{a,b\}),\ (\epsilon,\{b,c\}),\ (\epsilon,\{a,c\}),\ (b,\{a,c\}),\ (c,\{a,b\})\}$. Then we know that after the history a the actions b and c are not concurrent. However $abc \approx_L bac \approx_L bca \approx_L cba \approx_L cab \approx_L acb$.

To illustrate this we have depicted the quasi-prefix graph of the local trace $[abc\rangle_L$ in figure 56.

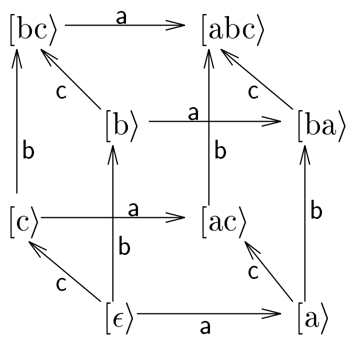


Figure 56: $G(A, (R_L)_r)([abc\rangle_L)$

Thus we can not conclude that a "diamond" of two directed edges imply that the two actions involved are concurrent. This leads to the introduction of diagonals in the quasi-prefix graph.

Definition

Let A be an alphabet and L a local independence relation. The *quasi-prefix graph with diagonals* is the elgraph with initial node $Gd(A, (R_L)_r) = (A^*/ \approx_L, A^*, \rightarrow_{d\approx_L}, [\epsilon\rangle_L)$, where $[u\rangle_L \xrightarrow{y}_{d\approx_L} [v\rangle_L$ for some $u, v, y \in A^*$ if there exist $(x, S) \in L$, $S', S'' \subseteq S$, and $y' \in A^*$ such that $S' \cap S'' = \emptyset$, $alph(y) = S'$, $alph(y') = S''$, $[u\rangle_L = [x\rangle_L \Diamond y'$, and $[v\rangle_L = [u\rangle_L \Diamond y$.

Now we can construct the quasi-prefix graph with diagonals.

<u>Example continued</u>

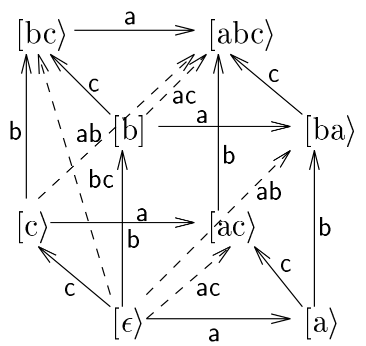


Figure 57: $Gd(A, (R_L)_r)([abc\rangle_L)$

We will say that diagonals are useful if the diagonals in the quasi-prefix graph add information to the interpretation of the quasi-prefix graph. There appears to be a connection between the property $G(A,(R_L)_r)(p)$ has the cube and inverse cube axiom and the property diagonals are useful in the quasi-prefix graph.

Conjecture *Let A be an alphabet and L a local independence relation. $G(A,(R_L)_r)$ has the cube and inverse cube axiom if and only if diagonals in the quasi-prefix graph are not useful.*

The proof of the if-direction is easy. When diagonals are not useful, the quasi-prefix graph with diagonals has the forward and backward diamond property. From theorem 8 and 9 follows that $G(A,(R_L)_r)$ satisfies the cube and inverse cube axiom.

8.1.6 Conclusions

The local trace theory is a restriction of the crop trace theory, but the quasi-prefix graph restricted to a local trace has no special properties. However the difference between the local trace theory and the crop trace theory has been made clear in the first example of this subsection. Thus the local trace theory is a restriction of the crop trace theory since not all the trace equivalences described by crop trace equivalence can be described by local trace equivalences. An interesting aspect of the local trace theory is the possibility of creating a graph of local traces which shows only the part of the quasi-prefix graph with concurrency.

8.2 Commutations with a context

Bauget and Gastin describe in [BG95] trace equivalences generated by left-context commutation relations. In subsection 8.3 the equivalences generated by a left-context commutation relation is handled. In that subsection the equivalences generated by a right-context commutation relation are also investigated. First however we examine a trace equivalence generated by a context commutation relation.

8.2.1 Cc trace equivalence

Let A be an alphabet and C an event-preserving relation over A^*. The relation C is a *context commutation relation over* A if for all $(v, w) \in C$ there exist $a, b \in A$ and $x, y \in A^*$ such that $v = xaby$ and $w = xbay$.

Definition

Let A be an alphabet and C a context commutation relation over A. Let $x \in A^*$.

1. The *cc trace equivalence* induced by C is defined by $\equiv_C$, the congruence induced by C.
2. $\langle x \rangle_C = \{z \in A^* |\ z \equiv_C x\}$ the equivalence class containing x is the *cc trace* containing x.

8.2.2 Difference between cop and cc traces

It is easy to see that a cc trace equivalence is a cop trace equivalence. Thus we have the well defined concatenation $\cdot$ between cc traces and a trace monoid $M(A, C)$. Between the cc traces we have a prefix ordering $\preceq_C$ such that $p \preceq_C q$ if and only if there exists $w \in A^*$ such that $p\langle w \rangle_C = q$.

We have a prefix graph $G(A, C) = (M(A, C), A, \rightarrow_{\equiv_C}, \langle \epsilon \rangle_C)$, similar to cop traces such that $G(A, C)$ is a reldepgraph and $p \xrightarrow{a}_{\equiv_C} q$ for some $a \in A$ if and only if $p \prec\!\cdot_C\, q$.

The difference between cop and cc traces is the relation by which the congruence is induced. The next example illustrates this.

<u>Example</u>

Let $A = \{a, b, c\}$ and $R = \{(abc, cba)\}$. Let $p = \langle abc \rangle$. Then $p = \{abc, cba\}$.

The congruence $\equiv_R$ can not be described by a context commutation relation.

Mtraces are cc traces, since the relation C_I is a context commutation relation.

8.2.3 Properties of cc traces

First some examples are given which illustrate the fact that cc traces have no special properties concerning the prefix graph.

Example

Let $A = \{a, b, c\}$ and $C_1 = \{(ac, ca), (cab, cba), (abc, acb)\}$. Let $p = \langle abc \rangle_{C_1}$. Then $p = \{abc, acb, cab, cba\}$. The prefix graph restricted to the cc trace p is depicted in figure 58.

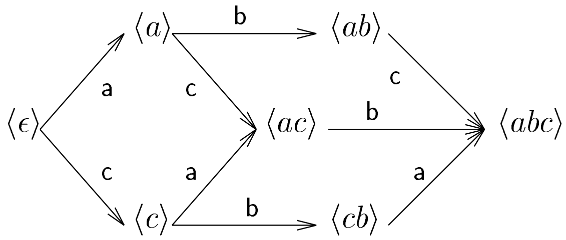


Figure 58: $G(A, C_1)(p)$

The prefix graph restricted to p has the forward diamond property, but not the backward diamond propery. The prefix graph restricted to p is also co-deterministic.

Example

Let $A = \{a, b, c\}$ and $C_2 = \{(ac, ca), (bc, cb), (cba, cab)\}$. Let $p = \langle abc \rangle_{C_2}$. The prefix graph restricted to p is depicted in figure 59.

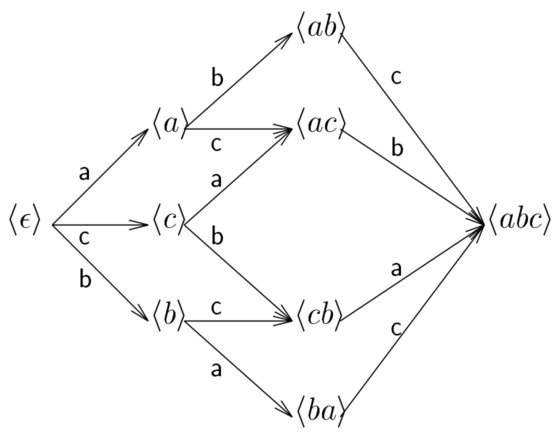


Figure 59: $G(A, C_2)(p)$

The prefix graph restricted to p does not have the forward diamond property, and is not co-deterministic.

Thus now we know that the prefix graph of a cc trace does not always have the (compatible) forward or the backward diamond property. We also know that the prefix graph of a cc trace is not always co-deterministic.

The translate function restricted to the set $Pre(p)$, denoted by ζ_p as defined in section 7.1.4, is the morphism from $G(A, R)(p)$ to $\mathcal{C}_{Po(p)}$. A cc trace equivalence is a cop trace equivalence. Thus we can use the results of section 7.2.3. We can directly conclude that following theorems hold.

From theorem 71:

Theorem 84 *Let A be an alphabet and C a context commutation relation over A. Let $p \in M(A, C)$. If $p = LE(Po(p))$ then ζ_p is surjective and full.*

From theorem 72:

Theorem 85 *Let A be an alphabet and C a context commutation relation over A. Let $p \in M(A, C)$.*
If ζ_p is injective then $G(A, C)(p)$ is co-deterministic.

From theorem 73:

Theorem 86 *Let A be an alphabet and C a context commutation relation over A. Let $p \in M(A, C)$. If $FD(G(A, C)(p))$ or $G(A, C)(p)$ is co-deterministic then $G(A, C)(p) \subseteq \mathcal{C}_{Po(p)}$ and ζ_p is full.*

From theorem 74

Theorem 87 *Let A be an alphabet and C a context commutation relation over A. Let $p \in M(A, C)$. $G(A, C)(p)$ and $\mathcal{C}_{Po(p)}$ are isomorphic if and only if $G(A, C)(p)$ has the forward diamond property.*

The function ψ_p as defined in section 7.1.5 from the configuration graph of a cc trace to the prefix graph restricted to the cc trace can be a morphism. Since the cc trace equivalence is a crop trace equivalence we can use the result from section 7.1.5.

From theorem 75:

Theorem 88 *Let A be an alphabet and C a context commutation relation over A. Let $p \in M(A, C)$. If ψ is is a morphism from $\mathcal{C}_{Po(p)}$ to $G(A, C)(p)$ then $\mathcal{C}_{Po(p)}$ and $G(A, C)(p)$ are isomorphic.*

8.2.4 Represented by partial orders

The next theorem has a result which is very important, since it implies that all congruences which can be represented by partial orders can be generated by a context commutation relation.

Theorem 89 *Let A be an alphabet and R an independence relation over A^*. If $\equiv_R$ is a congruence which can be represented by partial orders, then $\equiv_R$ can be generated by a context commutation relation.*

Proof:

Suppose we have $u \in A^*$. Let C_u be the context commutation relation induced by the labelled poset $Po(\langle u \rangle_{\equiv_R})$. By lemma 13 and 14 we know that $LE(Po(\langle u \rangle_{\equiv_R})) = \langle u \rangle_{C_u}$. Since $\equiv_R$ is represented by partial orders we have $LE(Po(\langle u \rangle_{\equiv_R})) = \langle u \rangle_{\equiv_R}$. Thus $\langle u \rangle_{\equiv_R} = \langle u \rangle_{C_u}$. Let $C = \bigcup_{u \in A^*} C_u$ then if $u \equiv_R v$ then $u \equiv_C v$.

Conversely, suppose $u \equiv_C v$, then there exist $u_0, \ldots, u_k \in A^*$ with $k \geq 0$ such that $u = u_0$, $v = u_k$, and $u_i = u_{i_1} x_i u_{i_2}$ and $u_{i+1} = u_{i_1} y_i u_{i_2}$ with $(x_i, y_i) \in R$ for $0 \leq i \leq k-1$. $\equiv_R$ is the congruence generated by R, thus for all $u_i \equiv_R u_{i+1}$ $0 \leq i \leq k-1$. Then $u \equiv_R v$.

Thus $u \equiv_R v \Leftrightarrow u \equiv_C v$.

Because C is a contex commutation relation we know that $\equiv_R$ can be generated by a context commutation relation. □

The converse of the theorem does not hold. The next example illustrates this.

<u>Example continued</u>

Let $A = \{a, b, c,\}$ and $C_1 = \{(ac, ca), (abc, acb), (cab, cba)\}$ We have $p = \langle abc \rangle_{C_1} = \{abc, acb, cab, cba\}$. The partial order of p is depicted in figure 60.

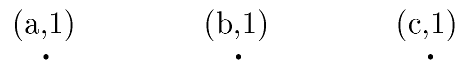


Figure 60: $Po(p)$

It is clear that $LE(Po(p)) = \{abc, acb, bac, bca, cab, cba\}$. Thus C_1 is a context commutation relation but $\equiv_{C_1}$ can not be represented by partial orders.

<u>Example</u>

Let $A = a, b$ and $C = \{aab, aba\}$ a context commutation relation. The congruence $\equiv_C$ can be generated by partial orders. Let $p = [a]_C = \{a\}$ and $q = [ba]_C = \{ba\}$.

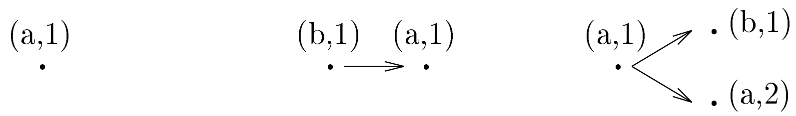


Figure 61: $Po(p)$, $Po(q)$, and $Po(pq)$

The Hasse diagrams of the partial orders of p, q, and pq are depicted in figure 61. $\equiv_C$ can be represented by partial orders, however $\equiv_C$ can not be modularly represented by partial orders. Since $\prec_{pq} \cap t_p(E_q) \times t_p(E_q) \neq t_p(\prec_q)$.

If we compare this result with the result from the Mazurkiewicz trace theory, $\equiv_I$ can be modularly represented by partial orders, we can conclude that cc traces form a strict generalization of Mazurkiewicz traces.

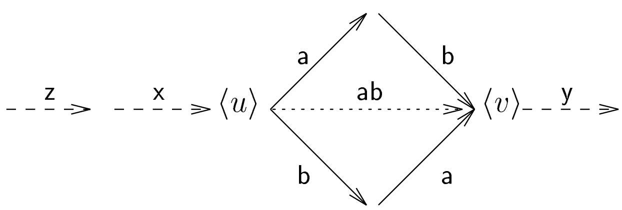
8.2.5 Diagonals in the prefix graph

Let A be an alphabet and C a context commutation relation.

As with local traces, diagonals are sometimes useful. First we define the prefix-graph with diagonals.

Definition

Let A be an alphabet and C a context commutation relation. The *prefix graph with diagonals* is the elgraph with initial node $Gd(A, C) = (M(A, C), A^*, \rightarrow_{d\equiv_C}, \langle\epsilon\rangle_C)$, where $\langle u\rangle_C \xrightarrow{ab}_{d\equiv_C} \langle v\rangle_C$ for some $u, v \in A^*$ and $a, b \in A$ if there exist $x, y, z \in A^*$ such that $u = zxab$, $v = zxba$, $(xaby, xbay) \in C$, and $u, v \in Pre(zxaby)$.



The role of diagonals may be connected to the cube and inverse cube axioms.

Conjecture *Let A be an alphabet and C a context commutation relation. $Gd(A, C)$ has the cube and inverse cube axiom if and only if diagonals are not useful.*

8.2.6 Conclusions

If we have a congruence that can be represented by a partial order then this congruence can be generated by a context commutation relation. This conclusion makes the subclass of cc trace equivalences interesting for investigation.

Another important property of cc traces is that we can construct a prefix-graph with diagonals such that if a vertex can be reached from a vertex through two concurrent events, then there exists a diagonal in the prefix graph. The use of diagonals is sometimes not needed. I suspect that diagonals in the prefix graph are only useful whenever the prefix graph satisfies the cube and inverse cube axioms.

8.3 Left and right context commutations

In [BG95] Bauget and Gastin describe a trace equivalence which has an event-preserving relation with a restriction. First the event-preserving relation is a context commutation relation and second the relation has only a left context. We have extended this with a context commutation relation with only a right context.

8.3.1 Lcc trace equivalence and rcc trace equivalence

Let A be an alphabet and C an event-preserving relation over A^*. The relation C is a *left-context commutation relation over* A if for all $(v, w) \in C$ there exist $x \in A$ and $a, b \in A$ such that $v = xab$ and $w = xba$.

The relation C is a *right-context commutation relation over* A if for all $(v, w) \in C$ there exist $x \in A$ and $a, b \in A$ such that $v = abx$ and $w = bax$.

Definition

Let A be an alphabet, C_l a left-context commutation relation over A, and C_r a right-context commutation relation over A. Let $x \in A^*$.

1. The *lcc trace equivalence* induced by C_l is defined by $\equiv_{C_l}$, the congruence induced by C_l.

2. $\langle x\rangle_{C_l} = \{z \in A^* | z \equiv_{C_l} x\}$ the equivalence class containg x is the *lcc trace* containing x.

3. The *rcc trace equivalence* induced by C_r is defined by $\equiv_{C_r}$, the congruence induced by C_r.

4. $\langle x\rangle_{C_r} = \{z \in A^* | z \equiv_{C_r} x\}$ the equivalence class containg x is the *rcc trace* containing x.

8.3.2 Difference between cc and rcc and lcc traces

The defined rcc and lcc trace equivalences are also cc trace equivalences, since a right-context commutation relation is a context commutation relation and a left-context commutation relation is a context commutation relation. We have the well defined concatenation $\cdot$ between rcc and lcc traces and a trace monoid $M(A, C)$. Between the rcc and lcc traces we have a prefix ordering $\preceq_C$ such that $p \preceq_C q$ if and only if there exists $w \in A^*$ such that $p\langle w\rangle_C = q$.

We have a prefix graph $G(A, C) = (M(A, C), A, \rightarrow_{\equiv_C}, \langle\epsilon\rangle_C)$, similar to cc traces such that $G(A, C)$ is a reldepgraph and $p \xrightarrow{a}_{\equiv_C} q$ for some $a \in A$ if and only if $p \prec\!\!\cdot_C\, q$.

The next example shows a congruence induced by a context commutation relation which can not be induced by a left-context commutation relation nor a right-context commutation relation.

<u>Example</u>

Let $A = \{a, b, c\}$ and $R = \{(abc, acb), (abc, bac)\}$. Let $p = \langle abc\rangle$. Then $p = \{abc, bac, acb\}$. The congruence $\equiv_R$ can not be described by a left-context commutation relation nor a right-context commutation relation.

Mtraces are rcc and lcc traces, since the relation C_I is a context commutation relation without left context and right context.

The translate function restricted to the set $Pre(p)$, denoted by ζ_p as defined in section 7.1.4, is the morphism from $G(A, R)(p)$ to $\mathcal{C}_{Po(p)}$. Theorem 84 to theorem 88 also hold for rcc traces and lcc traces.

8.3.3 The link with local traces

Let A be an alphabet and C be a left-context commutation relation over A.

Local independence relations and left-context commutation relations can be related. This relation is expressed by the function $\alpha : (A^* \times A^*) \rightarrow (A^* \times P_f(A))$ and is defined in the following way:
$\alpha((uab, uba)) = \{(xu, \{a, b\}) | x \in A^*\}$ and
$\alpha(C) = \bigcup_{r\in C} \alpha(r)$.

With this function we can prove the next lemma the result of which is used in the proof of theorem 91.

Lemma 90 *Let A be an alphabet, let C be a left-context commutation relation* *over A. Let $x, y \in A^*$ and $a, b \in A$. $xaby \doteq_C xbay$ if and only if $xaby \dot{\approx}_{\alpha(C)} xbay$.*

Proof:

If $xaby \doteq_C xbay$ then $(uab, uba) \in C$ for some $u \in A^*$ such that $x = vu$. Thus $(vu, \{a, b\}) \in \alpha(C)$. Therefore $xaby \dot{\approx}_{\alpha(C)} xay$.

If $xaby \dot{\approx}_{\alpha(C)} xbay$ then $(x, \{a, b\}) \in \alpha(C)$. From the function α we know there exists $(uab, uba) \in C$, where $u \in A^*$ is such that $x = vu$. Thus $xaby \doteq_C xbay$.

We can conclude that the lemma holds. □

Theorem 91 *Let A be an alphabet, let C a left-context commutation relation over A. Then $\equiv_C = \approx_{\alpha(C)}$.*

Proof:

Let $w, w' \in A^*$ then $w \equiv_C w'$ if and only if there exist $w_0, \ldots, w_k \in A^*$ such that $w = w_0 \doteq_C w_1 \doteq_C \ldots \doteq_C w_k = w'$.

With lemma 90 we know this holds if and only if $w = w_0 \dot{\approx}_{\alpha(C)} w_1 \dot{\approx}_{\alpha(C)} \ldots \dot{\approx}_{\alpha(C)} w_k = w'$.

Thus if and only if $w \approx_{\alpha(C)} w'$. □

Now we know that all the lcc traces are local traces, however there exist local traces which are not lcc traces. The next example illustrates this.

Example

Let $A = \{a, b, c\}$ and $L = \{(\epsilon, \{a, b\})\}$. Let $p = [abab\rangle_L$. Then $p = \{abab, baab\}$. Now we want to define a left-context commutation relation C such that the lcc trace $q = \langle abab\rangle_C$ is equal to the local trace p.

Such $C \subseteq \{(ab, ba)\}$. Then $q \subseteq \{abab, abba, baab, baba\}$. But $abba \notin p$, thus there exist no left-context commutation relation C such that $\equiv_C = \approx_L$.

Now we can compare the rcc traces with the local traces.

Example

Let $A = \{a, b, c\}$ and $C = \{(abc, bac)\}$. C is a right-context commutation relation. Let $p = \langle abc\rangle_C$. Then $p = \{abc, bac\}$. Suppose $L \subseteq \{(\epsilon, \{a, b\})\}$. Let $q = [abc\rangle_L$. Then $q \subseteq \{abc, bac\}$. Now compare $p' = \langle ab\rangle_C$ and $q' = [ab\rangle_L$. Then $p' = \{ab\}$ and $q' = \{ab, ba\}$. It is clear that exists no local independence relation L such that $\approx_L = \equiv_C$.

Thus there are rcc traces which are not local traces.

Example

Let $A = \{a, b, c\}$ and $L = \{(\epsilon, \{a, b\})\}$. Let $p = [ab\rangle_L$. Then $p = \{abba\}$. Now we have to define a right-context commutation relation. Suppose $C \subseteq \{(ab, ba)\}$. Then $q = \langle ab\rangle_C = \{ab, ba\}$. Now compare $p' = [aab\rangle_L$ and $q' = \langle aab\rangle$. Then $q' \subseteq \{aab, aba\}$ and so $q' \neq p = \{aab\}$. Since C is a right-context commutation relation we can not define that the commutation (ab, ba) only takes place after an empty word $\langle\epsilon\rangle_C$.

Thus there exist local traces which are not rcc traces.

8.3.4 Properties of lcctraces and rcctraces

The lcc traces have no special properties for the prefix graph restricted to a vertex. The next examples, previously used in subsection 8.2, illustrate this.

Example

Let $A = \{a, b, c\}$ and $C_1 = \{(ac, ca), (cab, cba), (abc, acb)\}$. Let $p = \langle abc \rangle_{C_1}$. Then $p = \{abc, acb, cab, cba\}$. The prefix graph restricted to the lcc trace p is depicted in figure 62.

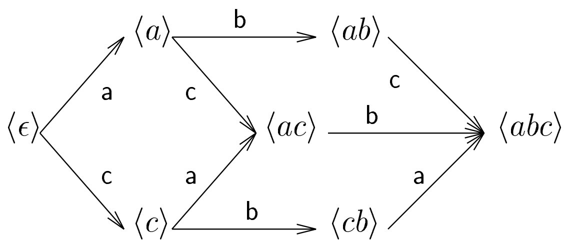


Figure 62: $G(A, C_1)(p)$

The prefix graph restricted to p has the forward diamond property, but not the backward diamond propery. The prefix graph restricted to p is co-deterministic.

Example

Let $A = \{a, b, c\}$ and $C_2 = \{(ac, ca), (bc, cb), (cba, cab)\}$. Let $p = \langle abc \rangle_{C_2}$. The prefix graph restricted to the lcc trace p is depicted in figure 63.

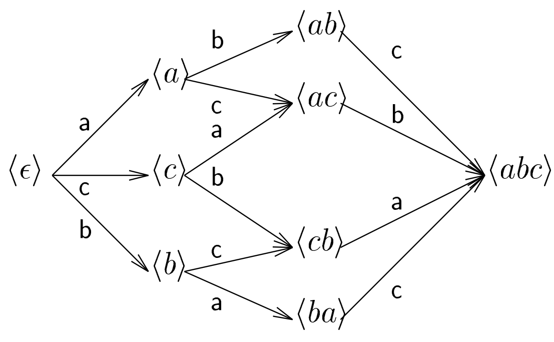


Figure 63: $G(A, C_2)(p)$

The prefix graph restricted to p has not the forward diamond property, and is not co-deterministic.

The rcc traces have also no special properties for the prefix graph restricted to a vertex. This is illustrated in the next examples.

Example

Let $A = \{a, b, c\}$ and $C_3 = \{(ac, ca), (abd, bad), (bcd, cbd)\}$. Let $p = \langle abcd \rangle_{C_3}$. Then $p = \{abcd, acbd, cabd, cbad\}$. The prefix graph restricted to the rcc trace p is depicted in figure 64.

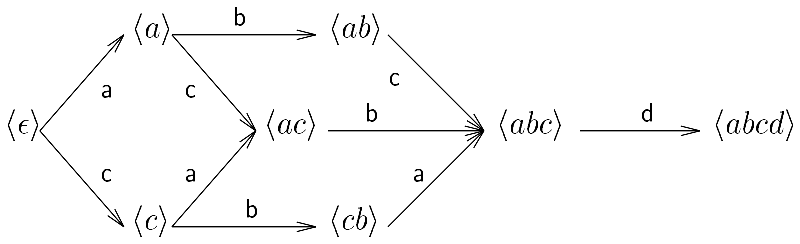


Figure 64: $G(A, C_3)(p)$

The prefix graph restricted to p has the forward diamond property, but not the backward diamond propery. The prefix graph restricted to p is co-deterministic.

Example

Let $A = \{a, b, c, d\}$ and $C_4 = \{(ac, ca), (bc, cb), (abd, bad)\}$. Let $p = \langle abcd \rangle_{C_4}$. The prefix graph restricted to the rcc trace p is depicted in figure 65.

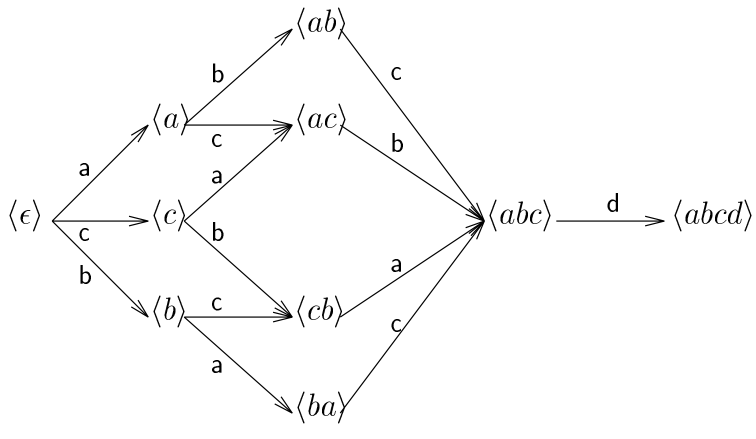


Figure 65: $G(A, C_4)(p)$

The prefix graph is not co-deterministic and does not have the forward diamond property.

8.3.5 Represented by partial orders

We know that if a trace equivalence can be represented by partial orders then the trace equivalence can be generated by a commutation relation. The next theorem shows the connection between congruences which can be represented by partial orders and are right-cancellative and congruences which can be generated by left-context commutation relations.

Theorem 92 *[BG95] Let A be an alphabet. If $\equiv$ is a congruence over A which can be represented by partial orders and is right-cancellative, then $\equiv$ can be generated by a left-context commutation relation.*

Proof:

We know from theorem 89 that $\equiv$ can be generated by a context commutation relation C.
Suppose $(xaby, xbay) \in C$ then because $\equiv$ is right-cancellative, we know that $(xab, xba) \in \equiv_C$. Since $\equiv_C$ is a congruence $(xaby, xbay) \in C$ is irrelevant and can be replaced by (xab, xba) without changing $\equiv_C$. Therefore $\equiv$ can be generated by a left-context relation. □

The next example shows that there exist lcc trace equivalences which can not be represented by partial orders.

Example

Let $A = \{a, b, c\}$ and $C = \{(ac, ca), (cab, cba), (abc, acb)\}$. C is a left context commutation relation. Let $p = \langle abc \rangle_C$. Then $p = \{abc, acb, cab, cba\}$. The Hasse diagram of the partial order of p is depicted in figure 66.

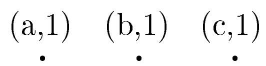


Figure 66: $Po(p)$

It is clear that $LE(Po(p)) = \{abc, acb, bac, bca, cab, cba\}$ and thus that $\equiv_C$ can not be represented by partial orders.

Similarly a congruence which can be represented by partial orders and which is left-cancellative can be generated by a right-context commutation relation.

Theorem 93 *[BG95] Let A be an alphabet.If $\equiv$ is a congruence over A which is represented by partial orders and is left-cancellative, then $\equiv$ can be generated by a right-context commutation relation.*

The next example shows that there exist rcc trace equivalences which can not be represented by partial orders.

Example

Let $A = \{a, b, c\}$ and $C = \{(acb, cab), (abd, bad), (bcd, cbd)\}$. C is a right-context commutation relation. Let $p = \langle abcd \rangle_C$. Then $p = \{abcd, acbd, cabd, cbad\}$. The Hasse diagram of the partial order is depicted in figure 67.

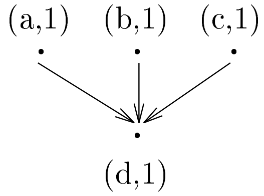


Figure 67: $Po(p)$

It is clear that $LE(Po(p)) = \{abcd, acbd, bacd, bcad, cabd, cbad\}$ and thus that $\equiv_C$ can not be represented by partial orders.

Let A be an alphabet and C an independence relation over A^*. The relation C is a *commutation relation over* A if for all $(v, w) \in C$ there exist $a, b \in A$ such that $v = ab$ and $w = ba$.

The next step is to examine a congruence which can be represented by partial orders and is left and right-cancellative. The next theorem shows that then the congruence can be generated by a commutation relation.

Theorem 94 *[BG95] Let A be an alphabet. Let $\equiv$ be a congruence over A. If $\equiv$ can be represented by partial orders and is cancellative then $\equiv$ can be generated by a commutation relation.*

Proof:

We know from theorem 89 that $\equiv$ can be generated by a context commutation relation C.

Suppose $(xaby, xbay) \in C$ then because $\equiv$ is cancellative we know $(ab, ba) \in \equiv_C$. Since $\equiv$ is a congruence $(xaby, xbay) \in C$ is irrelevant and can be replaced by (ab, ba) without changing $\equiv_C$. Therefore $\equiv$ can be generated by a commutation relation. $\square$

If a congruence can be modularly represented by partial orders then the congruence is cancellative.

Theorem 95 *Let A be an alphabet and $\equiv$ a congruence over A. If $\equiv$ can be modularly represented by partial orders then $\equiv$ is cancellative.*

Proof:

First we will prove that if $\equiv$ can be modularly represented by partial orders then $\equiv$ is right-cancellative.

Let $u, v, w \in A^*$ such that $uw \equiv vw$. Then $E_{uw} = E_{vw}$. Thus $E_u = E_v$.

$uw \equiv vw$ implies $Po(\langle uw \rangle) = Po(\langle vw \rangle)$, thus $\leq_{\langle uw \rangle} = \leq_{\langle vw \rangle}$. Since $\equiv$ is modularly represented by partial orders we know $\leq_{\langle uw \rangle} \cap (E_u \times E_u) = \leq_{\langle u \rangle}$ and $\leq_{\langle vw \rangle} \cap (E_v \times E_v) = \leq_{\langle v \rangle}$. We have $E_u = E_v$ and $\leq_{\langle uw \rangle} = \leq_{\langle vw \rangle}$ thus $\leq_{\langle u \rangle} = \leq_{\langle v \rangle}$. Then $Po(\langle u \rangle) = Po(\langle v \rangle)$. Now we can conclude that $\langle u \rangle = l_A(LE(Po(\langle u \rangle))) = l_A(LE(Po(\langle v \rangle))) = \langle v \rangle$. Thus $u \equiv v$.

Similarly we can prove that $\equiv$ is left-cancellative. □

From theorem 94 and theorem 95 follows that a congruence which can be modularly represented by partial orders can be generated by a commutation relation. On the other hand we also know that a commutation relation can de described as a independence relation. Thus a commutation relation generates a Mtrace equivalence. In section 6 we concluded that a Mtrace equivalence can be modularly represented by partial orders. Thus we can conclude that the following theorem holds.

Theorem 96 *$\equiv$ can be modularly represented by partial orders if and only if $\equiv$ is a Mazurkiewicz trace equivalence.*

8.3.6 Conclusions

In this subsection we have concluded that the cc trace theory is a generalization of the rcc trace theory and the lcc trace theory. On the other hand the Mazurkiewicz trace theory is a specialization of the rcc and lcc trace theory. The rcc and lcc trace equivalences have no special properties, like the (compatible) forward or backward diamond property. The difference between rcc and lcc traces is the fact that lcc traces are local traces and rcc traces not. This also implies that cc traces are dfferent from local traces.

An important result is theorem 96, which states that any congruence which can be modularly represented by partial orders is a Mazurkiewicz trace equivalence.

8.4 Commutations with a limited left context

As defined in section 6.2 of [BR95]there are also context commutation relations which have a limited left context. Biermann and Rozoy only describe a context commutation relation where the left context consists of one letter. In section 8.4.1 we extend this by allowing the left context to exist of words of a certain length. In the next section the resulting k-context traces will be compared with the local traces from section 8.1. In section 8.4.4 we define ≤k-context traces which have a limited left context the length of which may vary between a limit k and 0. And of course there will be some properties about the relation between the prefix graph and the graph of configurations.

8.4.1 K-context trace equivalence

A special case of the left context commutation relation is the left context commutation relation with a fixed length of the context.

Let A be an alphabet and C a left-context commutation relation over A. Let $k \geq 0$. C is a *k-context commutation relation over* A if $C \subseteq A^{k+2} \times A^{k+2}$.

The lcc trace equivalence induced by a k-context commutation relation over A is a *k-context trace equivalence.*

Example

A well-known example of a k-context commutation relation is the Producer/Consumer Paradigm. This example is also described in [BR95].

The principle behind the Producer/Consumer Paradigm is that a consumer can consume after a producer has produced. However a consumer does not have to consume immediately after the production. The consumer can build a reserve. The independence relation over A^*, where A is an alphabet containing p, for produce, and c, for consume, is therefore:
$C = \{(ppc, pcp)\}$.
Note that C is a 1-context commutation relation over $\{p, c\}$ leading to a 1-context trace equivalence $\equiv_C$. The prefixgraph $G(A, C)(p)$ where p is the 1-context trace $\langle pcpcpcpc \rangle_{\equiv_C}$ is depicted in figure 68.

8.4.2 Difference between lcc and k-context traces

It is easy to see that the k-context commutation relation is a left-context commutation relation. Thus k-context traces are lcc traces. However there exist lcc traces which are not k-context traces. The next example illustrates this.

Example

Let $A = \{a, b, c\}$ and $C = \{(ac, ca), (cab, cba), (abc, acb)\}$. Let $p = \langle abc \rangle_C$. C is a left-context commutation relation. Since $(ac, ca), (cab, cba) \in C$ C is not a k-context commutation relation. The prefix-graph restricted to the lcc trace p is depicted in figure 62. Let $p' \langle abc \rangle_{C'}$.

Suppose $C' = \{(ac, ca), (ab, ba), (cb, bc)\}$. Then $p' = \{abc, acb, bac, bca, cab, cba\} \neq p$. Suppose $C' = \{(cab, cba), (abc, acb)\}$. Then $p' = \{abc, acb\} \neq p$. We want $cab \in p'$. Since C' is a 1-context commutation relation this is not possible.

It is clear that there does not exists a k-context commutation relation C' such that $\equiv_{C'} = \equiv_C$.

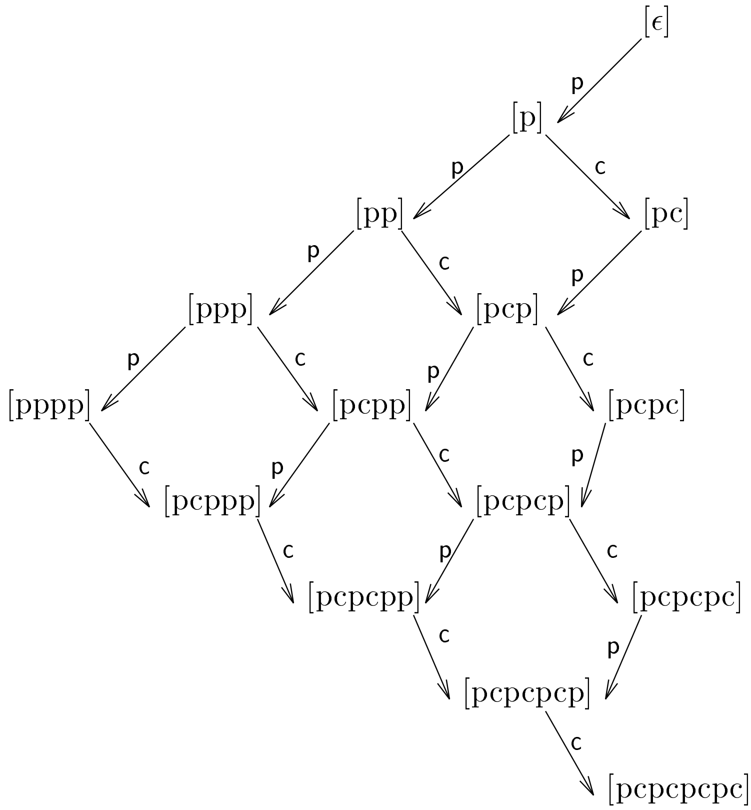


Figure 68: $G(A, C)(p)$

8.4.3 The link with local traces

Let A be an alphabet, $k \geq 0$ and C be a k-context commutation relation over A.

Local independence relations and k-context commutation relations can be related. This relation is expressed by the function $\alpha : (A^* \times A^*) \rightarrow (A^* \times P_f(A))$ and is defined in section8.3.3.

The next example shows that there exist local traces which can not be described by a k-context trace equivalence.

Example

Let $A = \{a, b, c\}$ and $L = \{(c, \{a, b\})\}$. Let p be the local trace containing $cabcab$. Then $p = \{cabcab, cbacab\}$.

Let $C = \{(cab, cba)\}$. C is a 1-context commutation relation. Let p' be the k-context trace containing $cabcab$ then
$p' = \{cabcab, cbacab, cbacba, cabcba\}$.
It is clear that the k-context trace equivalence $\equiv_C$ is different from the local trace equivalence $\approx_L$ and there does not exists a k-context commutation relation C' such that $\equiv_{C'}$ is equal to $\approx_L$.

Example

Let $A = \{a, b, c\}$ and $C = \{(abc, acb), (acd, adc), (cbd, cdb), (dcb, dbc)\}$. Let $p = \langle abcd \rangle_C$. The prefix graph restricted to p is depicted in figure 69.

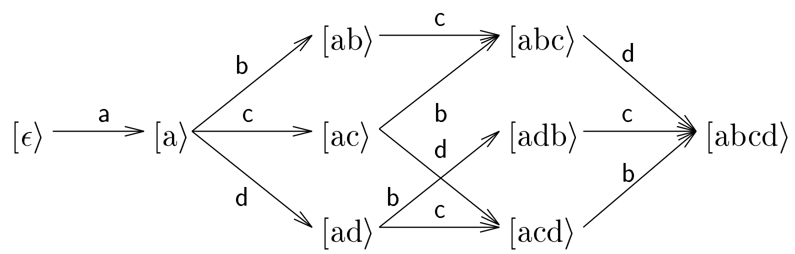


Figure 69: $G(A, C)(p)$

The relation C is a k-context commutation relation. We have defined a function α which relates the k-context commutation relation with a local independence relation. We get the following local independence relation $L' = \alpha(C)$:
$L' = \{(xa, \{b, c\})|x \in A^*\} \cup \{(xa, \{c, d\}|x \in A^*\} \cup \{(xc, \{b, d\}|x \in A^*\} \cup \{(xd, \{b, c\}|x \in A^*\}$.

This example shows that the prefix graph restricted to a k-context trace has no special properties. The prefix graph has not the forward nor the backward diamond property. If we add to the relation C the pair (bcd, bdc) we have a prefix graph which is not co-deterministic. We can therefore conclude that the prefix graph has no properties. Since the k-context commutation relation is also a left-context commutation relation we have the same results as in section 8.3.4.

8.4.4 ≤K-context trace equivalence

Let A be an alphabet, $k \geq 0$ and C a commutation relation over A. Then C is a *≤k-context commutation relation over A* if for all $(v, w) \in C$ there exists $x \in A^*$ and $a, b \in A$ such that $v = xab$ and $w = xba$ and $|x| \leq k$.

The *≤k-context trace congruence generated by C* is defined by $\equiv_C$.

All the terminology and notations for k-context trace equivalences will be used for ≤k-context trace equivalences.

Example

Let $A = \{a, b, c\}$ and $C = \{(ab, ba), (bac, bca)\}$. The prefix graph restricted to $\langle abc \rangle_C$ is depicted in figure 70.

This example shows that the prefix graph can have some properties, like the forward and backward diamond property. From the definition of a ≤k-context commutation relation follows that a k-context commutation relation is a ≤k-

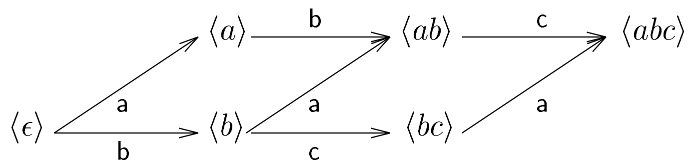


Figure 70: $G(A, C)(\langle abc \rangle)$

context commutation relation. Thus a k-context trace is a ≤k-context trace. The ≤k-context commutation relation can be rewritten into a k-context commutation relation by the following function. $\beta((uab, uba)) = \{(xuab, xuba) | xu \in A^k\}$ $\beta(C) = \bigcup_{r \in C} \beta(r)$. However if we compare the congruence induced by $\beta(C)$, where C is a k-context commutation relation, with the congruence induced by C, we can conclude that there exists a difference between the congruences. In the next example this is illustrated.

Example continued

The resulting k-context commutation relation is $C' = \{(aab, aba), (bab, bba), (cab, cba), (bac, bca)\}$. But now the prefix graph restricted to $\langle abc \rangle_{C'}$, depicted in figure 71, is not isomorphic to the prefix graph depicted in figure 70.

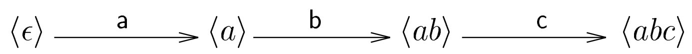


Figure 71: $G(A, C')(\langle abc \rangle)$

Thus there exist congruences generated by a ≤k-context commutation relation, who can not be generated by a k-context commutation relation. The next example illustrates this.

Example

Let $A = \{a, b, c\}$ and $C = \{(ab, ba), (abc, acb)\}$. C is a ≤1-context commutation relation. Let $p = \langle ab \rangle_C$. Then $p = \{ab, ba\}$. If we would like to describe the congruence generated by C, $\equiv_C$, by a k-context commutation relation C', then (ab, ba) is an element of C'. Thus C' has to be a 0-context commutation relation. But then the restriction b and c are only concurrent after c can not be described in the 0-context commutation relation C'.

8.4.5 Difference between lcc and ≤k-context traces

It is easy to see that the ≤k-context commutation relation is a left-context commutation relation. Thus ≤k-context traces are lcc traces. The difference between lcc traces and ≤k-context traces is the restriction on the relation by which the congruence is induced. Of course a finite left-context commutation relation can always be defined as a $\leq$k-context commutation relation, where k is the length of the largest context in the relation left-context commutation relation. In general the class of congruences induced by $\leq$k-context commutation relations is not equal to the class of congruences induced by left-context commutation relations.

8.4.6 The link with Mtraces

It is easy to see that the Mtraces, which can be generated by a commutation relation, are a special kind of k-context traces and $\leq$k-context traces and also of rcc traces. The Mtraces are 0-context traces and $\leq$0-context traces.

8.4.7 Conclusions

The k-context trace theory is a restriction of the $\leq$k-context trace theory. Thus this implies that there exist local traces which are not $\leq$k-context traces. The Mtrace trace theory is a restriction of the k-context trace theory, and thus of the $\leq$k-context trace theory. The k-context traces and thus $\leq$k-context traces have no properties this in contrast with Mtraces.

9 Conclusion

We have described several trace equivalences and can conclude that there is a certain ordering between the different trace equivalences. This ordering is visualized in figure 72. In fact the diagram is complete: all inclusions are strict and if two nodes are not connected, they are incomparable. This follows from examples in the cited subsubsections and with two more examples. The first example shows that a k-context trace is not a rcc trace and the second example shows that a local trace is not a cop trace.

I suspected that some of the generalizations would have some properties like the backward diamond property, but this is not true. All the generalizations of the Mazurkiewicz trace theory have examples in which the (quasi-)prefix graph does not have the (compatible) forward diamond property, does not have the backward diamond property, and is not co-deterministic. For all the generalizations we have as result that the (quasi-)prefix graph restricted to a trace is isomorphic to the configuration graph of the partial order of the trace if and only if the (quasi-)prefix graph restricted to a trace has the forward and backward diamond property. This result was described by Biermann and Rozoy in [BR95].

When we investigate the theory described by Bauget and Gastin we can conclude that all the congruences which can be represented by partial orders can be generated by context commutation relations. However there exist cc trace equivalences which can not be represented by partial orders. Thus there exists a strict subset of cc trace equivalences which can be represented by partial orders. In subsection 8.3 theorem 96 states that a congruence which can be modularly represented by partial orders is a Mazurkiewicz trace equivalence. The congruences which can be represented by partial orders and are left-cancellative (or right-cancellative) are a strict subset of the rcc trace equivalences (or lcc trace equivalences). Congruences which are cancellative and can be represented by partial orders are Mazurkiewicz trace equivalences and only these congruences can be modularly represented by partial orders.

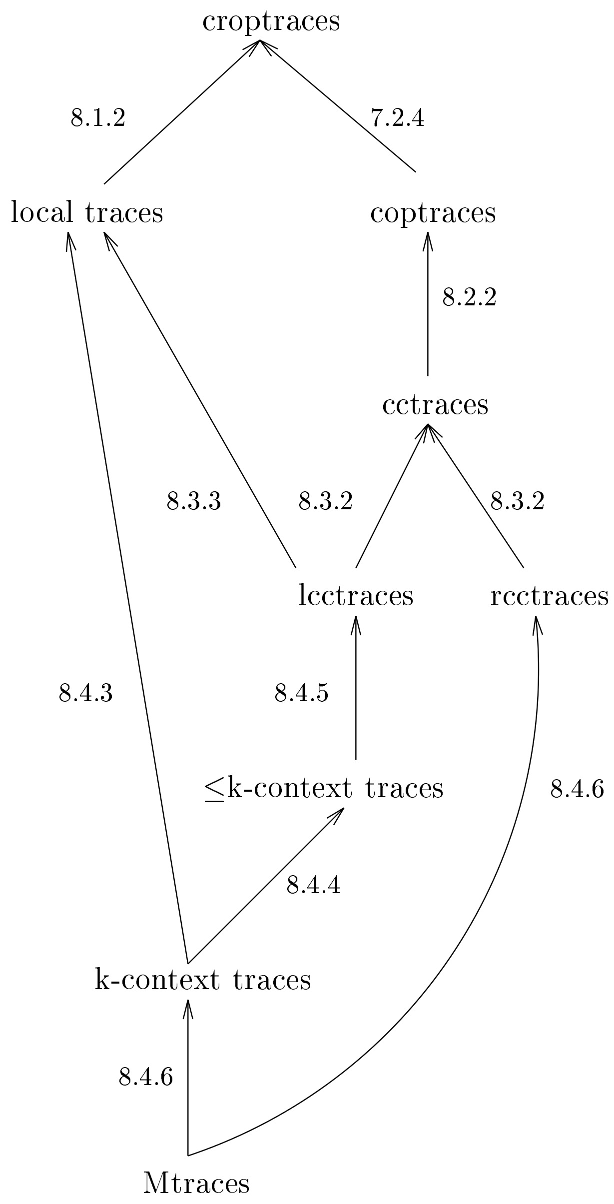


Figure 72: The ordering between the trace equivalences

References

[BR95] I. Biermann and B. Rozoy. Graphs for generalized traces. Proceedings STRICT 1995, p101-115, Workshops in Computing, Springer.

[K95] H.C.M. Kleijn. Local traces and local event structures. Proceedings WASAC 1995, p153-161.

[KR95] H.C.M. Kleijn and B. Rozoy. Local traces and event structures again. Proceedings WASAC 1995, p163-171.

[BG95] S. Bauget and P. Gastin. On congruences and partial orders. Proceedings WASAC 1995, p45-54 .

[DK95] M. Droste and D. Kuske. Automata with concurrency relations -a survey- , Proceedings WASAC 1995, p119-139.

[H94] P.W. Hoogers. Behavioural aspects of Petri nets, Ph.D Thesis, Leiden University 1994.

[DP90] B.A. Davey and H.A. Priestley. Introduction to Lattices and Order. Cambridge Mathematical textbooks 1990.

[M87] A. Mazurkiewicz. Trace theory. Lecture Notes in Computer Science 255 1987, p279-296.