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Graded Semantics in a Fibration

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Abstract

Graded semantics provides a framework for modelling behavioural equivalences at varying levels of granularity for general types of systems by combining coalgebra with graded monads. On the other hand, combining coalgebras with fibrations provides a powerful approach to representing various notions of branching-time behavioural conformance, which includes behavioural preorders and distances. However, there is currently no systematic account of the interaction between these two frameworks. This thesis takes a first step towards bridging this gap by providing a fibrational viewpoint on graded behavioural equivalences.

To achieve this, we introduce a notion of a graded fibration, which consists of a family of fibrations indexed by the natural numbers. Intuitively, the fibration at level n models a finite-depth approximation of behavioural equivalence at precisely depth n. We develop the basic theory of graded fibrations and associate a graded fibration with a given graded semantics on coalgebras for a functor on the category of sets and functions. In particular, we exhibit a graded fibration such that depth-n and finite-depth graded behavioural equivalences can be recovered as graded relations and their fibred limit, respectively. We also provide a fibrational account of infinite-depth graded behavioural equivalence [15]. We illustrate these constructions on bisimilarity, trace equivalence, and simulation equivalence for labelled transition systems.

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1 Introduction

Behavioural equivalence is a concept of central importance in the semantics of sequential and concurrent programming languages. It identifies systems based on their observable behaviour rather than their internal structure. For example, behavioural equivalence provides a basis for formal methods such as model checking and software verification, which support the construction and analysis of correct, reliable, and trustworthy computational systems. Prominent examples include language equivalence of finite-state automata and Park–Milner bisimulation equivalence between states in labelled transition systems.

The variation in system types and in the granularity of behavioural equivalences motivates the development of general approaches to behavioural semantics of state-based systems. Regarding system types, *universal coalgebra*, initiated by Aczel and Mendler [1] and systematised by Rutten [45], has emerged as a unifying framework for modelling state-based systems and their behaviours. In detail, system types are represented as endofunctors on a suitable category, often the category of sets and functions, while behavioural equivalence is typically characterized by the existence of coalgebra homomorphisms mapping the two states to the same state. If a final coalgebra exists, it provides a canonical domain of behaviours where behaviourally equivalent states are identified. This perspective has since been extended to probabilistic and quantitative systems [49], including notions of behavioural equivalence [13] and (pseudo)metrics [4, 52]. The study of behavioural equivalences within this setting has led to a rich body of theory, which includes coinductive proof principles via final semantics [45] and logical characterizations via Hennessy–Milner style theorems based on suitable modal logics [36, 37], which have led to model checking and minimization tools over general system types [9, 54].

One of the key challenges for coalgebraists lies in the modelling and analysis of coalgebras under more coarse-grained notions of behavioural equivalence such as trace semantics of LTS. These notions of behavioural equivalence, which we generally refer to as the *semantics* of systems, form spectra with different levels of observational granularity, most prominently the linear-time branching-time spectrum introduced by van Glabbeek [53]. Similar distinctions arise for probabilistic and quantitative systems [10, 13, 39, 46]. The variation in behavioural semantics of the systems has been addressed in coalgebraic generality through a variety of approaches [7, 21, 28, 29, 33]. Notably, *graded semantics* [12, 14, 43] uses *graded monads* to model behavioural semantics in which grades represent observation depth, providing a uniform treatment of a large part of the linear-time branching-time spectrum, including intermediate notions such as ready and failure semantics. Subsequent developments have extended this framework to behavioural preorders [14], graded semantics for Eilenberg-Moore coalgebras [18], quantitative graded semantics and spectra of

behavioural metrics [17], as well as game-theoretic characterizations via graded behavioural equivalence games [15]. More recently, this approach has been extended further to topological base categories, providing a unified treatment of behavioural preorders, metrics, and topologies as notions of behavioural conformance [16].

At the same time, fibrational methods have long provided a natural setting for coalgebraic notions of behavioural relations, particularly infinitary ones such as infinite trace equivalence. Following the seminal work of Hermida and Jacobs [23], notions such as (bi)simulation can be uniformly formulated by lifting system type functors from a base category to suitable fibrations of predicates, relations, and related structures. This approach has enabled a systematic treatment of behavioural equivalences and other quantitative behavioural notions, yielding abstract accounts of (bi)simulations, behavioural metrics [8], and logical predicates [22]. Subsequent developments have extended this perspective to coalgebraic modal logic [5, 35, 37], codensity lifting and games [32, 34], and quantitative reasoning [50], demonstrating the flexibility of fibrations as a general tool for transporting coalgebraic constructions to richer semantic settings.

These two lines of work provide complementary perspectives on behavioural semantics: graded semantics captures behavioural equivalences at varying granularities, while fibrational methods provide a general framework for behavioural conformance. This motivates the central research question of this thesis.

Research Question. Can we investigate the interaction between these two perspectives by providing a fibrational account of the graded behavioural equivalence induced by a graded semantics?

This thesis addresses this question by investigating graded semantics from the perspective of fibrations and studying how graded behavioural relations can be organized in a fibred setting. Whereas graded semantics is traditionally formulated in the base category, fibrations provide a natural language for describing different behavioural structures over underlying systems. This suggests that graded semantics and fibrational techniques should complement one another, combining the inductive flavour of graded semantics with the coinductive perspective of fibrational methods to treat observational depth and behavioural comparison uniformly. Motivated by this perspective, we introduce a notion of *graded fibration*, consisting of a family of fibrations indexed by grades with suitable coherence structure connecting different grades. This structure allows graded constructions to be lifted to the fibres and graded behavioural relations to be formulated across different observational depths. We establish the basic theory of graded fibrations, investigate their interaction with graded monads, and study the resulting finite-depth and infinite-depth behavioural relations. The resulting framework unifies three dimensions of behavioural semantics: system types,

captured coalgebraically; observational granularity, captured by graded semantics; and behavioural comparison, captured fibrationally, laying the foundation for a fibrational treatment of coinductive predicates for coalgebras modulo graded semantics.

1.1 Organization

The remainder of this thesis is organized as follows. Section 2 recalls the required background on categories, coalgebras, fibrations, graded monads and semantics, and fixes notations used throughout the thesis. Section 3 presents graded semantics and predeterminization, and defines graded behavioural equivalence. Section 4 introduces graded fibrations and develops their basic categorical properties. Section 5 studies the graded behavioural equivalence formulated within graded fibrations and establishes the main theoretical results. Section 6 discusses applications of the framework and instantiate it on several running examples.

2 Preliminaries

We assume familiarity with the basic concepts of category theory and refer the reader to Mac Lane and Riehl [40, 44] for a comprehensive exposition. In this section, we review the necessary background on coalgebra [27, 45], graded monads [19, 48], and fibrations [23, 32].

2.1 Relations in a Category

We begin with a categorical viewpoint of relations [38, 51]; for a more comprehensive treatment, we refer the reader to Milius [42]. Typically, a relation in **Set** between sets X and Y is understood as a subset $R \subseteq X \times Y$. More abstractly, the relation R is given by a span

$$X \xleftarrow{p_1} R \xrightarrow{p_2} Y,$$

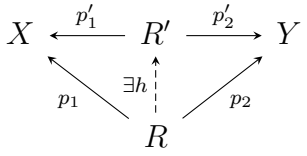
where p_1, p_2 are the evident projections. Such a diagram is called a *span from X to Y*. However, not every span behaves like a genuine relation. To capture the idea that R embeds into the product $X \times Y$ in **Set**, one imposes a joint injectivity condition on the projections.

Definition 2.1 (Jointly Monic Spans)**.** A span $X \xleftarrow{p_1} R \xrightarrow{p_2} Y$ is *jointly monic* if, for any pair of morphisms $f, g : S \to R$, the equalities $p_1 \circ f = p_1 \circ g$ and $p_2 \circ f = p_2 \circ g$ together imply that $f = g$.

Jointly monic spans generalize the notion of subobjects of a product and therefore serve as the appropriate notion of a *relation* in a category $\mathbf{C}$, which we fix for the remainder of this subsection. For objects $X, Y \in \mathbf{C}$, we denote by $\mathcal{R}_{\mathbf{C}}(X, Y)$ the set of jointly monic spans from X to Y. This set carries a preorder defined on a pair of jointly monic spans

$$X \xleftarrow{p_1} R \xrightarrow{p_2} Y \quad \text{and} \quad X \xleftarrow{p_1'} R' \xrightarrow{p_2'} Y,$$

by $R \leq R'$ if R factors through R', that is, if there exists a morphism $h : R \to R'$ making the following diagram commute:



Lemma 2.2. *The relation $\leq$ defines a preorder on $\mathcal{R}_{\mathbf{C}}(X, Y)$.*

Proof. Reflexivity is immediate by taking $R' = R$ and $h = \mathrm{id}_R$. Transitivity follows from composition of the mediating morphisms. □

Jointly monic spans are considered equivalent whenever they represent the same relation. This equivalence can be characterized by mutual factorization of the corresponding spans.

Definition 2.3 (Span isomorphism)**.** Given two jointly monic spans

$$X \xleftarrow{p_1} R \xrightarrow{p_2} Y \quad \text{and} \quad X \xleftarrow{p_1'} R' \xrightarrow{p_2'} Y,$$

the spans are said to be *isomorphic*, denoted as $R \cong R'$, if $R \leq R'$ and $R' \leq R$.

The following proposition shows that this equivalence coincides with isomorphism in the category $\mathbf{Span}(\mathbf{C})$ of spans in $\mathbf{C}$.

Proposition 2.4. *Given two jointly monic spans*

$$X \xleftarrow{p_1} R \xrightarrow{p_2} Y \quad \textit{and} \quad X \xleftarrow{p_1'} R' \xrightarrow{p_2'} Y,$$

we have $R \cong R'$ if and only if there exists an isomorphism $i \colon R \to R'$ in $\mathbf{C}$ such that $p_1 = p_1' \circ i$ and $p_1 = p_1' \circ i$.

Proof. Suppose $i\colon R \to R'$ is an isomorphism so that

$$p_1 = p'_1 \circ i \quad \text{and} \quad p_2 = p'_2 \circ i.$$

Composing with i^{-1} yields

$$p'_1 = p_1 \circ i^{-1} \quad \text{and} \quad p'_2 = p_2 \circ i^{-1}.$$

Hence i witnesses $R \leq R'$, while i^{-1} witnesses $R' \leq R$.

Conversely, suppose $R \leq R'$ and $R' \leq R$. Then there exists morphisms $f\colon R \to R'$ and $g\colon R' \to R$ such that

$$p_i = p'_i \circ f \quad \text{and} \quad p'_i = p_i \circ g, \quad i = 1, 2.$$

Therefore $p'_i = p'_i \circ (f \circ g)$ for $i = 1, 2$. Since the span R' is jointly monic, it follows that $f \circ g = \mathrm{id}_{R'}$. Similarly, $p_i = p_i \circ (g \circ f)$ for $i = 1, 2$, and the joint monicity of R implies $g \circ f = \mathrm{id}_R$. Thus f and g are inverse isomorphisms, so $R \cong R'$ as spans. □

Next, we show the quotient $\mathcal{R}_{\mathbf{C}}(X, Y)/{\cong}$ forms a complete lattice under additional assumptions that $\mathbf{C}$ is complete and well-powered. Given $X, Y \in \mathbf{C}$, let

$$q\colon \mathcal{R}_{\mathbf{C}}(X, Y) \to \mathcal{R}_{\mathbf{C}}(X, Y)/{\cong}$$

denote the canonical quotient map. Fix, once and for all, a section

$$s\colon \mathcal{R}_{\mathbf{C}}(X, Y)/{\cong} \to \mathcal{R}_{\mathbf{C}}(X, Y)$$

in the sense that $q \circ s = \mathrm{id}$. Then the relation defined by $[R] \leq [S] :\Leftrightarrow s([R]) \leq s([S])$ endows $\mathcal{R}_{\mathbf{C}}(X, Y)/{\cong}$ with the structure of a complete lattice. Indeed, the meet of a family of relations is represented, up to isomorphism, by the wide pullback of their underlying morphisms over the common codomain Y. Completeness and well-poweredness then ensure that these meets, together with the corresponding joins, exist in the quotient.

In what follows, we refer to an equivalence class of jointly monic spans from X to Y simply as a *relation from X to Y*.

2.2 Universal Coalgebra

Universal coalgebra [27, 45] provides a categorical framework for modelling state-based systems by abstracting the branching type of systems as an endofunctor G on a concrete category $\mathbf{C}$, which we fix for the remainder of the section.

Definition 2.5. A *G-coalgebra* is a pair (X, γ) consisting of an object $X \in \mathbf{C}$, called the *carrier*, together with a morphism $\gamma : X \to GX$ called the *structure map*. A *homomorphism* from (X, γ) to another G-coalgebra (Y, δ) is a morphism $h\colon X \to Y$ such that $\delta \circ h = Gh \circ \gamma$. We write $\mathbf{Coalg}(G)$ for the category of G-coalgebras and their homomorphisms. A terminal object in $\mathbf{Coalg}(G)$ is called final.

In a G-coalgebra (X, γ), the underlying set $|X|$ of the carrier X is interpreted as a set of *states* and the structure map γ assigns to each state $x \in |X|$ a structured collection of *successors* $\gamma(x)$. The functor G therefore determines the type of the transition system under consideration.

Example 2.6. Let A be a finite set of actions.

1. **Labelled transition systems.** Let $\mathcal{P}$ denote the covariate powerset functor, and $\mathcal{P}_\omega, \mathcal{P}^+$ denote its finitary and non-empty variant respectively. An *A-labelled transition system* (LTS) can be identified with a coalgebra

   $$\gamma : X \to \mathcal{P}(A \times X).$$

   For a state $x \in X$, the successor set $\gamma(x)$ is defined as

   $$\gamma(x) := \{(a, y) \mid x \xrightarrow{a} y\}.$$

   We call (X, γ) finitely branching (resp. serial) if $\gamma(x)$ is finite (resp. non-empty) for all $x \in X$. Finitely branching (resp. serial) LTS are coalgebras for the functor $\mathcal{P}_\omega(A \times -)$ (resp. $\mathcal{P}^+(A \times -)$).

2. **Generative probabilistic systems.** Let $\mathcal{D}$ denote the distribution functor, which maps a set X to the set $\mathcal{D}X$ of probability distributions over X, i.e., functions $u\colon X \to [0, 1]$ such that $\sum_{x \in X} u(x) = 1$. A *(generative) probabilistic transition system* (PTS) is a coalgebra

   $$\gamma\colon X \to \mathcal{D}(A \times X).$$

   For a state $x \in X$, the value $\gamma(x)(a, y)$ is interpreted as the probability of performing action a and moving to state y. A PTS (X, γ) is finitely branching if $\gamma(x)$ is finitely supported for all $x \in X$; then, finitely branching PTS are coalgebras for $\mathcal{D}_\omega(A \times -)$.

Coalgebras carry a canonical notion of behavioural equivalence. Given two G-coalgebras (X, γ) and (Y, δ), states $x \in X$ and $y \in Y$ are said to be *behaviourally equivalent* if there exist coalgebra morphisms $f\colon (X, \gamma) \to (Z, \zeta)$ and $g\colon (Y, \delta) \to (Z, \zeta)$ such that $f(x) = g(y)$. Behavioural equivalence can be approximated along the *final sequence* of

G, of which we consider only the initial ω-segment. Let 1 denote a terminal object of $\mathbf{C}$. Starting from $G^0 1 = 1$, one constructs the sequence

$$1 \leftarrow G1 \leftarrow G^2 1 \leftarrow \cdots .$$

Associated with every coalgebra (X, γ) is a canonical cone $\gamma_n \colon X \to G^n 1$, defined inductively by

$$\gamma_0 := !_X, \text{ and } \gamma_{n+1} := G\gamma_n \circ \gamma,$$

where $!_X \colon X \to 1$ is the unique map. Given G-coalgebras (X, γ) and (Y, δ), states $x \in X$ and $y \in Y$ are *finite-depth behaviourally equivalent* if $\gamma_n(x) = \delta_n(y)$ for all $n \in \mathbb{N}$.

Intuitively, the map γ_n records the behaviour observable up to depth n. For finitary functors on **Set**, a theorem of Worrell [55] states that finite-depth behavioural equivalence coincides with behavioural equivalence. Consequently, this applies in particular to the finitely supported distribution and finite powerset functors underlying the monads considered in Example 2.6. Thus, for these classes of transition systems, behavioural equivalence can be characterized entirely through finite approximations, a fact that underlies the graded approach to behavioural semantics developed in the following sections.

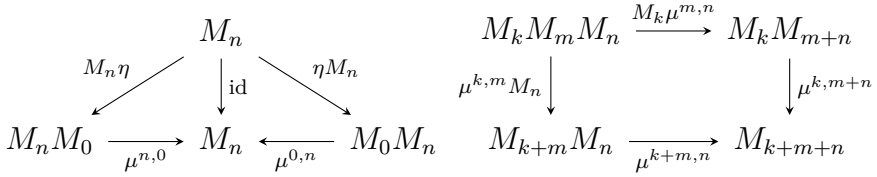
2.3 Graded Monads and Graded Algebras

Graded monads provide a refinement of ordinary monads in which computations are indexed by a notion of *grade*, typically representing resources such as depth, length, or stages of observation. In the setting relevant to behavioural equivalences, the grades are taken from the additive monoid $(\mathbb{N}, +, 0)$, which we will later view as the transition depth in a coalgebra.

Definition 2.7. An $\mathbb{N}$-*graded monad* $\mathbb{M}$ on a category $\mathbf{C}$ is a family of endofunctors $\{M_n \colon \mathbf{C} \to \mathbf{C}\}_{n \in \mathbb{N}}$ equipped with natural transformations $\eta \colon \mathrm{id}_{\mathbf{C}} \to M_0$ and

$$\{\mu^{m,n} \colon M_m M_n \to M_{m+n}\}_{m,n \in \mathbb{N}}$$

(the *unit* and the *multiplication*, respectively), satisfying the unit and associativity laws



for all $k, m, n \in \mathbb{N}$.

The notion above was introduced by Smirnov [48] and later generalized to accomodate grade domains given by monoidal categories in the work of Katsumata [31], Fujii et al. [19], and Melliès [41]. In particular, taking the grading category to be the discrete monoidal category induced by $(\mathbb{N}, +, 0)$ recovers the present definition. In what follows, we will refer to $\mathbb{N}$-graded monads simply as graded monads, unless otherwise specified.

Example 2.8. We describe several examples of subsequent interest.

1. Every endofunctor $G\colon \mathbf{Set} \to \mathbf{Set}$ induces a graded monad $M_n = G^n$, where G^n denotes the n-fold composite of G. The unit and multiplication are identities.

2. Let (T, η, μ) be a monad and let F be an endofunctor on **Set**, equipped with a distributive law
$$\lambda\colon FT \to TF.$$
Writing $\lambda_n\colon F^nT \to TF^n$ for the iterated distributive law such that $\lambda_0 := \mathrm{id}_T$, and $\lambda_{n+1} := \lambda^n F \circ F^n\lambda$. One obtains a graded monad with $M_n = TF^n$, whose unit is η, and multiplication is given by
$$\mu^{m,n} := (TF^mTF^n \xrightarrow{T\lambda_m F^n} TTF^{m+n} \xrightarrow{\mu F^{m+n}} TF^{m+n}).$$
These graded monads underlie Kleisli-style coalgebraic trace semantics [21].

3. Given a monad (T, η, μ) and a set of actions A, the previous construction yields the graded monad $M_nX = TF^n = T(A^n \times X)$, which models traces together with effects specified by T. For instance, taking T to be the distribution monad gives the graded monad corresponding to probabilistic trace semantics.

Graded monads on **Set** can be presented in terms of graded equational theories [43], where operations and equations are typically restricted to depth at most one, giving rise to the following notion of depth-1 graded monads:

Definition 2.9 (Depth-1 graded monads)**.** A graded monad is *depth-1* if for all $n \in \mathbb{N}$, the morphism $\mu^{1,n}$ is a coequalizer of the following diagram:
$$M_1M_0M_nX \underset{\mu^{1,0}M_n}{\overset{M_1\mu^{0,n}}{\rightrightarrows}} M_1M_nX \xrightarrow{\mu^{1,n}} M_{1+n}X. \tag{2.1}$$

As for ordinary monads, graded monads are equivalently presented via a suitable graded adaptation of Kleisli triples [31]. We shall make notational use of the graded Kleisli extension operator, which we review next. Given $k \in \mathbb{N}$ and a morphism $f\colon X \to M_mY$, its *graded Kleisli extension* is the morphism $f^*_k := \mu^{k,m} \circ M_kf\colon M_kX \to M_{k+m}Y$.

The graded Kleisli operation satisfies analogues of the usual monad laws. For any $k \in \mathbb{N}$ and morphisms $f\colon X \to M_m Y$ and $g\colon Y \to M_n Z$, one has

$$f_0^* \circ \eta_X = f, \tag{2.2}$$
$$(\eta_X)_m^* = \mathrm{id}_{M_m X}, \tag{2.3}$$
$$(g_m^* \circ f)_k^* = g_{m+k}^* \circ f_k^*. \tag{2.4}$$

These identities generalise the familiar laws for ordinary monads and provide a convenient algebraic calculus for graded computations.

Finally, the Eilenberg–Moore construction for ordinary monads extends to the graded setting. Its restricted form considered here accounts only for operations up to a specified finite depth, leading to the following notion of *graded algebra* [19, 43]:

Definition 2.10 (Graded algebra)**.** Let $\mathbb{M}$ be a graded monad on $\mathbf{C}$. A M_n*-algebra*

$$A := ((A_i)_{i \le n}, (a^{i,j})_{i+j \le n})$$

consists of a family of $\mathbf{C}$-objects $(A_i)_{i \le n}$ and morphisms $a^{i,j}\colon M_i A_j \to A_{i+j}$ for $i + j \le n$ satisfying

$$\forall j \le n.\ a^{0,j} \circ \eta_{A_j} = \mathrm{id}_{A_j},$$

and

$$\begin{array}{ccc} M_i M_j A_k, & \xrightarrow{M_n a^{m,r}} & M_i A_{j+k} \\ {\scriptstyle \mu^{i,j}}\downarrow & & \downarrow{\scriptstyle a^{i,j+k}} \\ M_{i+j} A_k & \xrightarrow{a^{i+j,k}} & A_{i+j+k} \end{array} \tag{2.5}$$

for all $i, j, k \in \mathbb{N}$ such that $i + j + k \le n$. We refer to the M_0-algebra $(A_i, a^{0,i})$ as the i*-th part* of an M_n-algebra.

A *homomorphism* from A to another M_n-algebra B is a family of $\mathbf{C}$-morphisms $\{f_i\colon A_i \to B_i\}_{i \le n}$ such that whenever $i + j \le n$, then

$$f_{i+j} \circ a^{i,j} = b^{i,j} \circ M_i f_j.$$

We write $\mathbf{Alg}_n(\mathbb{M})$ for the category of M_n-algebras and their homomorphisms.

2.4 CLat$_\sqcap$-Fibrations

Let $p\colon \mathbf{E} \to \mathbf{C}$ be a functor. If $p(R) = X$, then we say R is *above* X, and similarly for morphisms. We write $\mathbf{E}_X$ for the category of all objects R above a given object X and arrows above the identity id_X, and refer to it as the *fibre* above X.

Definition 2.11. A functor $p\colon \mathbf{E} \to \mathbf{C}$ is a *poset fibration* if

- each fibre $\mathbf{E}_X$ is a poset category, that is, at most one arrow between every two objects, the corresponding order on objects is denoted by $\sqsubseteq$;
- for every $f\colon X \to Y$ in $\mathbf{C}$ and object S above Y there is a Cartesian morphism $\tilde{f}_S\colon f^\bullet(S) \to S$ above f, with the property that for every arrow $g\colon Z \to X$, every object R above Z and arrow $h\colon R \to S$ above $f \circ g$, there is a unique arrow $k\colon R \to f^\bullet(S)$ above g such that $\tilde{f}_S \circ k = h$.

For a morphism $f\colon X \to Y$ in $\mathbf{C}$, the assignment $R \mapsto f^\bullet(R)$ gives rise to a pullback functor $f^\bullet\colon \mathbf{E}_Y \to \mathbf{E}_X$, called *reindexing along* f. Following [34, 50], we use a strengthening of poset fibration.

Definition 2.12 ($\mathbf{CLat}_\sqcap$-Fibration, [34, Definition II.6])**.** A poset fibration $p\colon \mathbf{E} \to \mathbf{C}$ is called a $\mathbf{CLat}_\sqcap$-fibration if each fiber is a complete lattice, and reindexing preserves all meets.

Proposition 2.13 ([34, Proposition II.9])**.** *Let $p\colon \mathbf{E} \to \mathbf{C}$ be a $\mathbf{CLat}_\sqcap$-fibration.*

1. *p is split, and faithful as a functor.*
2. *Each arrow $f\colon X \to Y$ has its pushforward $f_\bullet\colon \mathbf{E}_X \to \mathbf{E}_Y$, so that an adjunction $f_\bullet \dashv f^\bullet$ is formed. This is a consequence of Freyd's adjoint functor theorem; it makes p a bifibration [26].*
3. *$p^{\mathrm{op}}\colon \mathbf{E}^{\mathrm{op}} \to \mathbf{C}^{\mathrm{op}}$ is also a $\mathbf{CLat}_\sqcap$-fibration.*
4. *The change-of-base [26, Lemma 1.5.1] of p along any functor $H\colon \mathbf{D} \to \mathbf{C}$ is also a $\mathbf{CLat}_\sqcap$-fibration.*
5. *If $\mathbf{C}$ is (co)complete, then the total category $\mathbf{E}$ is also (co)complete. This follows from [26, Proposition 9.2.1].*

Example 2.14 (See [34, Table V])**.** Various well-known categories can be seen as categories that equip sets with certain indistinguishability structures. The evident forgetful functors from the total categories to **Set** are all $\mathbf{CLat}_\sqcap$-fibrations.

1. **PMet** $\to$ **Set**, where **PMet** is the category of 1-bounded pseudometric spaces and non-expansive maps.
2. **Pre** $\to$ **Set**, where **Pre** is the category of preordered sets and monotone maps.

3. **Rel** $\to$ **Set** $\times$ **Set**, where **Rel** is the category of binary relations whose objects are triples $(X, Y, R \subseteq X \times Y)$ of two sets and a relation between them, and whose morphisms $(X, Y, R) \to (Z, W, S)$ are pairs of relation-preserving functions $(f\colon X \to Z, g\colon Y \to W)$ such that $(x, y) \in R$ implies $(f(x), g(y)) \in S$.

2.5 Notations and Assumptions

We make the following assumption throughout the remainder of this thesis.

Assumption 2.15. We fix a locally small category **C** which is complete, well-powered, and comes equipped with a proper orthogonal factorisation system (E, M) (cf. [2, Chapter IV]) unless explicitly mentioned otherwise. We further assume that **C** is equipped with a monad T which preserves E, and a $\mathbf{CLat}_\sqcap$-fibration $p\colon \mathbf{E} \to \mathbf{C}$.

In particular, these assumptions imply that the category $\mathbf{Alg}(T)$ of Eilenberg-Moore algebras of T is complete, well-powered, and inherits a proper orthogonal factorization system (E, M) preserved by T.

3 Graded Semantics

Graded semantics provide a uniform treatment for a variety of behavioural equivalence granularity through graded monads. Intuitively, the n-th component of a graded monad describes the space of observations that can be made in the n-step unfolding of a given coalgebra, while a natural transformation specifies how one-step transitions are translated into these observations.

Definition 3.1 (Graded semantics)**.** Let $G\colon \mathbf{C} \to \mathbf{C}$ be an endofunctor on a category **C**. A *(depth-1) graded semantics* for G consists of a depth-1 graded monad $\mathbb{M}$ on **C** and a natural transformation $\alpha\colon G \to M_1$.

Given a G-coalgebra (X, γ), the graded semantics induces a sequence of *n-step α-pretrace* maps $\gamma^{(n)}\colon X \to M_n X$ by

$$\gamma^{(0)} := \eta_X, \text{ and } \gamma^{(n+1)} := \mu_X^{1,n} \circ M_1\gamma^{(n)} \circ (\alpha_X \circ \gamma).$$

Definition 3.2 (Graded behavioural equivalence)**.** States $x \in X, y \in Y$ in G-coalgebras (X, γ) and (Y, δ) are *depth-n $(\alpha, \mathbb{M})$-behaviourally equivalent*, denoted as $x \equiv_n^{(\alpha,\mathbb{M})} y$, if

$$M_n !_X \circ \gamma^{(n)}(x) = M_n !_Y \circ \gamma^{(n)}(y),$$

and *finite-depth* $(\alpha, \mathbb{M})$*-behaviourally equivalent*, denoted as $x \equiv_{\omega}^{(\alpha,\mathbb{M})} y$, if

$$\forall n \in \mathbb{N}.\ x \equiv_{n}^{(\alpha,\mathbb{M})} y.$$

Thus, the object $M_n 1$ serves as the domain of observations (the α-traces) available after n steps, and the map $\mathrm{tr}_n := M_n! \circ \gamma^{(n)}$ drops poststates from α-pretraces and assigns to each state its observable α-traces at depth n. Two states are considered equivalent at depth n (resp. every finite depth) when their α-traces agree at depth n (resp. every finite depth).

We provide several examples of subsequent interest. These examples follow the presentation in [12]; for a comprehensive collection of graded semantics, see [12, Section 4].

Example 3.3. Let A be a finite set of actions, and consider a finitely-branching LTS $\gamma\colon X \to \mathcal{P}_\omega(A \times X)$.

1. *Bisimilarity.* For each **Set**-endofunctor G, there is a graded monad $\mathbb{M}_G$ given by $M_n = G^n$, with the unit and multiplication be identities. Following the discussion of the final sequence of a functor G in Section 2.2, we take $\mathbb{M}_G$ and $\alpha = \mathrm{id}$. The resulting graded semantics records the n-step approximations of coalgebraic behaviour. It therefore yields finite-depth behavioural equivalences, whose limit coincides with behavioural equivalence when G is finitary. In particular, for finitely branching LTSs, this semantics yields bisimilarity.

2. *Trace equivalence.* For $x \in X$, let $\mathrm{Tr}_{<\omega}(x) = \{w \in A^* \mid \exists y \in X.\ x \xrightarrow{w} y\}$ be the set of finite traces of x. States are trace equivalent when they have the same trace set. This semantics is obtained from the graded monad $M_n X = \mathcal{P}_\omega(A^n \times X)$ as in Example 2.8.3, with $\alpha = \mathrm{id}$ again. Inductively, $\gamma^{(n)}(x)$ consists precisely of the pairs (w, y) where $w \in A^n$ labels a path from x to y. After projecting away the final state, the n-th component therefore records exactly the traces of length n, and the resulting α-trace equivalence coincides with ordinary trace equivalence.

3. *Simulation equivalence.* Two states $x, y \in X$ are considered equivalent, written $x \leq\geq y$, if x can be simulated by y and y can be simulated by x. Simulation equivalence can be modelled using a depth-1 graded equational theory with the same signature as for trace equivalence but replacing join preservation by monotonicity (see [12, Section 4] for details). The arising graded monad is defined inductively by

$$M_0 X = \mathcal{P}_\omega(X), \text{ and } M_{n+1} X = \mathcal{P}_\omega^{\downarrow}(A \times M_n X),$$

ordered by inclusion, where $\mathcal{P}_\omega^{\downarrow}$ denotes the set of finitely generated down-closed subsets. The semantic transformation is given by

$$\alpha_X\colon \mathcal{P}(A \times X) \to \mathcal{P}_\omega^{\downarrow}(A \times \mathcal{P}_\omega X);\ S \mapsto {\downarrow}\{(a, \{x\}) \mid (a, x) \in S\},$$

where $\downarrow$ denotes downset.

The graded semantics for coalgebras over **Set** can also be lifted to other ambient categories, where the additional structure of the base category allows one to capture more quantitative notions of behaviour, such as behavioural preorders and distances. We illustrate this extension with an additional example.

Example 3.4. Consider a finitely-branching LTS $\gamma\colon X \to GX$ where $G = \mathcal{P}_\omega(A \times -)$. For $x, y \in X$, the *trace inclusion* $\mathrm{Tr}_{<\omega}(x) \subseteq \mathrm{Tr}_{\leq\omega}(y)$ can be characterized by lifting the graded semantics for trace equivalence to the category **Pre** of preorders and monotone maps, where we equip the state space with the discrete preorder, so that the transition map is automatically monotone. More precisely, we define $\overline{\mathbb{M}}$ with components

$$\overline{M}_n = \overline{\mathcal{P}}_\omega(A^n \times -),$$

where A is considered a discrete preorder, and $\overline{\mathcal{P}}_\omega\colon$ **Pre** $\to$ **Pre** is the lifting of the finite powerset functor that equips a preorder $(X, \leq)$ with the one-sided Egli–Milner ordering

$$A \leq B \iff \forall x \in A.\ \exists y \in B.\ a \leq b.$$

This gives a graded semantics $(\mathrm{id}, \overline{\mathbb{M}})$ for the functor $\overline{\mathcal{P}}_\omega(A \times -)$. The resulting behavioural preorder satisfies

$$x \leq y \iff \forall n \in \mathbb{N}.\ \mathrm{Tr}_n(x) \subseteq \mathrm{Tr}_n(y),$$

where $\mathrm{Tr}_n(X)$ denotes the set of traces of length n from x. Thus, the induced behavioural preorder is precisely trace inclusion.

3.1 Pre-determinization

Graded semantics can be extended from finite-depth observations to capture infinite-depth behaviour via *pre-determinization* [15], generalizing the classical powerset construction for LTS, as well as the coalgebraic determinization construction of Silva et al. [47]. We briefly recall the construction and refer to Ford et al. [15] for the underlying technical details.

Let $G\colon$ **Set** $\to$ **Set** be an endofunctor and $(\alpha, \mathbb{M})$ a graded semantics for G. If $\mathbb{M}$ is depth-1 [15], the graded monad induces a functor

$$\overline{M}_1\colon\ \mathbf{Alg}_0(\mathbb{M}) \to \mathbf{Alg}_0(\mathbb{M}),$$

which can be viewed as the semantic counterpart of one additional observation step. Let

$$F \dashv U\colon\ \mathbf{Alg}_0(\mathbb{M}) \rightleftarrows \mathbf{Set}$$

be the free-forgetful adjunction. For a G-coalgebra (X, γ), the composite

$$X \xrightarrow{\gamma} GX \xrightarrow{\alpha_X} M_1 X$$

induces, by adjoint transposition, an $\overline{M}_1$-coalgebra

$$\gamma^\sharp \colon FX \to \overline{M}_1 FX,$$

called the *pre-determinization* of γ.

Example 3.5. We give a description of the functor $\overline{M}_1$ and the pre-determinization construction in some of the running examples.

1. For the graded monad with $M_n = G^n$, we have $M_0 = \mathrm{id}$ and multiplications are all identities, so M_0-algebras are just sets. It follows that $\overline{M}_1 = G$ and $\gamma^\sharp = \gamma$ for each G-coalgebra $\gamma \colon X \to GX$.

2. For finitely-branching LTS $G = \mathcal{P}_\omega(A \times -)$, the grade-0 monad is the finite powerset monad $M_0 = \mathcal{P}_\omega$, and one has

   $$\overline{M}_1(\mathcal{P}_\omega(X), \mu_{\mathcal{P}_\omega(X)}) = (\mathcal{P}_\omega(A \times X), \mu_{\mathcal{P}_\omega(A \times X)})$$

   This implies $\overline{M}_1 = (-)^A$ when A is finite, and the pre-determinization of a G-coalgebra $\gamma \colon X \to \mathcal{P}_\omega(X)$ is given by

   $$\gamma^\sharp \colon (\mathcal{P}_\omega(X), \mu_{\mathcal{P}_\omega(X)}) \to (\mathcal{P}_\omega(A \times X), \mu_{\mathcal{P}_\omega(A \times X)}) \cong (\mathcal{P}_\omega(X)^A, \mu_{\mathcal{P}_\omega(X)^A}).$$

   Consequently, the underlying map

   $$U\gamma^\sharp : \mathcal{P}_\omega(X) \to (\mathcal{P}_\omega(X))^A$$

   is exactly the classical powerset construction, which transforms a nondeterministic transition system into a deterministic one. Importantly, each state x is bisimilar to $\eta_X(x) = \{x\}$.

Intuitively, pre-determinization turns a branching system into a deterministic one with M_0-structured states. Under the additional assumption that M_0 is affine, i.e. $M_0 1 \cong 1$, behavioural equivalence of the resulting coalgebra coincides with graded behavioural equivalence of the original system, motivating the following definition.

Definition 3.6 (Infinite-depth graded behavioural equivalence)**.** Let (X, γ) and (Y, δ) be G-coalgebras, with pre-determinized coalgebras

$$\gamma^\sharp \colon FX \to \overline{M}_1 FX, \quad \delta^\sharp \colon FY \to \overline{M}_1 FY.$$

The *infinite-depth* $(\alpha, \mathbb{M})$*-behavioural equivalence* of states $x \in X$ and $y \in Y$ is defined by

$$x \equiv_{\infty}^{(\alpha,\mathbb{M})} y \iff \eta_X(x) \equiv \eta_Y(y),$$

where $\equiv$ denotes ordinary coalgebraic behavioural equivalence on the pre-determinized coalgebras.

Thus, infinite-depth graded behavioural equivalence is obtained by applying ordinary behavioural equivalence to the pre-determinized coalgebras.

4 Graded Fibrations

In this section, we introduce *graded fibrations* which are tailored towards the treatment of graded coalgebraic semantics in the sense of Section 3. Briefly, graded fibrations are a refinement of vanilla fibrations by an explicit grading structure. This enables logical predicates and behavioural observations at different observation depths to be organized within a uniform fibred framework, and is particularly suited to settings where observations or computational resources are bounded. In detail, we define graded fibrations, develop their basic categorical properties, and establish the constructions that will be used later to model graded behavioural equivalences.

To motivate our construction, comparing the examples in Example 3.3.2 and Example 3.4 suggests that the graded semantics is, to some extent, independent of how we intend to compare states. The distinction arises only when interpreting these observations. For example, one may define *depth-*n *trace equivalence* through equality:

$$x \equiv_n y :\Leftrightarrow \mathrm{tr}_n(x) = \mathrm{tr}_n(y),$$

or define *trace inclusion*, by working with a lifting of $\mathbb{M}$ to the category **Pre** of preorders and monotone maps, as the behavioural preorder on states in preordered LTS by

$$x \sqsubseteq_n y :\Leftrightarrow \mathrm{tr}_n(x) \leq \mathrm{tr}_n(y).$$

Thus, the distinction lies not in the construction of the graded observations themselves, but in the structure available for comparing them. More generally, graded semantics provides observations at each grade, while the ambient category determines the structure available for comparing those observations. Indeed, every depth induces its own behavioural relation R_n on X from a relation $R_{M_n 1}$ on $M_n 1$ (cf. [16, Definition II.11.]):

$$x \, R_n \, y :\Leftrightarrow \mathrm{tr}_n(x) \, R_{M_n 1} \, \mathrm{tr}_n(y).$$

The relation R_n can be characterized using the relation fibration **Rel** $\to$ **Set**, whose fibres are given by the pullback squares of the form

$$\begin{array}{ccc} (X, R_n) & \longrightarrow & (M_n 1, R_{M_n 1}) \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{\mathrm{tr}_n} & M_n 1, \end{array}$$

so that $R_n = (\mathrm{tr}_n)^\bullet R_{M_n 1}$, where $(\mathrm{tr}_n)^\bullet$ denotes reindexing along tr_n as defined in Section 2.4. These relations are usually studied independently with no intrinsic categorical connection, i.e., one needs a fibration for each individual n. We seek an integrated fibration that cohesively packages state behaviour at each finite depth together. Ideally, our definition should also capture full finite-depth equivalence in the sense that we obtain a relation R_ω computed by the meet $\bigcap_{n<\omega} R_n$.

4.1 Graded Structure

Let **I** be a small discrete category. An **I**-graded family of objects $\{A_i\}_{i \in \mathbf{I}}$ in **C**, is a functor $\mathbf{I} \to \mathbf{C}$, thus an object in the functor category $\mathbf{C}^\mathbf{I}$. Our graded fibration construction starts with a $\mathbf{CLat}_\sqcap$-fibration $p \colon \mathbf{E} \to \mathbf{C}$ and the induced fibration $p^\mathbf{I} \colon \mathbf{E}^\mathbf{I} \to \mathbf{C}^\mathbf{I}$ given as follows.

Definition 4.1 (**I**-graded fibration)**.** Let **I** be a small discrete category, and let **C** be a category with **I**-limits, and $p \colon \mathbf{E} \to \mathbf{C}$ be a $\mathbf{CLat}_\sqcap$-fibration. The **I**-*graded fibration* $\Delta_\mathbf{I}^* p^\mathbf{I} \colon \Delta_\mathbf{I}^* \mathbf{E}^\mathbf{I} \to \mathbf{C}$ induced by p is the pullback

$$\begin{array}{ccc} \Delta_\mathbf{I}^* \mathbf{E}^\mathbf{I} & \longrightarrow & \mathbf{E}^\mathbf{I} \\ {\scriptstyle \Delta_\mathbf{I}^* p^\mathbf{I}} \downarrow & \lrcorner & \downarrow {\scriptstyle p^\mathbf{I}} \\ \mathbf{C} & \xrightarrow{\Delta_\mathbf{I}} & \mathbf{C}^\mathbf{I} \end{array} \tag{4.1}$$

where $\Delta_\mathbf{I} \colon \mathbf{C} \to \mathbf{C}^\mathbf{I}$ is the diagonal functor, which sends each object X to the constant functor $k_X \colon \mathbf{I} \to \mathbf{C}$ and each morphism $f \colon X \to Y$ to the natural transformation $k_f \colon k_X \Rightarrow k_Y$ whose components are all given by f.

This definition is a limit-based variant of [3, Definition 6], where the corresponding notion is formulated in terms of **I**-colimits; it is likewise derived from Definition 4.3.

Since $p^\mathbf{I} = p \circ (-)$ is a $\mathbf{CLat}_\sqcap$-fibration, it follows from Proposition 2.13 that $\Delta_\mathbf{I}^* p^\mathbf{I}$ is also a $\mathbf{CLat}_\sqcap$-fibration. Here, the pullback $\Delta_\mathbf{I}^* \mathbf{E}^\mathbf{I}$ is a category whose objects consist of an object $X \in \mathbf{C}$ together with an **I**-diagram P in the fibre $\mathbf{E}_X$. We identify such a diagram with its

family of components $\{P_i\}_{i\in\mathbf{I}}$, where $p(P_i) = X$ for all $i \in \mathbf{I}$. We write $\hat{\Delta}_{\mathbf{I}} \colon \mathbf{E} \to \Delta_{\mathbf{I}}^{*}\mathbf{E}^{\mathbf{I}}$ for the mediating map induced by $p \colon \mathbf{E} \to \mathbf{C}$ and $\tilde{\Delta}_{\mathbf{I}} \colon \mathbf{E} \to \mathbf{E}^{\mathbf{I}}$, which maps each object $P \in \mathbf{E}$ to the constant $\mathbf{I}$-diagram at P, regarded as an object of the pullback category.

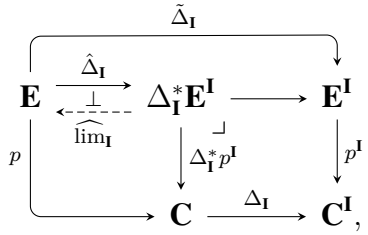
In general, the functor category $\mathbf{C}^{\mathbf{I}}$ consists of $\mathbf{I}$-diagrams in $\mathbf{C}$. For example, when $\mathbf{C} = \mathbf{Set}$, it can be viewed as the category of $\mathbf{I}$-graded sets, while $\mathbf{E}^{\mathbf{I}}$ is considered as the category of $\mathbf{I}$-graded predicates or relations fibred over it.

Example 4.2 (Graded relation fibration, cf. [3, Example 2])**.** Consider the classical *relation fibration* $p \colon \mathbf{Rel} \to \mathbf{Set}$. Let $\mathbb{N}$ be the discrete category of natural numbers. The category $\Delta_{\mathbb{N}}^{*}\mathbf{Rel}^{\mathbb{N}}$ has as objects $\mathbb{N}$-graded relations $(X, \{R_n\}_{n\in\mathbb{N}})$ and a morphism $f \colon (X, \{R_n\}_{n\in\mathbb{N}}) \to (Y, \{R'_n\}_{n\in\mathbb{N}})$ is a map $f \colon X \to Y$ that preserves relations for each grade. The fibration $\Delta_{\mathbb{N}}^{*}p^{\mathbb{N}} \colon \Delta_{\mathbb{N}}^{*}\mathbf{Rel}^{\mathbb{N}} \to \mathbf{Set}$ is the forgetful functor, which maps a set X equipped with an $\mathbb{N}$-graded family of relations $\{R_n\}_{n\in\mathbb{N}}$ to its underlying set X, and a relation-preserving map to its underlying function.

A desired feature of the graded fibration is the existence of quantifiers between fibrations $\Delta_{\mathbf{I}}^{*}p^{\mathbf{I}} \to p$ that enables reasoning across grades and provides a uniform modelling of both depth-n and depth-ω relations. This is defined via the notion of *fibred limits*, as described below.

Definition 4.3 (Fibred $\mathbf{I}$-limits, [24, Definition 3.3.4])**.** For any small category $\mathbf{I}$, a fibration $p \colon \mathbf{E} \to \mathbf{C}$ has *fibred* $\mathbf{I}$*-limits* (resp. colimits) if and only if the fibred functor $\hat{\Delta}_{\mathbf{I}} \colon p \to \Delta_{\mathbf{I}}^{*}p^{\mathbf{I}}$ has a fibred right (resp. left) adjoint $\hat{\Delta}_{\mathbf{I}} \dashv \widehat{\lim}_{\mathbf{I}}$.

This definition can be visualized by the following diagram, which builds upon Eq. (4.1):

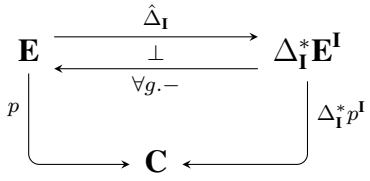


(4.2)

where $\Delta_{\mathbf{I}}$ and $\tilde{\Delta}_{\mathbf{I}}$ are both diagonal functors. The notion of fibred limits can also be characterized as follows.

Corollary 4.4 ([24, Corollary 3.3.6])**.** *Let* $\mathbf{I}$ *be a small category and* $p \colon \mathbf{E} \to \mathbf{C}$ *be a fibration such that* $\mathbf{C}$ *has* $\mathbf{I}$*-limits. Then* p *has fibred* $\mathbf{I}$*-limits if* $\mathbf{E}$ *has and* p *strictly preserves* $\mathbf{I}$*-limits.*

Since $\mathbf{C}$ has $\mathbf{I}$-limits, it follows that the $\mathbf{CLat}_{\sqcap}$-fibration p has fibred $\mathbf{I}$-limits [26, Exercise 9.2.4]. This implies that the fibred functor $\hat{\Delta}_{\mathbf{I}}$ has a fibred right adjoint, which we shall denote by $\forall g.-$, where g stands for individual grades.



For instance, for the fibration given in Example 4.2, The functor $\forall g.-$ maps a graded relation $\{R_n\}_{n\in\mathbb{N}}$ to its intersection $\bigcap_{n\in\mathbb{N}} R_n$. To see this characterize the fibred right adjoint, notice that for any relation $S \subseteq X \times X$, $S \subseteq \bigcap_{n\in\mathbb{N}} R_n$ if and only if $\forall n \in \mathbb{N}.\ S \subseteq R_n$, thus $S \leq \forall g.(\{R_n\}_{n\in\mathbb{N}}) \Leftrightarrow \hat{\Delta}_{\mathbb{N}}(S) \leq \{R_n\}_{n\in\mathbb{N}}$.

4.2 Semantic Reindexing

The general graded fibration organizes relations across different grades, but does not by itself specify how these relations are induced by the semantics of a particular coalgebra. In our setting, behavioural relations are obtained by pulling back graded relations along the sequence of pretrace maps induced by the coalgebra structure and the chosen graded semantics. We therefore refine the graded fibration construction by making this semantic dependence explicit through a coalgebra-dependent semantic map.

In the reminder of this section, we fix an endofunctor $G\colon \mathbf{C} \to \mathbf{C}$. Let (X, γ) be a G-coalgebra equipped with a graded semantics $(\alpha, \mathbb{M})$. Behaviour is observed through a family of depth-indexed unfoldings

$$\gamma^{(n)}\colon X \to M_n X, \qquad (n \in \mathbb{N})$$

where $\gamma^{(n)}$ denotes the depth-n pretrace map. Passing to a general grading category $\mathbf{I}$, these maps yield a morphism of $\mathbf{I}$-diagrams

$$\tau_\gamma\colon \Delta_{\mathbf{I}} X \to \mathbb{M}_{\mathbf{I}} X, \tag{4.3}$$

whose i-th component is $\tau_\gamma^{(i)} = \gamma^{(i)}$. Here, $\mathbb{M}_{\mathbf{I}} : \mathbf{C} \to \mathbf{C}^{\mathbf{I}}$ is the graded functor sending $X \mapsto \{M_i X\}_{i\in\mathbf{I}}$, and $\Delta_{\mathbf{I}}$ denotes the diagonal functor. Since the pretrace maps are determined by the coalgebra structure and the graded semantics $(\alpha, \mathbb{M})$, they assemble into a natural transformation

$$\tau\colon \Delta_{\mathbf{I}} U \to \mathbb{M}_{\mathbf{I}} U,$$

where $U\colon$ $\mathbf{Coalg}(G) \to \mathbf{C}$ is the forgetful functor. Naturality of τ follows from the fact that $\tau_\delta \circ \Delta_{\mathbf{I}} U(f) = \mathbb{M}_{\mathbf{I}} U(f) \circ \tau_\gamma$ for every coalgebra homomorphism $f\colon (X, \gamma) \to (Y, \delta)$.

Induced reindexing. Let $p\colon \mathbf{E} \to \mathbf{C}$ be a fibration, and let $p^{\mathbf{I}}\colon \mathbf{E}^{\mathbf{I}} \to \mathbf{C}^{\mathbf{I}}$ be its $\mathbf{I}$-graded extension. By functoriality of reindexing, the morphism τ_γ induces a reindexing functor

$$\tau_\gamma^\bullet\colon \mathbf{E}^{\mathbf{I}}_{\mathbb{M}_{\mathbf{I}}X} \to \mathbf{E}^{\mathbf{I}}_{\Delta_{\mathbf{I}}X},$$

which acts componentwise as

$$\{P'_i\}_{i\in\mathbf{I}} \mapsto \{(\tau_\gamma^{(i)})^\bullet P'_i\}_{i\in\mathbf{I}}.$$

Thus, any graded family of relations $\{P'_i\}_{i\in\mathbf{I}}$ on the semantic domains $\{M_iX\}_{i\in\mathbf{I}}$ is transported, via the coalgebra-dependent semantic map τ_γ, to the graded family of pullback relations $\{(\tau_\gamma^{(i)})^\bullet P'_i\}_{i\in\mathbf{I}}$ on the carrier X.

4.3 Graded Fibrations From Graded Monads

Now let $(\mathbf{I}, \otimes, e)$ be a monoid, viewed as a discrete strict monoidal category, and consider an $\mathbf{I}$-graded monad $\mathbb{M} = \{M_i\}_{i\in\mathbf{I}}$ on $\mathbf{C}$. We write

$$\mathbb{M}_{\mathbf{I}}\colon \mathbf{C} \to \mathbf{C}^{\mathbf{I}};\ X \mapsto \{M_iX\}_{i\in\mathbf{I}},$$

for the $\mathbf{I}$-indexed functor induced by $\mathbb{M}$. We define an $\mathbf{I}$-graded fibration by applying a change-of-base of $p^{\mathbf{I}}\colon \mathbf{E}^{\mathbf{I}} \to \mathbf{C}^{\mathbf{I}}$ in Definition 4.1 along $\mathbb{M}_{\mathbf{I}}U\colon$ $\mathbf{Coalg}(G) \to \mathbf{C}$. In detail, we have the following.

Definition 4.5. Let $\mathbf{I}$ be a small discrete category, $\mathbf{C}$ be a category with $\mathbf{I}$-limits, $\mathbb{M}$ be an $\mathbf{I}$-graded monad on $\mathbf{C}$, and $p\colon \mathbf{E} \to \mathbf{C}$ be a $\mathbf{CLat}_\sqcap$-fibration. The $\mathbf{I}$-*graded fibration induced by* $\mathbb{M}$, denoted as $(\mathbb{M}_{\mathbf{I}}U)^*p^{\mathbf{I}}\colon \mathbb{M}_{\mathbf{I}}^*\mathbf{E}^{\mathbf{I}} \to \mathbf{Coalg}(G)$, of p is defined by the following pullback.

$$\begin{array}{ccc} (\mathbb{M}_{\mathbf{I}}U)^*\mathbf{E}^{\mathbf{I}} & \longrightarrow & \mathbf{E}^{\mathbf{I}} \\ {\scriptstyle (\mathbb{M}_{\mathbf{I}}U)^*p^{\mathbf{I}}}\downarrow & \lrcorner & \downarrow{\scriptstyle p^{\mathbf{I}}} \\ \mathbf{Coalg}(G) & \xrightarrow{\mathbb{M}_{\mathbf{I}}U} & \mathbf{C}^{\mathbf{I}}, \end{array} \tag{4.4}$$

An object of $(\mathbb{M}_{\mathbf{I}}U)^*\mathbf{E}^{\mathbf{I}}$ is a pair $((X,\gamma), P)$ of a G-coalgebra (X,γ) and a family $P := \{P_i\}_{i\in\mathbf{I}}$ with $P_i \in \mathbf{E}_{M_iX}$. Thus, predicates in this graded fibration live over the semantic domains M_iX.

Analogous to Definition 4.3, the fibration $U^*p\colon U^*\mathbf{E} \to \mathbf{Coalg}(G)$ has fibred $\mathbf{I}$-limits if and only if the fibred functor $\widehat{\mathbb{M}_{\mathbf{I}}U}\colon U^*p \to (\mathbb{M}_{\mathbf{I}}U)^*p^{\mathbf{I}}$, uniquely determined by $U^*p\colon U^*\mathbf{E} \to$

Coalg(G) and an appropriately chosen $\widetilde{\mathbb{M}_{\mathbf{I}}U}\colon U^*\mathbf{E} \to \mathbf{E}^I$, has a fibred right adjoint $\widehat{\mathbb{M}_{\mathbf{I}}U} \dashv \widehat{\lim}_{\mathbf{I}}$.

Now, given a natural transformation

$$\tau\colon \Delta_{\mathbf{I}}U \to \mathbb{M}_{\mathbf{I}}U,$$

whose component τ_γ at each $(X,\gamma) \in \mathbf{Coalg}(G)$ is determined by both the coalgebra structure and a chosen graded semantics $(\alpha, \mathbb{M})$ for G. Such component induces a reindexing functor $\tau_\gamma^\bullet$ defined fibrewise by

$$\tau_\gamma^\bullet\colon (\mathbb{M}_{\mathbf{I}}U)^*\mathbf{E}^{\mathbf{I}} \to (\Delta_{\mathbf{I}}U)^*\mathbf{E}^{\mathbf{I}};\ ((X,\gamma),\{P'_i\}_{i\in\mathbf{I}}) \mapsto ((X,\gamma),\{(\tau_\gamma^{(i)})^\bullet P'_i\}_{i\in\mathbf{I}}).$$

We write $\tau^\bullet$ for the family of all these induced reindexing functors.

Definition 4.6. The *reindexed* **I**-*graded fibration along* τ is

$$(\mathbb{M}_{\mathbf{I}}U)^\tau p^{\mathbf{I}}\colon (\mathbb{M}_{\mathbf{I}}U)^\tau\mathbf{E}^{\mathbf{I}} \to \mathbf{Coalg}(G), \tag{4.5}$$

where the total category is the image $(\mathbb{M}_{\mathbf{I}}U)^\tau\mathbf{E}^{\mathbf{I}} := \mathrm{im}(\tau^\bullet) \hookrightarrow (\Delta_{\mathbf{I}}U)^*\mathbf{E}^{\mathbf{I}}$.

Thus, objects of $(\mathbb{M}_{\mathbf{I}}U)^\tau\mathbf{E}^{\mathbf{I}}$ are pairs consisting of a G-coalgebra (X,γ) together with a family of predicates $\{P_i\}_{i\in\mathbf{I}}$ on X, obtained via semantic reindexing along τ as $\{P_i = (\tau_\gamma^{(i)})^\bullet P'_i\}_{i\in\mathbf{I}}$ for some family of predicates $\{P'_i\}_{i\in\mathbf{I}}$ living over the semantic domains $\{M_iX\}_{i\in\mathbf{I}}$. Naturality of τ gives the compatibility $(\Delta_{\mathbf{I}}U(f))^\bullet \circ \tau_\delta^\bullet = \tau_\gamma^\bullet \circ (\mathbb{M}_{\mathbf{I}}U(f))^\bullet$, for any G-coalgebra homomorphism $f\colon (X,\gamma) \to (Y,\gamma)$. Hence, the image is stable under reindexing, and equivalently, one may view $(\mathbb{M}_{\mathbf{I}}U)^\tau\mathbf{E}^{\mathbf{I}} \to \mathbf{Coalg}(G)$ as a subfibration of $(\Delta_{\mathbf{I}}U)^*p^{\mathbf{I}}\colon (\Delta_{\mathbf{I}}U)^*\mathbf{E}^{\mathbf{I}} \to \mathbf{Coalg}(G)$.

Moreover, reindexing restricts the domain of the fibred functor $(\widehat{\mathbb{M}_{\mathbf{I}}U})^\tau$ to a full subcategory $\Gamma^*\mathbf{E} \hookrightarrow U^*\mathbf{E}$, whose fibre over (X,γ) is given by $\Gamma^*\mathbf{E}_{(X,\gamma)} := \bigcap_{i\in\mathbf{I}} \mathrm{im}((\tau_\gamma^{(i)})^\bullet)$. By choosing appropriate liftings $\widetilde{M}_i$ of M_i, such as the saturation lifting

$$\widetilde{M}_i(P) := \{(\tau_\gamma^{(i)}(x), \tau_\gamma^{(i)}(y)) \mid (x,y) \in P\},$$

the fibred functor $(\widehat{\mathbb{M}_{\mathbf{I}}U})^\tau\colon \Gamma^*\mathbf{E} \to (\mathbb{M}_{\mathbf{I}}U)^\tau\mathbf{E}^{\mathbf{I}}$ behaves exactly as the diagonal functor and therefore again admits a fibred right adjoint, as shown in the following proposition.

Proposition 4.7. *Let* $p\colon \mathbf{E} \to \mathbf{C}$ *be a* $\mathbf{CLat}_\sqcap$*-fibration, and let*

$$(\mathbb{M}_{\mathbf{I}}U)^\tau\mathbf{E}^{\mathbf{I}} \hookrightarrow (\Delta_{\mathbf{I}}U)^*\mathbf{E}^{\mathbf{I}}$$

be the reindexed graded fibration defined as the image of $\tau^\bullet$ as in Eq. (4.5)*. Then the fibred functor $\widehat{(\mathbb{M}_{\mathbf{I}}U)}^\tau\colon \Gamma^*\mathbf{E} \to (\mathbb{M}_{\mathbf{I}}U)^\tau\mathbf{E}^{\mathbf{I}}$ admits a fibred right adjoint*

$$\forall g.-\colon (\mathbb{M}_{\mathbf{I}}U)^\tau\mathbf{E}^{\mathbf{I}} \to \Gamma^*\mathbf{E}$$

defined on objects by

$$\forall g.((X,\gamma),\{P_i\}_{i\in\mathbf{I}}) := ((X,\gamma),\bigcap_{i\in\mathbf{I}} P_i).$$

Proof. Since $p\colon \mathbf{E}\to\mathbf{C}$ is a $\mathbf{CLat}_\sqcap$-fibration, the change-of-base of p along the forgetful functor $U\colon$ $\mathbf{Coalg}(G)\to\mathbf{Set}$ is also a $\mathbf{CLat}_\sqcap$-fibration, following Proposition 2.13. Let $((X,\gamma),\{P_i\}_{i\in\mathbf{I}})$ be an object of $(\mathbb{M}_{\mathbf{I}}U)^\tau\mathbf{E}^{\mathbf{I}}$. Since each $P_i \in U^*\mathbf{E}_{(X,\gamma)}$ and each fiber $U^*\mathbf{E}_{(X,\gamma)}$ is a complete lattice, the object $\bigcap_{i\in\mathbf{I}} P_i$ is well-defined in $U^*\mathbf{E}_{(X,\gamma)}$. Moreover, it belongs to $\Gamma^*\mathbf{E}$, since $P_i \in \mathrm{im}((\tau^{(i)}_\gamma)^\bullet)$ for every $i\in\mathbf{I}$.

We first show that $\forall g.-$ is fibred. Let $f\colon (X,\gamma)\to(Y,\delta)$ be a G-coalgebra homomorphism in $\mathbf{Coalg}(C)$. Since $U^*p\colon U^*\mathbf{E}\to\mathbf{Coalg}(G)$ is a $\mathbf{CLat}_\sqcap$-fibration, reindexing preserves meets so that

$$f^\bullet\Big(\bigcap_{i\in\mathbf{I}} P_i\Big) \;\cong\; \bigcap_{i\in\mathbf{I}} f^\bullet(P_i).$$

Thus $\forall g.-$ commutes with reindexing and defines a fibred functor.

We now verify the adjunction. Let $Q\in\Gamma^*\mathbf{E}_{(X,\gamma)}\subseteq U^*\mathbf{E}_{(X,\gamma)}$ and $((X,\gamma),\{P_i\}_{i\in\mathbf{I}})$ be an object of $(\mathbb{M}_{\mathbf{I}}U)^\tau\mathbf{E}^{\mathbf{I}}$ over the same base object (X,γ). Since morphisms in $(\mathbb{M}_{\mathbf{I}}U)^\tau\mathbf{E}^{\mathbf{I}}$ are computed component-wise in the fibers, we have $\widehat{(\mathbb{M}_{\mathbf{I}}U)}^\tau(Q)\le\{P_i\}_{i\in\mathbf{I}}$ if and only if $Q\le P_i,\ \forall i\in\mathbf{I}$. By the universal property of the meet in $U^*\mathbf{E}_{(X,\gamma)}$, this is equivalent to $Q\le\bigcap_{i\in\mathbf{I}}P_i$. Since the fibres, and hence $(\mathbb{M}_{\mathbf{I}}U)^\tau\mathbf{E}^{\mathbf{I}}$, are posetal, the above equivalence of inequalities is equivalent to a bijection of hom-sets:

$$\mathrm{Hom}_{(\mathbb{M}_{\mathbf{I}}U)^\tau\mathbf{E}^{\mathbf{I}}}(\widehat{(\mathbb{M}_{\mathbf{I}}U)}^\tau Q,\{P_i\}_{i\in\mathbf{I}}) \;\cong\; \mathrm{Hom}_{U^*\mathbf{E}}(Q,\bigcap_{i\in\mathbf{I}}P_i) = \mathrm{Hom}_{\Gamma^*\mathbf{E}}(Q,\bigcap_{i\in\mathbf{I}}P_i),$$

natural in both Q and $\{P_i\}_{i\in\mathbf{I}}$. Therefore $\widehat{(\mathbb{M}_{\mathbf{I}}U)}^\tau \dashv \forall g.-$.

Finally, since both functors are defined fibrewise and preserve reindexing, the adjunction is fibred. □

5 Graded Behavioural Equivalence via Fibrations

In this section, we recover graded behavioural equivalence and its finite-depth approximates in ungraded and graded fibrations, respectively. To reason about the pre-determinization, we

first need a notion of relations between algebras. In particular, we shall use these relations to lift the notion of behavioural equivalence from the underlying sets to the algebraic structure carried by the states, allowing us to formulate and analyse the preservation of relations by the pre-determinized structure.

5.1 Relations of Eilenberg-Moore Algebras

Recall from Section 2.1 that, relations in a category $\mathbf{C}$ are (isomorphism classes of) jointly monic spans. We now instantiate this construction to the case where the base category is replaced by the Eilenberg-Moore category $\mathbf{Alg}(T)$ of a monad $T\colon \mathbf{C} \to \mathbf{C}$, we first note the following:

Lemma 5.1. *If* $\mathbf{C}$ *is complete and well-powered, then so is* $\mathbf{Alg}(T)$.

Proof. We use standard facts about Eilenberg-Moore categories.

- **Completeness.** It is classical that the forgetful functor $U\colon \mathbf{Alg}(T) \to \mathbf{C}$ creates all limits in $\mathbf{C}$. In particular, since $\mathbf{C}$ is complete, then so is $\mathbf{Alg}(T)$.

- **Well-poweredness.** We prove by constructing an injective map from the class of subobjects $\mathrm{Sub}_{\mathbf{Alg}(T)}((B,b))$ to $\mathrm{Sub}_{\mathbf{C}}(B)$ for any algebra $(B,b) \in \mathbf{Alg}(T)$. Since $U^T\colon \mathbf{Alg}(T) \to \mathbf{C}$ is a right adjoint (with left adjoint $F^T(X) = (TX, \mu_X)$), it preserves limits, specifically pullbacks, and therefore preserves monomorphisms. Furthermore, an underlying monomorphism $m\colon A \rightarrowtail B$ in $\mathbf{C}$ admits at most one algebra structure $a\colon TA \to A$ making m a morphism in $\mathbf{Alg}(T)$. Indeed, any such structure must satisfy $m \circ a = b \circ Tm$; if a_1, a_2 both satisfy this, then $m \circ a_1 = m \circ a_2$, which yields $a_1 = a_2$ by left-cancellation of the mono m.

  Define

  $$\begin{aligned}\Phi\colon \mathrm{Sub}_{\mathbf{Alg}(T)}((B,b)) &\to \mathrm{Sub}_{\mathbf{C}}(B);\\ [m\colon (A,a) \rightarrowtail (B,b)] &\mapsto [m\colon A \rightarrowtail B].\end{aligned}$$

  To see Φ is injective, suppose two algebra subobjects $[m\colon (A,a) \rightarrowtail (B,b)]$ and $[m'\colon (A',a') \rightarrowtail (B,b)]$ map to the same subobject in $\mathbf{C}$. Then there exists a $\mathbf{C}$-isomorphism $i\colon A \to A'$ with $m' \circ i = m$. This i is automatically an algebra homomorphism because $m' \circ (i \circ a) = m \circ a = b \circ Tm = b \circ T(m' \circ i) = (m' \circ a') \circ Ti = m' \circ (\alpha' \circ Ti)$, which reduces to $i \circ a = a' \circ Ti$ upon canceling the mono m'. Hence i is an isomorphism in $\mathbf{Alg}(T)$, proving $[m] = [m']$ in $\mathrm{Sub}_{\mathbf{Alg}(T)}((B,b))$. Since Φ is an injection into the small set $\mathrm{Sub}_{\mathbf{C}}(B)$, $\mathrm{Sub}_{\mathbf{Alg}(T)}((B,b))$ is a small set, so $\mathbf{Alg}(T)$ is well-powered.

□

We regard $\mathbf{Alg}(T)$ as the setting of structured objects and define relations between such algebras by specialising the general notion of spans in $\mathbf{C}$ to $\mathbf{Alg}(T)$.

Definition 5.2. Let A, B be objects of $\mathbf{Alg}(T)$. A *T-algebra relation* from A to B is an isomorphism class of jointly monic spans

$$A \xleftarrow{p_1^R} R \xrightarrow{p_2^R} B$$

in $\mathbf{Alg}(T)$, where p_1^R and p_2^R are T-algebra homomorphisms. Two such spans are identified whenever there exists an isomorphism $i\colon R \xrightarrow{\cong} R'$ in $\mathbf{Alg}(T)$ such that $p_1^R = p_1^{R'} \circ i$ and $p_2^R = p_2^{R'} \circ i$.

Accordingly, we write R for the span (R, p_1^R, p_2^R) when the feet A and B are fixed, and $\mathcal{R}_{\mathbf{Alg}(T)}(A, B)$ for the collection of jointly monic spans from A to B with both legs being T-algebra homomorphisms, ordered by factorization in $\mathbf{Alg}(T)$. This order is well-defined by Lemma 2.2, and yields a preorder structure on $\mathcal{R}_{\mathbf{Alg}(T)}(A, B)$. Let $\mathcal{R}[A, B]$ denote the quotient $\mathcal{R}_{\mathbf{Alg}(T)}(A, B)/\cong$ by span isomorphism, and let

$$q\colon \mathcal{R}_{\mathbf{Alg}(T)}(A, B) \to \mathcal{R}[A, B]; R \mapsto [R]$$

denote the canonical quotient map. For the rest of the development, we refer to an equivalence class $[R]$ in the quotient as a *T-algebra relation from A to B*, and fix a section

$$s\colon \mathcal{R}[A, B] \to \mathcal{R}_{\mathbf{Alg}(T)}(A, B)$$

of the quotient map q, i.e. $q \circ s = \mathrm{id}$, and write $\tilde{R} := s([R])$ for the chosen representative. From the completeness and well-poweredness of $\mathbf{Alg}(T)$, we obtain the following corollary.

Corollary 5.3. *The quotient $\mathcal{R}[A, B]$, endowed with the order $\leq$ defined through factorization as in Section 2.1, carries the structure of a complete lattice.*

The Category of T-algebra Relations. We now define the categories of binary relations and endorelations over T-algebras.

Definition 5.4. The category $\mathbf{Rel}_{\mathbf{Alg}(T)}$ of binary relations over T-algebras is defined as follows:

- An object is a triple $(A, B, [R])$, where $A, B \in \mathbf{Alg}(T)$ and $[R]$ a T-algebra relation from A to B, with $\tilde{R}$ being the chosen representative span $A \xleftarrow{p_1^{\tilde{R}}} \tilde{R} \xrightarrow{p_2^{\tilde{R}}} B$ in $\mathbf{Alg}(T)$.

- A morphism $(f, g)\colon (A_1, B_1, [R]) \to (A_2, B_2, [S])$ is a pair of T-algebra homomorphisms $f\colon A_1 \to A_2$ and $g\colon B_1 \to B_2$ such that there exists a T-algebra homomorphism $h\colon \tilde{R} \to \tilde{S}$ making following diagram commutes.

$$\begin{array}{ccccc} A_1 & \xleftarrow{p_1^R} & \tilde{R} & \xrightarrow{p_2^R} & B_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ A_2 & \xleftarrow[p_1^S]{} & \tilde{S} & \xrightarrow[p_2^S]{} & B_2 \end{array}$$

Hereafter, we identify each relation $(A, B, [R])$ with the subalgebra

$$\langle p_1^{\tilde{R}}, p_2^{\tilde{R}} \rangle\colon \tilde{R} \hookrightarrow A \times B,$$

where the inclusion is the unique morphism $\langle p_1^{\tilde{R}}, p_2^{\tilde{R}} \rangle$ induced by the universal property of the product; this morphism is monic because the span $A \xleftarrow{p_1^{\tilde{R}}} \tilde{R} \xrightarrow{p_2^{\tilde{R}}} B$ is jointly monic. This identification is independent of the chosen splitting. To see this, suppose s' is another section, and $\tilde{R}' = s'([R])$. By definition, there is an isomorphism $i\colon \tilde{R} \xrightarrow{\cong} \tilde{R}'$. Let $m = \langle p_1^{\tilde{R}}, p_2^{\tilde{R}} \rangle$ and $m' = \langle p_1^{\tilde{R}'}, p_2^{\tilde{R}'} \rangle$, then $m' \circ i = \langle p_1^{\tilde{R}'}, p_2^{\tilde{R}'} \rangle \circ i = \langle p_1^{\tilde{R}'} \circ i, p_2^{\tilde{R}'} \circ i \langle = \langle p_1^{\tilde{R}}, p_2^{\tilde{R}} \rangle = m$.

Moreover, a pair of T-algebra homomorphisms

$$f\colon A_1 \to A_2 \quad \text{and} \quad g\colon B_1 \to B_2$$

defines a morphism of T-algebra relations

$$(A_1, B_1, [R]) \to (A_2, B_2, [S])$$

precisely when $R \leq (f \times g)^\bullet S$, where $(f \times g)^\bullet S \hookrightarrow A_1 \times B_1$ is the following pullback in $\mathbf{Alg}(T)$:

$$\begin{array}{ccc} (f \times g)^\bullet S & \longrightarrow & S \\ \downarrow & \lrcorner & \downarrow \\ A_1 \times B_1 & \xrightarrow[f \times g]{} & A_2 \times B_2. \end{array}$$

This condition is equivalent to the existence of an T-algebra homomorphism $h\colon R \to S$ making the defining diagram commute, by the universal property of the pullback.

Lemma 5.5. *The data specified in Definition 5.4 defines a category.*

Proof. The identity morphism on $(A, B, [R])$ is given by $(\mathrm{id}_A, \mathrm{id}_B)$, witnessed by $h = \mathrm{id}_R$. Closure under composition follows from the stability of subobject inclusion under pullback. In particular, one can easily check this by inspecting the following commuting diagram

$$\begin{array}{ccccc} (f_1 \times g_1)^\bullet (f_2 \times g_2)^\bullet R_3 & \longrightarrow & (f_2 \times g_2)^\bullet R_3 & \longrightarrow & R_3 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ A_1 \times B_1 & \xrightarrow[f_1 \times g_1]{} & A_2 \times B_2 & \xrightarrow[f_2 \times g_2]{} & A_3 \times B_3. \end{array}$$

Hence $R_1 \leq (f_1 \times g_1)^\bullet \big((f_2 \times g_2)^\bullet R_3 \big) \cong ((f_2 \circ f_1) \times (g_2 \circ g_1))^\bullet R_3$, and $(f_2 \circ f_1, g_2 \circ g_1)$ is a morphism. Associativity of composition is immediate from associativity in $\mathbf{Alg}(T)$, since composition is defined componentwise. It remains to verify that the definition of morphism is independent of the choice of section.

Let s and s' be two sections of the quotient map q, and write $\tilde{R} = s([R])$, $\tilde{R}' = s'([R])$ respectively, and similarly for S. Since $\tilde{R}$ and $\tilde{R}'$ represent the same equivalence class, there is an isomorphism of spans $i_R \colon \tilde{R} \cong \tilde{R}'$ commuting with the respective legs; likewise, there is an isomorphism $i_S \colon \tilde{S} \cong \tilde{S}'$. Thus, if a T-algebra homomorphism $h \colon \tilde{R} \to \tilde{S}$ makes the defining diagram commute, then $h' = i_S \circ h \circ i_R^{-1} \colon \tilde{R}' \to \tilde{S}'$ makes the corresponding diagram commute for the representatives chosen by s'. The converse follows by symmetry. Therefore, the existence of a witnessing homomorphism depends only on the equivalence classes $[R]$ and $[S]$, and hence the definition of morphism is independent of the chosen section. □

Theorem 5.6. *The forgetful functor*

$$U \colon \mathbf{Rel}_{\mathbf{Alg}(T)} \to \mathbf{Alg}(T) \times \mathbf{Alg}(T),$$

which sends an object $(A, B, [R])$ to (A, B) and a morphism (f, g) to its underlying pair of T-algebra homomorphisms, is a $\mathbf{CLat}_\sqcap$-fibration.

Proof. We show that the functor

$$U \colon \mathbf{Rel}_{\mathbf{Alg}(T)} \to \mathbf{Alg}(T) \times \mathbf{Alg}(T),$$

is a fibration whose fibers are complete lattices and whose reindexing functors preserve finite meets.

First, note that the fiber $(\mathbf{Rel}_{\mathbf{Alg}(T)})_{(A,B)}$ above (A, B) consists of all objects $(A, B, [R])$ together with morphisms lying over $(\mathrm{id}_A, \mathrm{id}_B)$. Indeed, by the definition of morphisms, such a morphism exists from $[R]$ to $[S]$ if and only if

$$R \leq (\mathrm{id}_A \times \mathrm{id}_B)^\bullet S \cong S,$$

that is, if and only if $R \leq S$ as subalgebras of $A \times B$. Thus the fiber can be identified (up to isomorphism) with the poset of subalgebras $R \hookrightarrow A \times B$, ordered by inclusion. In particular, each fiber is a complete lattice: given a family of subalgebras $\{R_i \hookrightarrow A \times B\}_{i \in I}$, their meet is represented by the wide pullback of the inclusions $R_i \hookrightarrow A \times B$, and hence corresponds to their intersection as subalgebras of $A \times B$. Their join is represented by the image of the canonical morphism $\coprod_{i \in I} R_i \hookrightarrow A \times B$, and hence corresponds to the subalgebra generated by their union.

Next, we show that U is a fibration. Let $(f, g) : (A_1, B_1) \to (A_2, B_2)$ be a morphism in $\mathbf{Alg}(T) \times \mathbf{Alg}(T)$, and let $(A_2, B_2, [S])$ be an object over (A_2, B_2). We claim that the pullback relation $(f \times g)^\bullet S \hookrightarrow A_1 \times B_1$ defines a Cartesian lifting of $(A_2, B_2, [S])$ along (f, g). Concretely, the morphism

$$(f, g) : (A_1, B_1, [(f \times g)^\bullet S]) \to (A_2, B_2, [S])$$

is well-defined, since by construction $(f \times g)^\bullet S \leq (f \times g)^\bullet S$. To verify the Cartesian property, suppose given a morphism

$$(f', g') \colon (A_0, B_0, [R]) \to (A_2, B_2, [S])$$

such that $(f, g) = (f, g) \circ (u, v)$ for some $(u, v) \colon (A_0, B_0) \to (A_1, B_1)$. By assumption, we have

$$R \leq (f' \times g')^* S = ((f \circ u) \times (g \circ v))^* S.$$

By functoriality of pullbacks, this is isomorphic to $(u \times v)^\bullet((f \times g)^\bullet S)$. Hence $R \leq (u \times v)^\bullet((f \times g)^\bullet S)$, which shows that (u, v) defines a morphism $(A_0, B_0, [R]) \to (A_1, B_1, [(f \times g)^\bullet S])$. Uniqueness is immediate since each fiber is a preorder. Thus, U is a fibration. To conclude the proof, it remains to be shown that reindexing preserves meets.

To this end, note that reindexing along (f, g) is given by pullback:

$$(f, g)^\bullet([S]) = [(f \times g)^\bullet S].$$

Since $\mathbf{Alg}(T)$ is complete and well-powered, the poset $\mathrm{Sub}_{\mathbf{Alg}(T)}(A \times B)$ has arbitrary meets given by intersections of subalgebras. In particular, for any family $(S_i \hookrightarrow A \times B)_{i \in I}$ of subalgebras, one has $\bigwedge_{i \in I} S_i = \bigcap_{i \in I} S_i$, where the intersection is taken in $\mathbf{Alg}(T)$. Since pullback (inverse image) preserves arbitrary intersections of subobjects, we obtain

$$\begin{aligned}(f \times g)^\bullet\Big(\bigwedge_{i \in I} [S_i]\Big) &= (f \times g)^\bullet\Big(\bigcap_{i \in I} S_i\Big) \\ &= \bigcap_{i \in I} (f \times g)^\bullet(S_i) \\ &= \bigwedge_{i \in I} (f \times g)^\bullet([S_i]).\end{aligned}$$

We conclude that reindexing preserves meets, as desired. □

5.2 Relation Liftings

Relations over T-algebras form the basis for modelling behavioural equivalences in fibrations over the pre-determinization construction of Section 3.1. To transport such relations along an endofunctor, we introduce the notion of a *relation lifting* along a functor $G\colon \mathbf{Alg}(T) \to \mathbf{Alg}(T)$ in the sense of the following.

Definition 5.7 (Relation lifting)**.** Let $A_1, A_2 \in \mathbf{Alg}(T)$, and let $[R]$ be a T-algebra relation represented by

$$A_1 \xleftarrow{p_1} R \xrightarrow{p_2} A_2.$$

The *relation lifting* of G, denoted by $\overline{G}$, assigns to R a T-algebra relation from GA_1 to GA_2 defined as the image of the following composite

$$GR \xrightarrow{G\langle p_1,p_2\rangle} G(A_1 \times A_2) \xrightarrow{\langle G\pi_1,G\pi_2\rangle} GA_1 \times GA_2, \tag{5.1}$$

where π_i is the projection $A_1 \times A_2 \to A_i$, for $i = 1, 2$.

Since $\mathbf{Alg}(T)$ has products and images, the composite Eq. (5.1) is a T-algebra homomorphism. Its image in $\mathbf{Alg}(T)$ therefore inherits a canonical T-algebra structure making it a T-subalgebra of $GA_1 \times GA_2$. We define $\overline{G}(R)$ to be this image subalgebra, equipped with the induced projections to GA_1 and GA_2.

We establish several basic properties of relation liftings needed for the graded fibrational framework developed later.

Proposition 5.8. *The relation lifting $\overline{G}$ is well-defined on T-algebra relations, i.e. it is independent of the choice of representatives of isomorphism classes.*

Proof. Suppose s' is another section, and let $R' = s'([R])$. By definition of the quotient, there exists an isomorphism

$$i\colon R \xrightarrow{\cong} R'$$

in $\mathbf{Alg}(T)$ commuting with the projections to A_1 and A_2. Since G is an endofunctor on $\mathbf{Alg}(T)$, $Gi\colon GR \to GR'$ is again an isomorphism, and the defining composites

$$GR \to GA_1 \times GA_2, \quad GR' \to GA_1 \times GA_2$$

for the relation lifting differ only by precomposition with Gi. Consequently, the corresponding image factorizations are isomorphic by the uniqueness of image factorizations up to unique isomorphism. Hence

$$\overline{G}(R) \;\cong\; \overline{G}(R').$$

Thus $\overline{G}$ is well-defined on $\mathcal{R}[A_1, A_2]$. □

Explicitly, factor the above composite Eq. (5.1) as

$$GR \twoheadrightarrow \widetilde{R} \hookrightarrow GA_1 \times GA_2,$$

where the second morphism is monic in $\mathbf{Alg}(T)$. Then $\overline{G}R$ is given by the jointly monic span:

$$GA_1 \leftarrow \widetilde{R} \rightarrow GA_2.$$

Lemma 5.9. *The relation lifting $\overline{G}$ is monotone: if $R \leq S$, then $\overline{G}R \leq \overline{G}S$.*

Proof. Suppose $R \leq S$, witnessed by a morphism $h\colon R \to S$ in $\mathbf{Alg}(T)$ such that $p_1^R = p_1^S \circ h$, and $p_2^R = p_2^S \circ h$, one then has $\langle p_1^R, p_2^R\rangle = \langle p_1^S, p_2^S\rangle \circ h$. Applying the functor G, we obtain a morphism $Gh\colon GR \to GS$, and by functoriality, $G\langle p_1^R, p_2^R\rangle = G\langle p_1^S, p_2^S\rangle \circ Gh$. Postcompose with $\langle G\pi_1, G\pi_2\rangle$, it follows that the following diagram commutes:

$$\begin{array}{ccc} GR & \xrightarrow{\quad Gh \quad} & GS \\ \downarrow & & \downarrow \\ GA_1 \times GA_2 & \xrightarrow{\ \mathrm{id}\ } & GA_1 \times GA_2, \end{array}$$

where the vertical maps are the composites defining the relation lifting. Now factor these vertical morphisms through their images:

$$GR \twoheadrightarrow \widetilde{R} \hookrightarrow GA_1 \times GA_2, \quad GS \twoheadrightarrow \widetilde{S} \hookrightarrow GA_1 \times GA_2.$$

By the commutativity of the above diagram and the universal property of images in $\mathbf{Alg}(T)$, there exists a (unique) morphism

$$\widetilde{h} : \widetilde{R} \to \widetilde{S}$$

making the following diagram commute:

$$\begin{array}{ccc} GR & \xrightarrow{\quad Gh \quad} & GS \\ \downarrow & & \downarrow \\ \widetilde{R} & \xrightarrow{\quad \widetilde{h} \quad} & \widetilde{S} \\ \downarrow & & \downarrow \\ GA_1 \times GA_2 & \xrightarrow{\ \mathrm{id}\ } & GA_1 \times GA_2. \end{array}$$

This shows that $\overline{G}R$ factors through $\overline{G}S$, i.e., $\overline{G}R \leq \overline{G}S$. □

In particular, the relation lifting of G defines a monotone map on fibers

$$\overline{G}\colon (\mathbf{Rel}_{\mathbf{Alg}(T)})_{(A_1,A_2)} \to (\mathbf{Rel}_{\mathbf{Alg}(T)})_{(GA_1,GA_2)}.$$

5.3 Infinite-depth Graded Behavioural Equivalence

In this section, we fix the endofunctors $F\colon \mathbf{Alg}(T) \to \mathbf{Alg}(T)$ and $G\colon \mathbf{C} \to \mathbf{C}$, together with a graded semantics $(\alpha, \mathbb{M})$ for G. We let (X, γ) and (Y, δ) be G-coalgebras, unless explicitly stated otherwise. Following Staton [51], we assume that M_0 is affine. This ensures that $M_0 1 = 1$, $\overline{M}_1$ preserves weak pullbacks, and that behavioural equivalence coincides with the corresponding coalgebraic bisimilarity. We are going to provide a fibration account of infinite-depth graded behavioural equivalence from Definition 3.6 via bisimilarity on a pre-determinized coalgebra.

Bisimilarity over T-algebras. We begin by defining bisimulation and bisimilarity via relation lifting for coalgebras in $\mathbf{Alg}(T)$. Let $F\colon \mathbf{Alg}(T) \to \mathbf{Alg}(T)$ be an endofunctor, and let (A_1, γ_1) and (A_2, γ_2) be F-coalgebras.

Definition 5.10. Let $[R]$ be a T-algebra relation from A_1 to A_2 represented by R, we say that $[R]$ (or simply, R) is a *bisimulation* if there exists a coalgebra homomorphism $R \to \overline{F}R$ making the following diagram commute:

$$\begin{array}{ccccc} A_1 & \longleftarrow & R & \longrightarrow & A_2 \\ {\scriptstyle\gamma_1}\downarrow & & \vdots\,{\scriptstyle\exists} & & \downarrow{\scriptstyle\gamma_2} \\ FA_1 & \longleftarrow & \overline{F}R & \longrightarrow & FA_2. \end{array}$$

If $\mathbf{C}$ is a concrete category, we also use the point-wise notation: a relation R is a bisimulation if, for every $(x, y) \in R$, we have $(\gamma_1(x), \gamma_2(y)) \in \overline{F}R$.

Since relations over T-algebras can be identified with subalgebras of the product (Section 5.1), the above condition can be equivalently expressed by the commutativity of the following square:

$$\begin{array}{ccc} R & \dashrightarrow & \overline{F}R \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\overline{F}\iota} \\ A_1 \times A_2 & \xrightarrow{\gamma_1 \times \gamma_2} & FA_1 \times FA_2. \end{array} \tag{5.2}$$

Lemma 5.11. *The diagram in Eq.* (5.2) *commutes iff* $R \leq (\gamma_1 \times \gamma_2)^{\bullet}\overline{F}R$.

Proof. Recall that the reindexing $(\gamma_1 \times \gamma_2)^\bullet \overline{F}R$ is defined as the pullback

$$\begin{array}{ccc} (\gamma_1 \times \gamma_2)^\bullet \overline{F}R & \xrightarrow{\;k\;} & \overline{F}R \\ {\scriptstyle \jmath}\downarrow & \lrcorner & \downarrow{\scriptstyle \overline{F}\imath} \\ A_1 \times A_2 & \xrightarrow{\gamma_1 \times \gamma_2} & FA_1 \times FA_2 \end{array}$$

'$\Rightarrow$': Assume the square Eq. (5.2) commutes, i.e., there exists $\rho\colon R \to \overline{F}R$ such that $(\gamma_1 \times \gamma_2) \circ \imath = \overline{F}\imath \circ \rho$. Then the pair $(\imath, \rho)$ forms a morphism from R into the cospan defining the pullback $(\gamma_1 \times \gamma_2)^\bullet \overline{F}R$. By the universal property of pullbacks, there exists a unique morphism $h\colon R \to (\gamma_1 \times \gamma_2)^\bullet \overline{F}R$ such that

$$\jmath \circ h = \imath \quad \text{and} \quad k \circ h = \rho.$$

Thus $\imath$ factors through $\jmath$, i.e. $R \leq (\gamma_1 \times \gamma_2)^\bullet \overline{F}R$.

'$\Leftarrow$': Assume $R \leq (\gamma_1 \times \gamma_2)^\bullet \overline{F}R$, i.e., there exists $h\colon R \to (\gamma_1 \times \gamma_2)^\bullet \overline{F}R$ such that $\jmath \circ h = \imath$. Define $\rho := k \circ h\colon R \to \overline{F}R$. Then using the pullback equality $(\gamma_1 \times \gamma_2) \circ \jmath = \overline{F}\imath \circ k$ we compute:

$$(\gamma_1 \times \gamma_2) \circ \imath = (\gamma_1 \times \gamma_2) \circ \jmath \circ h = \overline{F}\imath \circ k \circ h = \overline{F}\imath \circ \rho.$$

Hence the square Eq. (5.2) commutes.

□

Definition 5.12 (Bisimulation operator)**.** Let $\overline{F}$ denote the relation lifting of F as in Definition 5.7. We define the *bisimulation operator*

$$\Phi\colon (\mathbf{Rel}_{\mathbf{Alg}(T)})_{(A_1,A_2)} \to (\mathbf{Rel}_{\mathbf{Alg}(T)})_{(A_1,A_2)}$$

associated with F-coalgebras (A_1, γ_1) and (A_2, γ_2), on the fiber $(\mathbf{Rel}_{\mathbf{Alg}(T)})_{(A_1,A_2)}$ of the relation fibration $\mathbf{Rel}_{\mathbf{Alg}(T)} \to \mathbf{Alg}(T) \times \mathbf{Alg}(T)$ by

$$\Phi(R) := (\gamma_1 \times \gamma_2)^\bullet \overline{F}R, \tag{5.3}$$

where $(\gamma_1 \times \gamma_2)^\bullet$ denotes reindexing along $(\gamma_1 \times \gamma_2)$ in the relation fibration.

Proposition 5.13 (Well-definedness of Φ)**.** *The bisimulation operator Φ is well-defined on T-algebra relations, i.e. it is independent of the choice of splitting.*

Since relations over T-algebras can be identified with subobjects of the product, we make use of the following lemma.

Lemma 5.14. *Let $f\colon X \to Y$ in a category with pullbacks. If $S \cong S'$ are isomorphic subobjects of Y, then $f^\bullet(S) \cong f^\bullet(S')$ as subobjects of X.*

Proof. Let S and S' be subobjects of Y represented by monomorphisms

$$\jmath\colon S \hookrightarrow Y, \quad \jmath'\colon S' \hookrightarrow Y,$$

and suppose $S \cong S'$ as subobjects. This means there exists an isomorphism $i\colon S \xrightarrow{\cong} S'$ such that $\jmath = \jmath' \circ i$. Define reindexing via pullback:

$$\begin{array}{ccc} f^\bullet(S) & \xrightarrow{p} & S \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \jmath} \\ X & \xrightarrow[f]{} & Y \end{array} \qquad \begin{array}{ccc} f^\bullet(S') & \xrightarrow{p'} & S' \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle \jmath'} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Since $\jmath = \jmath' \circ i$, we have: $\jmath \circ p = \jmath' \circ i \circ p$. Thus the pair $(q, i \circ p)\colon f^\bullet(S) \to X \times_Y S'$ is a cone over the pullback defining $f^\bullet(S')$. By the universal property of the pullback $f^\bullet(S')$, there exists a unique morphism $\tilde{i}\colon f^\bullet(S) \to f^\bullet(S')$ such that:

$$p' \circ \tilde{i} = i \circ p, \quad q' \circ \tilde{i} = q.$$

Similarly, using $i^{-1}\colon S' \xrightarrow{\cong} S$, the same argument yields a morphism $\tilde{i}^{-1}\colon f^\bullet(S') \to f^\bullet(S)$ such that:

$$p \circ \tilde{i}^{-1} = i^{-1} \circ p', \quad q \circ \tilde{i}^{-1} = q'.$$

Notice that $\tilde{i}^{-1} \circ \tilde{i}$ and $\mathrm{id}_{f^\bullet(S)}$ have the same projections to S and X, by uniqueness in the universal property of the pullback, one has $\tilde{i}^{-1} \circ \tilde{i} = \mathrm{id}_{f^\bullet(S)}$. The same argument yields $\tilde{i} \circ \tilde{i}^{-1} = \mathrm{id}_{f^\bullet(S')}$, thus $\tilde{i}\colon f^\bullet(S) \to f^\bullet(S')$ is an isomorphism, and therefore $f^\bullet(S)$ and $f^\bullet(S')$ defines the same subobject of X. □

The well-definedness of the bisimulation operator Φ, as stated in Proposition 5.13, then follows from Proposition 5.8 and Lemma 5.14.

Theorem 5.15. *The bisimulation operator*

$$\Phi\colon \left(\mathbf{Rel}_{\mathbf{Alg}(T)}\right)_{(A_1,A_2)} \to \left(\mathbf{Rel}_{\mathbf{Alg}(T)}\right)_{(A_1,A_2)}$$

defined in Definition 5.12 is monotone with respect to the order on the fiber $\left(\mathbf{Rel}_{\mathbf{Alg}(T)}\right)_{(A_1,A_2)}$ defined by factorization in $\mathbf{Alg}(T)$.

Proof. This follows from the monotonicity of relation lifting together with the fact that inverse image along a morphism preserves the preorder on relations. □

Definition 5.16 (Bisimulation)**.** A T-algebra relation R from A_1 to A_2 is an F-bisimulation if it is a post-fixed point of Φ, i.e.

$$R \leq \Phi(R).$$

Since the forgetful functor $U\colon \mathbf{Rel}_{\mathbf{Alg}(T)} \to \mathbf{Alg}(T) \times \mathbf{Alg}(T)$ is a $\mathbf{CLat}_{\sqcap}$-fibration (Theorem 5.6), it admits greatest fixed points of monotone operators on fibers by the Knaster–Tarski theorem.

Definition 5.17 (Bisimilarity)**.** The bisimilarity relation from A_1 to A_2, denoted as $\sim$, is defined as the greatest fixed point of Φ, we write $\nu\Phi$ to indicate this relation. Equivalently, it is the greatest bisimulation:

$$\sim \;=\; \bigvee\{R \mid (A_1, A_2, [R]) \in (\mathbf{Rel}_{\mathbf{Alg}(T)})_{(A_1,A_2)},\ R \leq \Phi(R)\}.$$

Bisimilarity on the Pre-determinization. We now instantiate the notion of bisimilarity to the pre-determinized coalgebra. Let A_1, A_2 be M_0-algebras.

Recall that the pre-determinizations of the G-coalgebras (X, γ) and (Y, δ) are given by

$$\gamma^\sharp\colon FX \to \overline{M}_1 FX, \quad \delta^\sharp\colon FY \to \overline{M}_1 FY, \tag{5.4}$$

where $F \dashv U\colon \mathbf{Alg}_0(\mathbb{M}) \rightleftarrows \mathbf{Set}$ is the canonical free-forgetful adjunction for the Eilenberg-Moore category of M_0. We regard Eq. (5.4) as $\overline{M}_1$-coalgebras and apply the general theory of bisimilarity developed in Section 5.3.

Following Definition 5.7, the relation lifting $\overline{\overline{M}}_1$ of $\overline{M}_1$ is given as follows. For any M_0-algebra relation $[R]$ from A_1 to A_2, represented by R, we define

$$\overline{\overline{M}}_1 R := \operatorname{im}\Big(\overline{M}_1 R \to \overline{M}_1(A_1 \times A_2) \to \overline{M}_1 A_1 \times \overline{M}_1 A_2\Big).$$

When point-wise notation is feasible, this lifting admits the following characterization on underlying carriers:

$$\overline{\overline{M}}_1 R := \left\{(\overline{M}_1\pi_1(w), \overline{M}_1\pi_2(w)) \in \overline{M}_1 A_1 \times \overline{M}_1 A_2 \;\middle|\; w \in \overline{M}_1 R\right\},$$

where we implicitly apply the forgetful functor and omit the algebra structures from the notation.

Example 5.18. We give a description of the functor $\overline{\overline{M}}_1$ on M_0-algebra relations constructed above in some of the running examples.

1. For graded monads of the form $\mathbb{M}_G$, which capture finite depth graded behavioural equivalence, we have $M_0 = \mathrm{id}$, so M_0-algebras are just sets, and under this correspondence, $\overline{M}_1$ is the original functor G. The relation lifting $\overline{\overline{M}}_1$ is the canonical relation lifting $\overline{G}$ of the **Set** endofunctor G defined pointwise as:
$$\overline{G}R := \{(G\pi_1(w), G\pi_2(w)) \in GX \times GY \mid w \in GR\}.$$

2. For trace semantics of an LTS, where $M_nX = \mathcal{P}(A^n \times X)$, and $\overline{M}_1X = X^A$, as given by [15, Example 5.3.(2)]. The relation lifting $\overline{\overline{M}}_1$ of $\overline{M}_1$ is defined pointwise as:
$$\overline{\overline{M}}_1R = \left\{(f,g) \in X^A \times Y^A \mid \forall a \in A, (f(a), g(a)) \in R^A\right\}.$$
Two functions f, g are related by $\overline{\overline{M}}_1$ iff they are related component-wise for every label $a \in A$.

As an instance of Definition 5.16, an $\overline{M}_1$-*bisimulation* on FX is a relation $[R]$ satisfying
$$R \leq \Phi(R) := (\gamma^\sharp \times \gamma^\sharp)^\bullet(\overline{\overline{M}}_1 R),$$
where the bisimulation operator Φ is defined as in Definition 5.12, but now on the fiber $(\mathbf{Rel}_{\mathbf{Alg}_0(\mathbb{M})})_{(FX,FX)}$. The *bisimilarity* relation on FX, following Definition 5.17, is defined as the greatest fixed point of Φ, we write
$$\sim_{\mathrm{bis}} := \nu\Phi. \tag{5.5}$$

We then have the following fibrational characterization of the infinite-depth graded behaviour equivalence:

Proposition 5.19. *Let $G\colon \mathbf{C} \to \mathbf{C}$ be an endofunctor, let $(\alpha, \mathbb{M})$ be a graded semantics for G, and let (X, γ) be a G-coalgebra. Assume that the functor*
$$\overline{M}_1\colon \mathbf{Alg}_0(\mathbb{M}) \to \mathbf{Alg}_0(\mathbb{M})$$
preserves weak pullbacks. Then the infinite-depth $(\alpha, \mathbb{M})$-behavioural equivalence *on X, as defined in Definition 3.6, coincides with the bisimilarity relation on the pre-determinized coalgebra $\gamma^\sharp\colon FX \to \overline{M}_1FX$. The following are equivalent.*

1. $\equiv_\infty^{(\alpha,\mathbb{M})}$,

2. $\sim_{\mathrm{bis}}$,

3. $\nu\Phi$, *where* Φ *is the bisimulation operator associated with* $\gamma^\sharp$.

Thus, infinite-depth graded behavioural equivalence is characterized coinductively via the bisimulation operator living on the fiber $(\mathbf{Rel}_{\mathbf{Alg}_0(\mathbb{M})})_{(FX,FX)}$ of the fibration $\mathbf{Rel}_{\mathbf{Alg}_0(\mathbb{M})} \to \mathbf{Alg}_0(\mathbb{M}) \times \mathbf{Alg}_0(\mathbb{M})$.

5.4 Finite-depth Graded Behavioural Equivalence

Recall from Definition 3.2 that for a given G-coalgebra (X, γ), the sequence of α-pretraces is defined inductively by

$$\gamma^{(0)} := \eta_X, \text{ and } \gamma^{(n+1)} := \mu_X^{1,n} \circ M_1\gamma^{(n)} \circ (\alpha_X \circ \gamma).$$

For each $n \in \mathbb{N}$, we obtain the depth-n semantic map $\gamma^{(n)} : X \to M_nX$. Finite-depth graded behavioural equivalence is then defined defined by requiring the corresponding poststate-forgetting traces $M_n!_X \circ \gamma^{(n)}$ to agree for each $n \in \mathbb{N}$.

Reindexed $\mathbb{N}$-graded fibration of relations. We now instantiate the reindexed graded fibration framework Definition 4.6 over $\mathbb{N}$.

Let $p\colon \mathbf{Rel} \to \mathbf{Set}$ be the relation fibration, and consider its $\mathbb{N}$-indexed extension $p^{\mathbb{N}}\colon \mathbf{Rel}^{\mathbb{N}} \to \mathbf{Set}^{\mathbb{N}}$. Given a graded monad $\mathbb{M} = (M_n)_{n\in\mathbb{N}}$, we form the induced functor

$$\mathbb{M}_{\mathbb{N}} : \mathbf{Set} \to \mathbf{Set}^{\mathbb{N}};\ X \mapsto (M_nX)_{n\in\mathbb{N}}.$$

For a coalgebra (X, γ), the family of α-pretrace maps $\tau_\gamma := \{\tau_\gamma^{(n)} := \gamma^{(n)}\}_{n\in\mathbb{N}}$ defines a semantic morphism $\tau_\gamma\colon \Delta_{\mathbb{N}}UX \to \mathbb{M}_{\mathbb{N}}UX$. This induces a reindexing functor along τ_γ as

$$\tau_\gamma^\bullet\colon (\mathbb{M}_{\mathbb{N}}U)^*\mathbf{Rel}^{\mathbb{N}} \to (\Delta_{\mathbb{N}}U)^*\mathbf{Rel}^{\mathbb{N}},$$

given fibrewise by pullback of relations along $\gamma^{(n)}$.

Semantic predicates at each depth. For each $n \in \mathbb{N}$, let Δ_{M_n1} denote the diagonal relation on M_n1. We define the semantic equivalence relation on M_nX by

$$Q_n := (M_n!_X \times M_n!_X)^\bullet\Delta_{M_n1},$$

and, pulling this relation back along the depth-n semantic map $\gamma^{(n)}\colon X \to M_nX$, define

$$P_n := (\gamma^{(n)} \times \gamma^{(n)})^\bullet Q_n.$$

The relation P_n thus captures the agreement of the depth-n semantic unfoldings after forgetting their poststates.

Theorem 5.20. *For every $n \in \mathbb{N}$, the relation $P_n \subseteq X \times X$ coincides with the depth-n behavioural equivalence induced by the graded semantics $(\alpha, \mathbb{M})$. In particular, for all $x, y \in X$,*

$$x \equiv_n^{(\alpha,\mathbb{M})} y \quad \iff \quad (x, y) \in P_n.$$

Proof. For each $n \in \mathbb{N}$, consider the two consecutive reindexings represented by the following pullback diagram

$$\begin{array}{ccccc} P_n & \longrightarrow & Q_n & \longrightarrow & \Delta_{M_n 1} \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ X \times X & \xrightarrow[\gamma^{(n)} \times \gamma^{(n)}]{} & M_n X \times M_n X & \xrightarrow[M_n !_X \times M_n !_X]{} & M_n 1 \times M_n 1. \end{array}$$

By functoriality of reindexing,

$$\begin{aligned} P_n &= (\gamma^{(n)} \times \gamma^{(n)})^\bullet (M_n !_X \times M_n !_X)^\bullet \Delta_{M_n 1} \\ &= (M_n !_X \circ \gamma^{(n)} \times M_n !_X \circ !_X \circ \gamma^{(n)})^\bullet \Delta_{M_n 1} \end{aligned}$$

Thus P_n is the kernel relation of the depth-n trace map $M_n !_X \circ \gamma^{(n)} \colon X \to M_n 1$. By the characterization of depth-n behavioural equivalence for $(\alpha, \mathbb{M})$, one has $(x, y) \in P_n$ when $(M_n !_X \circ \gamma^{(n)}(x), M_n !_X \circ \gamma^{(n)}(y)) \in \Delta_{M_n 1}$, i.e. $M_n !_X \circ \gamma^{(n)}(x) = M_n !_X \circ \gamma^{(n)}(y)$, which is precisely $x \equiv_n^{(\alpha,\mathbb{M})} y$. Since n was arbitrary, the claim follows. □

Finite-depth equivalence via graded intersection. The reindexed graded fibration $(\mathbb{M}_{\mathbb{N}} U)^\tau \mathbf{Rel}^{\mathbb{N}}$ induces, fibrewise, a notion of conjunction via intersection. Using the fibred right adjoint to reindexing, we define:

$$x \equiv_\omega^{(\alpha,\mathbb{M})} y \;\; :\iff \;\; (x, y) \in \forall g.(\{P_n\}_{n\in\mathbb{N}}) = \bigcap_{n\in\mathbb{N}} P_n.$$

Equivalently, $\equiv_\omega^{(\alpha,\mathbb{M})}$ is the meet in the fiber of $(\Delta_{\mathbb{N}} U)^* \mathbf{Rel}^{\mathbb{N}}$ of the graded family $\{P_n\}_{n\in\mathbb{N}}$, and hence corresponds to the ω-stage approximation of behavioural equivalence in the reindexed graded fibration.

Thus, depth-n graded behavioural equivalence arises from the n-th component of the reindexed semantic lifting, while finite-depth equivalence is obtained as the limit (fibred meet) of this graded family inside the reindexed $\mathbb{N}$-graded fibration induced by τ.

6 Examples

In this section, we illustrate our framework on a selection of example behavioural equivalences spanning the linear-time branching-time spectrum [53], which organizes behavioural equivalences according to the amount of branching information they preserve. At one end of the spectrum are linear-time semantics, such as trace equivalence, which observe only the sequences of actions that a system can perform. At the other end are branching-time semantics, such as bisimilarity, which additionally distinguish how these actions are organized into branching behaviours. Between these extremes lie a range of intermediate equivalences that retain varying amounts of branching information.

We provide a detailed account of the graded fibration framework in the case of bisimilarity, trace equivalence, and simulation equivalence, illustrating how the framework recovers the behavioural equivalences at varying levels of observational granularity specified by different graded semantics. These examples follow [12]; a more comprehensive treatment of graded semantics across the spectrum can be found in [12, Section 4].

6.1 Bisimilarity

Bisimilarity is the finest notion of behavioural equivalence in the linear-time branching-time spectrum. For a finitely branching labelled transition system represented as a coalgebra $\gamma\colon X \to GX$ where $G = \mathcal{P}_\omega(A \times -)$, a relation $R \subseteq X \times X$ is a G-bisimulation if related states have matching one-step behaviour. That is, whenever $(x, y) \in R$, for every label $a \in A$,

$$x \xrightarrow{a} x' \implies \exists y' \in X.; y \xrightarrow{a} y' \text{ and } (x', y') \in R,$$
$$y \xrightarrow{a} y' \implies \exists x' \in X.; x \xrightarrow{a} x' \text{ and } (x', y') \in R.$$

Bisimilarity is the greatest relation satisfying these conditions. Generally, under mild assumptions on G, bisimilarity coincides with coalgebraic behavioural equivalence. Moreover, if G is finitary, this notion coincides with its finite-depth approximations, which are captured by the canonical cone into the final chain, as recalled in Section 2.2.

Finite-depth behavioural equivalence. Recall from Section 2.2 that the finite-depth approximations of coalgebraic behavioural equivalence are captured by the canonical cone $\gamma_n\colon X \to G^n 1$, defined inductively by

$$\gamma_0 := !_X, \text{ and } \gamma_{n+1} := G\gamma_n \circ \gamma.$$

We now recover these finite-depth approximations within the graded fibrational framework. We take $\mathbb{M}_{\mathbb{N}} := \mathbb{M}_G$, as defined in Example 2.8.1, and construct the corresponding graded

fibration as follows.

$$\begin{array}{ccc} (\mathbb{M}_G U)^* \mathbf{Rel}^{\mathbb{N}} & \longrightarrow & \mathbf{Rel}^{\mathbb{N}} \\ {\scriptstyle (\mathbb{M}_G U)^* p^{\mathbb{N}}}\downarrow & \lrcorner & \downarrow{\scriptstyle p^{\mathbb{N}}} \\ \mathbf{Coalg}(G) & \xrightarrow{\mathbb{M}_G U} & \mathbf{Set}^{\mathbb{N}}. \end{array}$$

Here, each object of $(\mathbb{M}_G U)^* \mathbf{Rel}^{\mathbb{N}}$ is a pair $((X, \gamma), \{R'_n\}_{n\in\mathbb{N}})$, where $R'_n \subseteq G^n X \times G^n X$ is a relation in the semantic domain at grade n. We then consider the semantic map

$$\tau \colon \Delta_{\mathbb{N}} U \to \mathbb{M}_G U,$$

whose component at the coalgebra $\gamma \colon X \to GX$ is the graded family $\{\tau_\gamma^{(n)}\}_{n\in\mathbb{N}}$ defined inductively by

$$\tau_\gamma^{(0)} := \mathrm{id}_X, \text{ and } \tau_\gamma^{(n+1)} := G^n \tau_\gamma^{(n)} \circ \gamma.$$

Reindexing along this semantic map yields, at each grade n, a relation $R_n \subseteq X \times X$ as the pullback of R'_n along the semantic map $\tau_\gamma^{(n)}$:

$$\begin{array}{ccc} R_n & \longrightarrow & R'_n \\ \downarrow & \lrcorner & \downarrow \\ X \times X & \xrightarrow[\tau_\gamma^{(n)} \times \tau_\gamma^{(n)}]{} & G^n X \times G^n X. \end{array}$$

Thus, the reindexed graded fibration $(\mathbb{M}_G U)^\tau \mathbf{Rel}^{\mathbb{N}} \to \mathbf{Coalg}(G)$ constructed according to Definition 4.6 has objects of the form $((X, \gamma), \{R_n\}_{n\in\mathbb{N}})$, where $R_n = (\tau_\gamma^{(n)} \times \tau_\gamma^{(n)})^\bullet R'_n$ for the corresponding graded family $\{R'_n\}_{n\in\mathbb{N}}$.

The preceding construction therefore provides the required setting for recovering the finite-depth graded behavioural equivalence. In particular, choosing the diagonal relation at each observation domain $G^n 1$ yields precisely the depth-n graded behavioural equivalences induced by the graded semantics $(\mathrm{id}, \mathbb{M}_G)$, as summarized in the following proposition.

Proposition 6.1. *Consider the reindexed graded fibration $(\mathbb{M}_G U)^\tau \mathbf{Rel}^{\mathbb{N}} \to \mathbf{Coalg}(G)$ constructed above. For each $n \in \mathbb{N}$ define*

$$R'_n = (G^n !_X \times G^n !_X)^\bullet \Delta_{G^n 1},$$

where $\Delta_{G^n 1}$ is the diagonal relation at the observation domain $G^n 1$. The corresponding family of relations on X is obtained by pulling back this graded family of relations along the semantic maps, such that for each grade $n \in \mathbb{N}$,

$$R_n = (\tau_\gamma^{(n)} \times \tau_\gamma^{(n)})^\bullet R'_n = (\tau_\gamma^{(n)} \times \tau_\gamma^{(n)})^\bullet (G^n !_X \times G^n !_X)^\bullet \Delta_{G^n 1}.$$

For the graded semantics $(\mathrm{id}, \mathbb{M}_G)$, the following hold:

1. *for every $n \in \mathbb{N}$, R_n is precisely the depth-n graded behavioural equivalence induced by $(\mathrm{id}, \mathbb{M}_G)$;*

2. *the fibred functor $\widehat{(\mathbb{M}_G U)}^{\tau} \colon \Gamma^*\mathbf{Rel} \to (\mathbb{M}_G U)^{\tau}\mathbf{Rel}^{\mathbb{N}}$ admits a fibred right adjoint $\forall g.-$, and the finite-depth graded behavioural equivalence is recovered as $R_\omega := \forall g.(\{R_n\}_{n\in\mathbb{N}})$.*

In particular, the graded fibrational construction recovers the finite-depth graded semantics specified by $(\mathrm{id}, \mathbb{M}_G)$ for G.

Proof. For (1), it follows directly from Theorem 5.20 by taking $\mathbb{M}_{\mathbb{N}} = \mathbb{M}_G$ and $\alpha = \mathrm{id}$. Indeed, under this specialization, the observation relation at grade n is $R'_n = (G^n!_X \times G^n!_X)^{\bullet}\Delta_{G^n 1}$, and the corresponding relation on X is $R_n = (\tau_\gamma^{(n)} \times \tau_\gamma^{(n)})^{\bullet} R'_n$. Hence R_n is precisely the depth-n graded behavioural equivalence induced by $(\mathrm{id}, \mathbb{M}_G)$.

For (2), since $p \colon \mathbf{Rel} \to \mathbf{Set}$ is a $\mathbf{CLat}_\sqcap$-fibration, Proposition 4.7 implies that $\widehat{(\mathbb{M}_G U)}^{\tau} \colon \Gamma^*\mathbf{Rel} \to (\mathbb{M}_G U)^{\tau}\mathbf{Rel}^{\mathbb{N}}$ admits a fibred right adjoint $\forall g.-$. Applying this right adjoint to the graded family $\{R_n\}_{n\in\mathbb{N}}$ yields

$$R_\omega := \forall g.(\{R_n\}_{n\in\mathbb{N}}) = \bigcap_{n\in\mathbb{N}} R_n,$$

given by fibrewise intersections. By (1), each R_n is the depth-n graded behavioural equivalence induced by $(\mathrm{id}, \mathbb{M}_G)$; hence R_ω is precisely the finite-depth graded behavioural equivalence. □

Infinite-depth behavioural equivalence. We now give a fibrational account of the infinite-depth graded behavioural equivalence induced by the graded semantics $(\mathrm{id}, \mathbb{M}_G)$. By Definition 3.6, the equivalence $\equiv_\infty^{(\mathrm{id},\mathbb{M}_G)}$ is defined as the coalgebraic behavioural equivalence of the pre-determinized coalgebra. We therefore first specialize the pre-determinization construction to the graded monad $\mathbb{M}_G$.

Following Ford et al. [15], since $M_0 = \mathrm{id}$, the category of M_0-algebras is canonically identified with **Set**. Moreover, the functor $\overline{M}_1$ induced by $M_1 = G$ on M_0-algebras is simply

$$\overline{M}_1 = M_1 = G.$$

Consequently, the pre-determinization of a G-coalgebra is the coalgebra itself:

$$\gamma^\sharp = \gamma \colon X \to GX.$$

Hence, in this particular instance, the infinite-depth graded behavioural equivalence is characterized by the usual greatest fixed-point construction for bisimilarity with in the classical relation fibration **Rel** $\to$ **Set**, following Proposition 5.19. We thus have the following characterization.

Proposition 6.2. *Assume that G preserves weak pullbacks. The infinite-depth $(\mathrm{id}, \mathbb{M}_G)$-behavioural equivalence is the greatest fixed point of the bisimulation operator*

$$\Phi := (\gamma \times \gamma)^\bullet \overline{G},$$

where $\overline{G}$ denotes the canonical relation lifting of G, defined by

$$\overline{G}R := \{(G\pi_1(w), G\pi_2(w)) \mid w \in GR\}\,.$$

6.2 Trace Equivalence

Trace equivalence is the coarsest linear-time notion of behavioural equivalence that abstracts away from the branching structure and records only the sequences of observable actions. Let A be a finite set of actions. For an finitely-branching LTS, recall from Example 3.3.2 that two states are *trace equivalent* if they admit the same action sequence. The corresponding graded semantics is given by the graded monad $\mathbb{M}_{\mathcal{P}_\omega}(A)$ with $M_n = \mathcal{P}_\omega(A^n \times -)$ and $\alpha = \mathrm{id}$.

In this section, we show that the graded fibrational framework recovers this graded trace semantics at finite depths, and that the infinite-depth behavioural equivalence of a serial variant can be characterized by a coinductive predicate in the relation fibration over M_0-algebras.

Finite-depth behavioural equivalence. Trace equivalence admits a natural finite-depth approximation. For each $n \in \mathbb{N}$, recall from Example 3.3.2 that the depth-n pretrace maps are defined inductively by

$$\gamma^{(0)} = \eta_X, \text{ and } \gamma^{(n+1)} = \mu_X^{1,n} \circ M_n\gamma^{(n)} \circ \gamma$$

with unit and multiplication induced by concatenation of action sequences. Explicitly, $\gamma^{(n)}(x)$ consists of all pairs (w, y) such that $w \in A^n$ labels a path from x to y.

We now recover this graded semantics within the graded fibrational framework. Let

$\mathbb{M}_{\mathbb{N}} := \mathbb{M}_{\mathcal{P}_\omega}(A)$ and consider the corresponding graded fibration

$$\begin{array}{ccc} (\mathbb{M}_{\mathcal{P}_\omega}(A)U)^*\mathbf{Rel}^{\mathbb{N}} & \longrightarrow & \mathbf{Rel}^{\mathbb{N}} \\ {\scriptstyle (\mathbb{M}_{\mathcal{P}_\omega}(A)U)^*p^{\mathbb{N}}}\downarrow & \lrcorner & \downarrow{\scriptstyle p^{\mathbb{N}}} \\ \mathbf{Coalg}(G) & \xrightarrow[\mathbb{M}_{\mathcal{P}_\omega}(A)U]{} & \mathbf{Set}^{\mathbb{N}}, \end{array}$$

An object of $(\mathbb{M}_{\mathcal{P}_\omega}(A)U)^*\mathbf{Rel}^{\mathbb{N}}$ is a pair $((X,\gamma),\{R'_n\}_{n\in\mathbb{N}})$ consists of a G-coalgebra (X,γ) together with a family of relations $R'_n \subseteq M_nX \times M_nX$ for $n \in \mathbb{N}$ on the semantic domains at each grade.

We reindex along the semantic map

$$\tau \colon \Delta_{\mathbb{N}}U \to \mathbb{M}_{\mathcal{P}_\omega}(A)U,$$

whose component at a coalgebra $\gamma \colon X \to GX$ is given by the family defined by

$$\tau_\gamma^{(n)} = \gamma^{(n)} \colon X \to M_nX.$$

Reindexing along τ therefore transforms a relation $R'_n \subseteq M_nX \times M_nX$ on the semantic domain into a relation $R_n \subseteq X \times X$ on states, obtained by pulling back R'_n along $\tau_\gamma^{(n)} \times \tau_\gamma^{(n)}$:

$$\begin{array}{ccc} R_n & \longrightarrow & R'_n \\ \downarrow & \lrcorner & \downarrow \\ X \times X & \xrightarrow[\tau_\gamma^{(n)} \times \tau_\gamma^{(n)}]{} & M_nX \times M_nX. \end{array}$$

Thus the reindexed graded fibration $(\mathbb{M}_{\mathcal{P}_\omega}(A)U)^\tau\mathbf{Rel}^{\mathbb{N}} \to \mathbf{Coalg}(G)$ provides a graded family of relations on states, where the relation at grade n compares their n-step semantic observations.

To recover trace equivalence, we again choose the diagonal relation on the observation domain M_n1. Together with the previous fibrational construction, we have the following result.

Proposition 6.3. *Consider the reindexed graded fibration $(\mathbb{M}_{\mathcal{P}_\omega}(A)U)^\tau\mathbf{Rel}^{\mathbb{N}} \to \mathbf{Coalg}(G)$ constructed above. For each grade $n \in \mathbb{N}$, choose the equivalence on semantic domain as*

$$R'_n = (M_n!_X \times M_n!_X)^\bullet \Delta_{M_n1},$$

where $\Delta_{M_n 1}$ is the diagonal relation at the observation domain $M_n 1$. Then the corresponding relations on states is

$$R_n = (\tau_\gamma^{(n)} \times \tau_\gamma^{(n)})^\bullet R'_n = (\tau_\gamma^{(n)} \times \tau_\gamma^{(n)})^\bullet (M_n!_X \times M_n!_X)^\bullet \Delta_{M_n 1}.$$

For the graded semantics $(\mathrm{id}, \mathbb{M}_{\mathcal{P}_\omega}(A))$, the following hold:

1. *for every $n \in \mathbb{N}$, R_n is precisely the depth-n graded behavioural equivalence induced by $(\mathrm{id}, \mathbb{M}_{\mathcal{P}_\omega}(A))$;*

2. *the fibred functor $(\widehat{\mathbb{M}_{\mathcal{P}_\omega}(A)U})^\tau \colon \Gamma^*\mathbf{Rel} \to (\mathbb{M}_{\mathcal{P}_\omega}(A)U)^\tau\mathbf{Rel}^{\mathbb{N}}$ admits a fibred right adjoint $\forall g.-$, and the finite-depth graded behavioural equivalence is recovered as $R_\omega := \forall g.(\{R_n\}_{n\in\mathbb{N}})$.*

Thus the reindexed graded fibration recovers the finite-depth graded semantics specified by $(\mathrm{id}, \mathbb{M}_{\mathcal{P}_\omega}(A))$ for $G = \mathcal{P}_\omega(A \times -)$.

Proof. For (1), it follows directly from Theorem 5.20 by taking $\mathbb{M}_{\mathbb{N}} = \mathbb{M}_{\mathcal{P}_\omega}(A)$ and $\alpha = \mathrm{id}$. Indeed, under this specialization, the observation relation at grade n is $R'_n = (M_n!_X \times M_n!_X)^\bullet \Delta_{M_n 1}$, and the corresponding relation on X is $R_n = (\tau_\gamma^{(n)} \times \tau_\gamma^{(n)})^\bullet R'_n$. Hence R_n is precisely the depth-n graded behavioural equivalence induced by $(\mathrm{id}, \mathbb{M}_{\mathcal{P}_\omega}(A))$.

For (2), since $p \colon \mathbf{Rel} \to \mathbf{Set}$ is a $\mathbf{CLat}_\sqcap$-fibration, Proposition 4.7 implies that $(\widehat{\mathbb{M}_{\mathcal{P}_\omega}(A)U})^\tau \colon \Gamma^*\mathbf{Rel} \to (\mathbb{M}_{\mathcal{P}_\omega}(A)U)^\tau\mathbf{Rel}^{\mathbb{N}}$ admits a fibred right adjoint $\forall g.-$. Applying this right adjoint to the graded family $\{R_n\}_{n\in\mathbb{N}}$ yields

$$R_\omega := \forall g.(\{R_n\}_{n\in\mathbb{N}}) = \bigcap_{n\in\mathbb{N}} R_n,$$

given by fibrewise intersections. By (1), each R_n is the depth-n graded behavioural equivalence induced by $(\mathrm{id}, \mathbb{M}_{\mathcal{P}_\omega}(A))$; hence R_ω is precisely the finite-depth graded behavioural equivalence. □

Infinite-depth behavioural equivalence. We now give a fibrational account of the infinite-depth equivalence induced by the graded trace semantics based on the pre-determinization construction. For trace semantics, however, the pre-determinization requires the underlying transition system to be *serial*, so that the assumption $M_0 1 \cong 1$ is satisfied. We therefore consider the graded monad $M_n = \mathcal{P}^+_\omega(A^n \times -)$, with $\alpha = \mathrm{id}$, where $\mathcal{P}^+_\omega$ denotes the non-empty finitary powerset functor.

Since A is finite, Example 3.5.2 gives $\overline{M}_1 = (-)^A$. Consequently, for a coalgebra $\gamma\colon X \to \mathcal{P}^+_\omega(A \times X)$, the pre-determinization yields a coalgebra

$$\gamma^\sharp\colon \mathcal{P}^+_\omega(X) \to \mathcal{P}^+_\omega(X)^A$$

on the corresponding M_0-algebra, with $\gamma^\sharp(S)(a) = \bigcup_{x\in S} \gamma(x)(a)$. Thus, the pre-determinized system collects, for each action a, all a-successors of states contained in a given subset S.

The infinite-depth behavioural equivalence can consequently be characterized as the greatest bisimulation of this pre-determinized coalgebra. The following proposition makes this characterization explicit and shows that it recovers the expected trace-language semantics.

Proposition 6.4. *The infinite-depth $(\mathrm{id}, \mathbb{M}_{\mathcal{P}^+_\omega}(A))$-behavioural equivalence is the greatest fixed point of the bisimulation operator*

$$\Phi := (\gamma^\sharp \times \gamma^\sharp)^\bullet \overline{(-)^A},$$

where $\overline{(-)^A}$ denotes the relation lifting of $\overline{M}_1 = (-)^A$ as defined in Definition 5.7.

Proof. First, the endofunctor $(-)^A\colon \mathbf{Set} \to \mathbf{Set}$ preserves weak pullbacks. Indeed, $(-)^A$ is right adjoint to $- \times A$ and therefore preserves all limits, in particular pullbacks. Hence it preserves weak pullbacks. We may therefore apply Proposition 5.19 to the pre-determinized coalgebra with $\overline{M}_1 = (-)^A$. □

We provide a pointwise description of the infinite-depth $(\mathrm{id}, \mathbb{M}_{\mathcal{P}^+_\omega}(A))$-behavioural equivalence. For any relation $R \subseteq X \times X$,

$$\overline{(-)^A}R = \{(f, g) \in X^A \times X^A \mid \forall a \in A.\ (f(a), g(a)) \in R\}.$$

Indeed, an element of R^A is an A-indexed family $\{(x_a, y_a)\}_{a\in A}$ with $(x_a, y_a) \in R$ for every $a \in A$, and the two projections map it to the corresponding pair of functions in X^A. Therefore, for every relation $R \subseteq X \times X$,

$$\begin{aligned}(x, y) \in \Phi(R) &\iff (\gamma^\sharp(x), \gamma^\sharp(y)) \in \overline{(-)^A}R \\ &\iff \forall a \in A.\ (\gamma^\sharp(x)(a), \gamma^\sharp(y)(a)) \in R.\end{aligned}$$

Hence the infinite-depth $(\mathrm{id}, \mathbb{M}_{\mathcal{P}^+_\omega}(A))$-behavioural equivalence is the greatest relation $R \subseteq X \times X$ satisfying

$$\forall x, y \in X.\ (x, y) \in R \implies \forall a \in A.\ (\gamma^\sharp(x)(a), \gamma^\sharp(y)(a)) \in R.$$

6.3 Simulation Equivalence

Simulation equivalence is a branching-time behavioural equivalence obtained by requiring mutual simulation of two states. It can be modelled by a depth-1 graded equational theory with the same signature as for trace equivalence, but replacing join preservation by monotonicity; see [12, Section 4] for details.

In this section, we show that the graded fibrational framework recovers the corresponding finite-depth graded simulation semantics. In particular, the semantic maps at grade n encode the depth-n simulation trees, and reindexing the equivalence on the semantic domains yields the depth-n simulation equivalence.

Finite-depth behavioural equivalence. Recall from Example 3.3.3 that, the graded construction yields a family of ordered sets defined inductively by

$$M_0X = \mathcal{P}_\omega(X), \text{ and } M_{n+1}X = \mathcal{P}^{\downarrow}_\omega(A \times M_nX),$$

ordered by inclusion, where $\mathcal{P}^{\downarrow}_\omega$ denotes the set of finitely generated down-closed subsets. These sets characterize simulation trees and give rise to a graded monad $\mathbb{M}_{\leq\geq}$, with unit and multiplication induced by the concatenation of simulation trees. The semantic transformation is given by

$$\alpha_X \colon \mathcal{P}_\omega(A \times X) \to \mathcal{P}^{\downarrow}_\omega(A \times \mathcal{P}_\omega X);\ S \mapsto {\downarrow}\{(a, \{x\}) \mid (a, x) \in S\}, \tag{6.1}$$

where $\downarrow$ denotes downset.

We recover this graded semantics within a graded fibration by choosing $\mathbb{M}_{\mathbb{N}} = \mathbb{M}_{\leq\geq}$. The corresponding graded fibration can then be constructed as

$$\begin{array}{ccc} (\mathbb{M}_{\leq\geq}U)^*\mathbf{Rel}^{\mathbb{N}} & \longrightarrow & \mathbf{Rel}^{\mathbb{N}} \\ {\scriptstyle (\mathbb{M}_{\leq\geq}U)^*p^{\mathbb{N}}}\downarrow & \lrcorner & \downarrow{\scriptstyle p^{\mathbb{N}}} \\ \mathbf{Coalg}(G) & \xrightarrow[\mathbb{M}_{\leq\geq}U]{} & \mathbf{Set}^{\mathbb{N}}. \end{array}$$

An object of $(\mathbb{M}_{\leq\geq}U)^*\mathbf{Rel}^{\mathbb{N}}$ is a pair $((X, \gamma), \{R'_n\}_{n\in\mathbb{N}})$ consists of a G-coalgebra (X, γ) together with a family of equivalence relations $R'_n \subseteq M_nX \times M_nX$ for $n \in \mathbb{N}$ on the semantic domains at each grade.

The semantic map used to reindex this graded fibration is defined analogously to that in the trace equivalence case:

$$\tau \colon \Delta_{\mathbb{N}}U \to \mathbb{M}_{\leq\geq}U,$$

whose component at a coalgebra $\gamma\colon X \to GX$ is the family

$$\tau_\gamma^{(n)} = \gamma^{(n)}\colon X \to M_n X,$$

where the maps $\gamma^{(n)}$ are defined inductively by

$$\gamma^{(0)} = \eta_X, \text{ and } \gamma^{(n+1)} = \mu_X^{1,n} \circ M_1\gamma^{(n)} \circ (\alpha_X \circ \gamma).$$

We therefore build the reindexed graded fibration $(\mathbb{M}_{\leq\geq}U)^\tau\mathbf{Rel}^{\mathbb{N}} \to \mathbf{Coalg}(G)$ accordingly, and have the following result.

Proposition 6.5. *Consider the reindexed graded fibration $(\mathbb{M}_{\leq\geq}U)^\tau\mathbf{Rel}^{\mathbb{N}} \to \mathbf{Coalg}(G)$ constructed above. For each grade $n \in \mathbb{N}$, choose the equivalence on semantic domain as*

$$R'_n = (M_n!_X \times M_n!_X)^\bullet \Delta_{M_n 1},$$

where $\Delta_{M_n 1}$ is the diagonal relation at the observation domain $M_n 1$. Then the corresponding equivalence relations on states is

$$R_n = (\tau_\gamma^{(n)} \times \tau_\gamma^{(n)})^\bullet R'_n = (\tau_\gamma^{(n)} \times \tau_\gamma^{(n)})^\bullet (M_n!_X \times M_n!_X)^\bullet \Delta_{M_n 1}.$$

For the graded semantics $(\alpha, \mathbb{M}_{\leq\geq})$, the following hold:

1. *for every $n \in \mathbb{N}$, R_n is precisely the depth-n graded behavioural equivalence induced by $(\alpha, \mathbb{M}_{\leq\geq})$;*
2. *the fibred functor $(\widehat{\mathbb{M}_{\leq\geq}U})^\tau\colon \Gamma^*\mathbf{Rel} \to (\mathbb{M}_{\leq\geq}U)^\tau\mathbf{Rel}^{\mathbb{N}}$ admits a fibred right adjoint $\forall g.-$, and the finite-depth graded behavioural equivalence is recovered as $R_\omega := \forall g.(\{R_n\}_{n\in\mathbb{N}})$.*

Thus the reindexed graded fibration recovers the finite-depth graded semantics specified by $(\alpha, \mathbb{M}_{\leq\geq})$ for $G = \mathcal{P}_\omega(A \times -)$.

Proof. For (1), it follows directly from Theorem 5.20 by taking $\mathbb{M}_{\mathbb{N}} = \mathbb{M}_{\leq\geq}$ and the α given in Eq. (6.1). Indeed, under this specialization, the observation relation at grade n is $R'_n = (M_n!_X \times M_n!_X)^\bullet \Delta_{M_n 1}$, and the corresponding relation on X is $R_n = (\tau_\gamma^{(n)} \times \tau_\gamma^{(n)})^\bullet R'_n$. Hence R_n is precisely the depth-n graded behavioural equivalence induced by $(\alpha, \mathbb{M}_{\leq\geq})$.

For (2), since $p\colon \mathbf{Rel} \to \mathbf{Set}$ is a $\mathbf{CLat}_\sqcap$-fibration, Proposition 4.7 implies that $(\widehat{\mathbb{M}_{\leq\geq}U})^\tau\colon \Gamma^*\mathbf{Rel} \to (\mathbb{M}_{\leq\geq}U)^\tau\mathbf{Rel}^{\mathbb{N}}$ admits a fibred right adjoint $\forall g.-$. Applying this right adjoint to the graded family $\{R_n\}_{n\in\mathbb{N}}$ yields

$$R_\omega := \forall g.(\{R_n\}_{n\in\mathbb{N}}) = \bigcap_{n\in\mathbb{N}} R_n,$$

given by fibrewise intersections. By (1), each R_n is the depth-n graded behavioural equivalence induced by $(\alpha, \mathbb{M}_{\leq\geq})$; hence R_ω is precisely the finite-depth graded behavioural equivalence. □

Infinite-depth behavioural equivalence. In this paragraph, we show that the infinite-depth simulation equivalence induced by the graded semantics $(\alpha, \mathbb{M}_{\leq\geq})$ admits a greatest fixed-point characterization in the relation fibration $\mathbf{Rel}_{\mathbf{Alg}_0(\mathbb{M}_{\leq\geq})} \to \mathbf{Alg}_0(\mathbb{M}_{\leq\geq}) \times \mathbf{Alg}_0(\mathbb{M}_{\leq\geq})$.

Following Ford et al. [15, Example 5.3.(3)], in the pre-determinization, one has the induced functor $\overline{M}_1 = \mathcal{P}^{\downarrow}_\omega(A \times -)$, and the pre-determinization

$$\gamma^\sharp \colon (\mathcal{P}_\omega(X), \mu^{0,0}) \to (\mathcal{P}^{\downarrow}_\omega(A \times \mathcal{P}_\omega(X)), \mu^{0,1})$$

is given by

$$\gamma^\sharp(S) = {\downarrow}\{(a, \bigcup_{x \in S} \{y \mid (a, y) \in \gamma(x)\}) \mid a \in A\}.$$

The infinite-depth simulation equivalence can then be characterized as follows.

Proposition 6.6. *The infinite-depth $(\alpha, \mathbb{M}_{\leq\geq})$-behavioural equivalence is the greatest fixed point of the bisimulation operator*

$$\Phi := (\gamma^\sharp \times \gamma^\sharp)^\bullet \overline{\overline{M}}_1,$$

where $\overline{\overline{M}}_1$ denotes the relation lifting of $\overline{M}_1 = \mathcal{P}^{\downarrow}_\omega(A \times -)$, sending a relation $R \subseteq X \times X$ between semilattices to the set

$$\{(U, V) \mid (\forall (a, x) \in U.\, \exists (a, y) \in V.\, xRy) \wedge (\forall (a, y) \in V.\, \exists (a, x) \in U.\, xRy)\}.$$

Proof. Assume A is equipped with the discrete order. We show that $\mathcal{P}^{\downarrow}_\omega(A \times -)$ preserves weak pullbacks. Let

$$\begin{array}{ccc} P & \xrightarrow{p_2} & Y \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

be a weak pullback in **Set**. Let $U \in \mathcal{P}^{\downarrow}_\omega(A \times X), V \in \mathcal{P}^{\downarrow}_\omega(A \times Y)$ satisfy

$$\mathcal{P}^{\downarrow}_\omega(A \times f)(U) = \mathcal{P}^{\downarrow}_\omega(A \times g)(V).$$

Choose finite generating sets $S \subseteq A \times X, T \subseteq A \times Y$ such that $U = {\downarrow} S, V = {\downarrow} T$. For each pair of generators $(a, x) \in S, (a, y) \in T$ having the same image in $A \times Z$, the weak-pullback property yields $w \in P$ such that $p_1(w) = x, p_2(w) = y$. Let W be the downset generated by all corresponding elements (a, w). Since S and T are finite, W is finitely generated. By construction,

$$\mathcal{P}_\omega^{\downarrow}(A \times p_1)(W) = U, \quad \mathcal{P}_\omega^{\downarrow}(A \times p_2)(W) = V.$$

Thus the induced square

$$\begin{array}{ccc} \mathcal{P}_\omega^{\downarrow}(A \times P) & \longrightarrow & \mathcal{P}_\omega^{\downarrow}(A \times Y) \\ \downarrow & & \downarrow \\ \mathcal{P}_\omega^{\downarrow}(A \times X) & \longrightarrow & \mathcal{P}_\omega^{\downarrow}(A \times Z) \end{array}$$

is a weak pullback, and $\mathcal{P}_\omega^{\downarrow}(A \times -)$ preserves weak pullbacks. The claim then follows from Proposition 5.19 by taking $\overline{M}_1 = \mathcal{P}_\omega^{\downarrow}(A \times -)$. □

We provide a pointwise description of the infinite-depth $(\alpha, \mathbb{M}_{\leq\geq})$-behavioural equivalence. Let $U, V \in \mathcal{P}_\omega^{\downarrow}(A \times \mathcal{P}_\omega X)$. Then for a relation $R \subseteq \mathcal{P}_\omega X \times \mathcal{P}_\omega X$,

$$(U, V) \in \overline{\mathcal{P}_\omega^{\downarrow}(A \times -)}\, R$$

if and only if there exists a finitely generated downset $W \in \mathcal{P}_\omega^{\downarrow}(A \times R)$ such that $\mathcal{P}_\omega^{\downarrow}(A \times \pi_1)(W) = U$ and $\mathcal{P}_\omega^{\downarrow}(A \times \pi_2)(W) = V$. In particular, the relation lifting $\overline{\mathcal{P}_\omega^{\downarrow}(A \times -)}$ takes R to the set

$$\left\{ (U, V) \;\middle|\; \begin{array}{l} \forall (a, U') \in U.\ \exists (a, V') \in V.\ (U', V') \in R, \\ \forall (a, V') \in V.\ \exists (a, U') \in U.\ (U', V') \in R \end{array} \right\}.$$

Hence the infinite-depth $(\alpha, \mathbb{M}_{\leq\geq})$-behavioural equivalence is the greatest relation R satisfying

$$(U, V) \in R \Longrightarrow \begin{cases} \forall (a, U') \in \gamma^\sharp(U).\ \exists (a, V') \in \gamma^\sharp(V).\ (U', V') \in R, \\ \forall (a, V') \in \gamma^\sharp(V).\ \exists (a, U') \in \gamma^\sharp(U) \wedge (U', V') \in R \end{cases}$$

7 Conclusion

The research question posed in Section 1 can be answered affirmatively: graded behavioural equivalence induced by graded semantics admits a fibrational account. We developed such an account based on the notion of graded fibrations. Starting from the observation that behavioural equivalences in state-based systems vary both in system structure and in observational granularity, we introduced graded fibrations as a categorical setting in which these two dimensions can be integrated. We studied their basic categorical properties, including their behaviour under reindexing and the interaction between graded structure and fibred logical structure.

Building on this foundation, we showed how graded fibrations provide a natural environment for modelling graded semantics arising from graded monads. In particular, we demonstrated how the graded behavioural equivalences induced by such semantics can be recovered within the fibrational setting through suitable liftings. This connects the algebraic notion of grading with fibrational approaches to coinduction and behavioural relations. Thus, the framework provides a uniform account in which observational depth is reflected by the graded structure of the fibration, while behavioural comparison is expressed through its fibred structure.

There are several directions for future work. A first direction is to consider richer notions of behavioural comparison within the graded fibrational framework, including behavioural preorders [14], distances [20], and topologies [34]. This would emphasize the power of fibrations in capturing diverse comparison structures on state spaces.

A second direction is to extend the framework to coalgebraic systems equipped with additional quantitative structure, such as costs, rewards, weights [3, 6, 25], or probabilistic information [30, 49]. This would allow graded observational depth to be combined with quantitative aspects of behaviour, potentially providing a unified framework for quantitative and graded behavioural semantics.

A third direction is to generalize the grading structure underlying the framework. In particular, rather than restricting grades to natural numbers, one could consider graded monads indexed by arbitrary monoids. This could also encompass graded coalgebraic systems arising from settings such as timed systems [11], where the grade may represent an elapsed time or other resource.

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