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Induction of Groupoid Representations via Kan Extensions

Bachelor thesis

26 April 1564

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Contents

C1 | Introduction

Representation theory is a branch of mathematics that aims to study abstract groups by representing them as a collection of linear isomorphisms on a vector space compatible with the group structure. Concretely, a representation of a group G is a group homomorphism $\rho\colon G \to \mathrm{Aut}(V)$ where V is a vector space. One of the main benefits of studying representations is being able to use results from linear algebra to help tackle problems in abstract algebra.

At the end of the nineteenth century, early pioneers such as Ferdinand Frobenius, William Burnside, Issai Schur, and Richard Brauer started research in the representation theory of finite groups. Though initially met with some opposition from the wider mathematical community, early results such as Burnside's proof of the $p^a q^b$-theorem in 1904 led to wider acceptance of the theory and spurred further research [2].

Studying the representations of a particularly large group G directly might prove difficult or computationally expensive. A way of simplifying the problem is by looking at representations of a subgroup $H \subset G$, which might be easier to understand and more efficient to perform computations on. A representation ρ of G naturally restricts to a representation $\rho|_H$ of H. In order to fully understand the structure of representation of G however, one might also want to construct a G-representation out of an H-representation. This can be achieved via *induction* and *coinduction* of representations. The relation between representations of G and representations of H was studied by Frobenius as early as 1898, though initially formulated in the language of character theory [4]. He proved a theorem relating the restriction and induction of representations, now known as Frobenius reciprocity.

A group is sometimes described as the symmetries of an object. A groupoid generalizes this concept by allowing multiple objects. A groupoid is a category in which every morphism is invertible. An important example comes from topology, namely the fundamental groupoid. Recall that the fundamental group $\pi_1(X, x_0)$ of a topological space X captures the structure of homotopy classes of loops with a fixed basepoint $x_0 \in X$. An extension of this is the fundamental groupoid $\Pi_1(X)$, whose objects are points in X and whose morphisms are homotopy classes of paths between them. The fundamental groupoid carries more information than the collection of fundamental groups as it also accounts for transformations between different points, not just transformations of individual points.

Just as the definition of a group naturally extends to the definition of a groupoid, so does the definition of a group representation naturally extend to the definition of a groupoid representation. A representation of a groupoid assigns to all objects of the groupoid a vector space. All morphisms between two particular objects act as linear transformations between the corresponding vector spaces in a way that respects composition and identities. Concretely, a representation of a groupoid $\mathcal{G}$ is a functor $T\colon \mathcal{G} \to \mathbf{Vec}$.

In this thesis, we explore how induction and coinduction of groupoid representations fit into a broader category-theoretic framework of adjoint functors. By using Frobenius reciprocity as a defining property of induced representations, rather than a result of any particular construction, we obtain a more generally applicable definition that agrees with the old one. One of the benefits of this approach is that many results about induced representations, such as commutativity with direct sums, induction in stages, and even the projection formula, are now direct corollaries of properties of adjoint functors. For the construction of adjoint functors we make use of Kan extensions, a well-known general construction in category theory.

The main goal of this thesis is to give an explicit description of induced and coinduced representations of groupoids in the more general setting of adjoint functors. We formulate criteria under which coinduction is finite-dimensional. Further, we study how induction and coinduction interact with the structure of the category of representations of a groupoid. Ultimately, we prove a slight generalization of a theorem known as the projection formula.

In chapter 2 we briefly review some standard results from category theory that we need throughout the rest of the thesis. We recall the definitions of limits and colimits and that of adjoint functors and how these two concepts are related. We further go over some of the general properties of adjoint functors that we will later apply to induced representations such as the commutativity with limits.

Before turning to representations of groupoids, we shall briefly review several basic notions and results from the representation theory of groups in chapter 3. Aside from introducing notation and terminology, this chapter serves two main purposes. First, part of the material covered here will later be used to draw comparisons with the more general theory of groupoid representations. Second, some results will be used as motivation for later definitions.

We motivate the study of groupoids in chapter 4 by taking a closer look at the fundamental groupoid of a topological space. We then briefly go over the basic structure of groupoids and the morphisms between them. We end the chapter by discussing subgroupoids and cosets which are natural generalisations of subgroups and cosets of subgroups.

Afterwards, we can finally turn to groupoid representations. In chapter 5, we focus on individual representations and their structure. We further discuss some constructions such as the direct sum and the tensor product of fixed representations and how these are straightforward generalizations of constructions for group representations.

The constructions from chapter 5 play an important role in the structure of $\mathbf{Rep}_{\mathcal{G}}$, the category of representations of a groupoid $\mathcal{G}$. In chapter 6 we discuss the structure of $\mathbf{Rep}_{\mathcal{G}}$. First, we see how the direct sum construction extends to a biproduct in $\mathbf{Rep}_{\mathcal{G}}$, making $\mathbf{Rep}_{\mathcal{G}}$ an abelian category. Further, the tensor product extends to a tensor product on the whole of $\mathbf{Rep}_{\mathcal{G}}$, which gives $\mathbf{Rep}_{\mathcal{G}}$ a monoidal structure. Indeed, this monoidal structure interacts with the *internal hom* of $\mathbf{Rep}_{\mathcal{G}}$ making the category monoidally closed.

At last, we can define induction of representations as an adjoint to restriction of representations. We start chapter 7 by applying the theory of Kan extensions to obtain explicit constructions of induced representations. We then verify that this construction coincides with the original construction of induced group representations. Subsequently, we provide criteria under which an induced representation is finite-dimensional. We end the chapter by proving a version of the projection formula which relates induction of representations with the tensor product of representations. We achieve this by combining the properties of adjoints with the the closed symmetric monoidal structure discussed in chapter 6.

Finally, chapter 8 outlines some possible future research directions.

1.1 Prerequisites

We shall assume some familiarity with basic category theory. Although Chapter 2 covers some elementary notions needed throughout the rest of this thesis, the reader is at the very least expected to know what categories and functors are and to be familiar with common categories such as **Vec**, **Set**, and **Ab**. Most of the category theory in this thesis is based on *Categories for the Working Mathematician* by Mac Lane [6], which serves as an excellent source for an introduction to category theory.

We further recommend some background knowledge on the representation theory of groups, though this is not strictly required. As groupoids may be understood as "groups with many objects", familiarity with group representations can help in understanding and motivating certain constructions for groupoids. For instance, the tensor product of groupoid representations is a natural extension of the tensor product of group representations. On the other hand, viewing induction as an adjoint functor as opposed to a certain construction is best appreciated when one is familiar with the old construction.

1.2 Notation

Finally, we briefly review the notation used in this thesis. Categories are written either in a calligraphic font or in boldface. Given a small category $\mathcal{C}$, we write $\mathrm{ob}(\mathcal{C})$ for the set of objects and $\mathrm{mor}(\mathcal{C})$ for the set of all morphisms. Given any two objects $x, y \in \mathrm{ob}(\mathcal{C})$, we write $\mathcal{C}(x, y)$ for the set of morphisms from x to y. We denote by $\mathcal{C}^{\mathrm{op}}$ the opposite category of $\mathcal{C}$. Write $s, t \colon \mathrm{mor}(\mathcal{C}) \to \mathrm{ob}(\mathcal{C})$ for the functions that assign to each morphism its source and target object respectively. Given a fixed field k, we write **Vec** for the category of k-vector spaces and **FinVec** for the subcategory of finite-dimensional k-vector spaces. We omit the dependence on k in the notation as the particular choice of field will play no role in this thesis.

C2 | Category Theory Preliminaries

In this chapter we recall some standard definitions and results from category theory which we will need in later chapters of this thesis. Of particular importance are the definitions of comma categories, limits and colimits, and adjoint functors, as these will play an essential role in defining induction and coinduction of representations in Chapter 7. We also go over some basic properties of adjoint functors that we will later apply to induced and coinduced representations.

This chapter is primarily based on *Categories for the Working Mathematician* [6]. Readers familiar with basic category theory may safely skip this chapter and refer back when needed.

2.1 General Category Theory

Although familiarity with categories and functors is assumed, we shall briefly recall the notion of natural transformations as these will be used extensively throughout the rest of the thesis.

Definition 2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and Let $F, G\colon \mathcal{C} \to \mathcal{D}$ functors. A **natural transformation** $\alpha\colon F \Rightarrow G$ is a collection of morphisms $\{\alpha_x\colon F(x) \to G(x)\}_{x \in \operatorname{ob}\mathcal{C}}$ such that for each morphism $f\colon x \to y$ in $\mathcal{C}$, the following diagram commutes.

$$\begin{tikzcd} F(x) \arrow[r, "\alpha_x"] \arrow[d, "F(f)"] & G(x) \arrow[d, "G(f)"] \\ F(y) \arrow[r, "\alpha_y"] & G(y) \end{tikzcd}$$

A **natural isomorphism** is a natural transformation whose components are all isomorphisms in $\mathcal{D}$.

One of the first examples of a natural transformation one might encounter is the relation between a vector space and its double dual. Recall that given a k-vector space V, its dual is defined by $V^* := \mathbf{Vec}(V, k)$ and any linear map $f\colon V \to W$ induces a map $f^*\colon W^* \to V^*$ given by $\varphi \mapsto \varphi \circ f$. Indeed, this construction extends to a contravariant functor $(-)^*\colon \mathbf{Vec} \to \mathbf{Vec}$. For finite-dimensional vector spaces, a basic dimension counting argument shows that V^* is isomorphic to V. The problem is that writing down such an isomorphism requires some arbitrary choice of basis of V. This makes the collection of isomorphisms between vector spaces and their duals **unnatural**. Contrast this to the double dual of a vector space. For a vector space V we define the *evaluation map* $\mathrm{ev}_V\colon V \to V^{**}$ by

$$\mathrm{ev}_V(v)(\varphi) = \varphi(v).$$

Intuitively, given $v \in V$ and $\varphi \in V^*$ there is only one logical way to produce an element in k: evaluating φ in v. As this construction does not depend on any arbitrary choices we might expect it to be natural. Indeed, given $f\colon V \to W$, a straightforward computation shows that the diagram below commutes.

$$\begin{tikzcd} V \arrow[r, "\mathrm{ev}_V"] \arrow[d, "f"] & V^{**} \arrow[d, "f^{**}"] \\ W \arrow[r, "\mathrm{ev}_W"] & W^{**} \end{tikzcd}$$

We have thus found a natural transformation $\mathrm{ev}\colon 1_{\mathbf{Vec}} \Rightarrow (-)^{**}$, where $1_{\mathbf{Vec}}$ denotes the identity functor on **Vec**. This natural transformation is a natural isomorphism when restricting to just finite-dimensional vector spaces.

2.2 Comma Categories

Later on we will properly define what it means to induce a representation to a category $\mathcal{G}$ from a subcategory $\mathcal{H} \subset \mathcal{G}$. Intuitively, an induced representation should be 'maximally free' whilst still respecting the relations from $\mathcal{H}$. We can formalize this idea of respecting existing relations using **comma categories**.

Definition 2.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between categories and let $d \in \operatorname{ob}\mathcal{D}$. Define the category of **objects of F under d**, denoted by $(d \downarrow F)$, as the category with objects $\langle f, y\rangle$ for $y \in \operatorname{ob}\mathcal{C}$ and $f \in \mathcal{D}(d, F(y))$ and morphisms $\langle h\colon f \to f'\rangle$ for $\langle f, y\rangle, \langle f', y'\rangle \in \operatorname{ob}(d \downarrow F)$ and $h \in \mathcal{C}(y, y')$ satisfying $F(h) \circ f = f'$. Diagrammatically, this category may be written as follows.

$$\text{Objects:} \quad \begin{tikzcd} d \arrow[d, "f"] \\ F(y) \end{tikzcd} \qquad \text{Morphisms:} \quad \begin{tikzcd} & d \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ F(y) \arrow[rr, "F(h)"'] & & F(y') \end{tikzcd}$$

To ease notation, we shall sometimes simply write $f \in (d \downarrow F)$ instead of $\langle f, y\rangle \in (d \downarrow F)$ whenever the codomain of f is not relevant or clear from context.

To demonstrate how comma categories formalize the notion of respecting existing relations, consider the following example. Suppose G is a group with subgroup $H \subset G$. We can **categorify** H and G by defining categories $\mathcal{H}$ and $\mathcal{G}$ with a single object '$*$' and hom-sets $\mathcal{H}(*, *) = H$ and $\mathcal{G}(*, *) = G$. The inclusion morphism $i\colon H \to G$ turns into an inclusion functor $I\colon \mathcal{H} \to \mathcal{G}$.

We find that the category $(* \downarrow I)$ consists of objects $\langle f, *\rangle$ for each $f \in G$, and morphisms $\langle h\colon f \to f'\rangle$ with $h \in H$ for which $hf = f'$. Thus, two elements f and f' are connected as objects of $(* \downarrow I)$ precisely when they lie in the same right coset of G. If G is finite, the category $(* \downarrow I)$ consists of precisely $[G : H]$ connected components, all of whom have $\#H$ objects.

Note that there is a natural projection functor $Q\colon (d \downarrow F) \to \mathcal{C}$, given by

$$\begin{aligned} Q\colon (d \downarrow F) &\to \mathcal{C}, \\ \langle f, y\rangle &\mapsto y, \\ \big(h\colon \langle f, y\rangle \to \langle f', y'\rangle\big) &\mapsto (h\colon y \to y'). \end{aligned}$$

Definition 2.3. Dually, define the category of **objects of F over c**, denoted $(F \downarrow c)$, by the following diagrams.

$$\text{Objects:} \quad \begin{tikzcd} F(y) \arrow[d, "f"] \\ c \end{tikzcd} \qquad \text{Morphisms:} \quad \begin{tikzcd} F(y) \arrow[rr, "F(h)"] \arrow[dr, "f"'] & & F(y') \arrow[dl, "f'"] \\ & d & \end{tikzcd}$$

One can verify that for categorified groups $\mathcal{H} \subset \mathcal{G}$, two morphisms $f, f' \in \mathcal{G}$ lie in the same connected component of $(I \downarrow *)$ precisely when they lie in the same left coset of G.

2.3 Cones and Limits

Many important constructions in mathematics can be described via universal properties. Take for example the direct product of vector spaces, usually defined as

$$V \times W := \{(v, w) : v \in V \text{ and } w \in W\}.$$

Category theory provides an alternative way of defining the direct product via a *universal property*. Concretely, given vector spaces V and W, their direct product is a vector space $V \times W$ along with projection morphisms $\pi_V \colon V \times W \to V$ and $\pi_W \colon V \times W \to W$ for which the following universal property holds: given any other vector space U with projection maps $p_V \colon U \to V$ and $p_W \colon U \to W$, there exists a unique map $u \colon U \to V \times W$ satisfying $\pi_V \circ u = p_V$ and $\pi_W \circ u = p_W$. Diagrammatically, this universal property may be described as follows.

$$\begin{tikzcd}
& U \arrow[dl, "p_V"'] \arrow[d, dashed, "\exists! u"] \arrow[dr, "p_W"] & \\
V & V \times W \arrow[l, "\pi_V"] \arrow[r, "\pi_W"'] & W
\end{tikzcd}$$

Though the existence of the direct product is not given, the universal property does guarantee that this object, if it exists, is unique up to unique isomorphism. Because of this, we still speak of *the* direct product. One can verify that the explicit construction of $V \times W$ as given above along with the obvious projection maps does satisfy the universal property.

There are several benefits of using universal properties over explicit constructions. For one, these definitions are very general and provide a way to speak of the product in an arbitrary category. Proving a certain property of the direct product is then directly applicable to many situations. Further, many proofs which would normally be done by explicit calculations can instead be proven more abstractly via the universal properties.

We can generalize the idea of universal properties quite a lot further. Instead of picking just two objects V and W with no relations, pick an arbitrary collection of objects and morphisms between them. We can concretely view this as as a functor $D \colon J \to \mathcal{C}$, where J is a small index category and $\mathcal{C}$ is an arbitrary category. Such a functor is called a **diagram**. Define a **cone of D** to be a tuple $(N, \{\pi_j\}_{j \in J})$ consisting of an object N and morphisms $\pi_j \colon N \to D(j)$ for each j in J such that for each morphism $f \colon j \to k$ in J we have $D(f) \circ \pi_j = \pi_k$.

Definition 2.4. Let $D \colon J \to \mathcal{C}$ be a functor between a small index category J and an arbitrary category $\mathcal{C}$. A **limit** of D is a cone $(L, \{\pi_j\}_j)$ satisfying the following universal property: for any other cone $(N, \{\tau_j\}_j)$ there exists a unique morphism $u \colon N \to L$ such that $\pi_j \colon u = \tau_j$ for all $j \in J$. Diagrammatically, the following commutes for all morphisms $f \colon j \to k$ in J.

$$\begin{tikzcd}
& N \arrow[ddl, bend right, "\tau_j"'] \arrow[d, dashed, "\exists! u"] \arrow[ddr, bend left, "\tau_k"] & \\
& L \arrow[dl, "\pi_j"'] \arrow[dr, "\pi_k"] & \\
D(j) \arrow[rr, "D(f)"] & & D(k)
\end{tikzcd}$$

The cone $(L, \{\pi_j\}_j)$ is called the **limiting cone** or the universal cone and L the **limiting object** or the inverse limit. This object is denoted by $L = \varprojlim D(j)$.

Again, a limit need not exist but if it does, the universal property ensures that it is unique up to unique isomorphism. Hence, we speak of *the* limit. Dually, given a diagram $D \colon J \to \mathcal{C}$ we may define a **cocone** to be a tuple $(N, \{\varphi_j\}_{j \in J})$ consisting of an object $N \in \mathcal{C}$ and a collection of morphisms $\varphi_j \colon D(j) \to N$ satisfying $\varphi_k \circ D(f) = \varphi_j$ for each morphism $f \colon j \to k$ in J.

Definition 2.5. Let $D\colon J \to \mathcal{C}$ be a diagram. A **colimit** of D is a cocone $(L, \{\varphi_j\}_j)$ satisfying the following universal property: for any other cocone $(N, \{\psi_j\}_j)$ of D, there exists a unique morphism $u\colon L \to N$ such that $u \circ \varphi_j = \psi_j$ for each $j \in J$. This can also be expressed by the following diagram.

$$\begin{tikzcd}
D(j) \arrow[rr, "D(f)"] \arrow[dr, "\varphi_j"] \arrow[ddr, bend right, "\psi_j"'] & & D(k) \arrow[dl, "\varphi_k"'] \arrow[ddl, bend left, "\psi_k"] \\
& L \arrow[d, dashed, "\exists! u"'] & \\
& N &
\end{tikzcd}$$

We call $(L, \{\varphi_j\}_j)$ the **limiting cocone** and L the **limiting object**. It is denoted by $L = \varinjlim D(j)$.

As alluded to before, many important constructions in mathematics can be given as limits and colimits. We already gave the example of the direct product being a limit over the diagram consisting of two points with no morphisms between them. For another example, consider the diagram $D\colon \{\bullet \rightrightarrows \bullet\} \to \mathbf{Vec}$ whose image is

$$\begin{tikzcd}
V \arrow[r, bend left, "f"] \arrow[r, bend right, "\mathbf{0}_{V,W}"'] & W
\end{tikzcd},$$

where $\mathbf{0}_{V,W}\colon V \to W$ is the zero map. We can define the kernel of f, denoted $\ker(f)$, to be the limit of this diagram. One can verify that the usual definition of the kernel, namely $\ker(f) = \{v \in V : f(v) = 0\}$, satisfies the universal property.

2.4 Adjoint Functors

A commonly used fact in linear algebra is the following. If V and W are vector spaces and B is a basis of V, then any linear function $f\colon V \to W$ is determined precisely by the image of $f(B)$. This fact can be restated as

$$\mathbf{Vec}(V, W) \cong_{\mathbf{Set}} \mathbf{Set}(B, W).$$

Effectively this gives us a way to relate the categories **Set** and **Vec** with one another and translate certain problems over. More formally, we may view this statement as an **adjunction** between the between the forgetful functor $\mathbf{Vec} \to \mathbf{Set}$ and the free vector space functor $\mathbf{Set} \to \mathbf{Vec}$. In some sense, the free vector space functor is the "best possible" way of going back to the category of vector spaces after forgetting their structure. **Adjoint functors** aim to formalize what it means to be the "best possible" way of going back.

Definition 2.6. An adjunction between the categories $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors $F\colon \mathcal{D} \to \mathcal{C}$ and $G\colon \mathcal{C} \to \mathcal{D}$ along with a family of bijections

$$\mathcal{C}(Fd, c) \cong_{\mathbf{Set}} \mathcal{D}(d, Gc)$$

natural in $c \in \mathcal{C}$ and $d \in \mathcal{D}$[a]. The functors F and G are called **adjoint** with F being the **left adjoint** and G the **right adjoint** as they appear respectively on the left and right side of the bijective hom-sets. We denote this adjunction as $F \dashv G$.

As alluded to before, the free vector space functor is a left adjoint of the forgetful functor. Of course we expect there to be only one "best possible" way of going back, meaning we expect the free vector space functor to be the only left adjoint. This is indeed the case.

[a] This definition only works if $\mathcal{C}$ and $\mathcal{D}$ are locally small. There exists a generalisation for general categories.

Theorem 2.7. Let $F, F' : \mathcal{D} \to \mathcal{C}$ and $G, G' : \mathcal{C} \to \mathcal{D}$ be functors and suppose $F \dashv G$. If F' is also a left adjoint of G, then $F \cong F'$. Similarly, if G' is also a right adjoint of F, then $G \cong G'$.

We omit the proof of this theorem as it makes use of the Yoneda lemma and is not particularly relevant for this thesis. Interested readers may find a full proof in Chapter IV of [6].

Perhaps the most important property of adjoints is their interplay with limits and colimits. Let us continue with the setup from Definition 2.6. Intuitively, a right adjoint G picks out, for each $c \in \mathcal{C}$, some object $Gc \in \mathcal{D}$ which can be mapped into in a uniquely best-possible way, but this is precisely what is captured by a limit. Indeed, let $(L, \{\pi_j\}_j)$ be a limiting cone of some diagram $D : J \to \mathcal{C}$. By applying G we obtain a cone $(G(L), \{G(\pi_j)\}_j)$ in $\mathcal{D}$ of the diagram $G \circ D$. Now suppose we are given another cone $(N, \{\tau_j\}_j)$ in $\mathcal{D}$. By applying the properties of adjoints we obtain a cone $(F(N), \{\overline{\tau_j}\}_j)$ in $\mathcal{C}$. By the universal property, there is a unique map $\overline{u} : F(N) \to L$ for which the diagram commutes. By applying the properties of adjoints once more, we find that there exists a unique map $u : N \to G(L)$ for which the required diagrams commute. The situation is pictured in the diagrams below.

$$\begin{tikzcd}
& N \arrow[d, dashed, "\exists! u"] \arrow[ddl, bend right, "\tau_j"'] \arrow[ddr, bend left, "\tau_k"] & & & & F(N) \arrow[d, dashed, "\exists! \overline{u}"] \arrow[ddl, bend right, "\overline{\tau_j}"'] \arrow[ddr, bend left, "\overline{\tau_k}"] & \\
& G(L) \arrow[dl, "G\pi_j"'] \arrow[dr, "G\pi_k"] & & \xleftrightarrow{\text{adjoints}} & & L \arrow[dl, "\pi_j"'] \arrow[dr, "\pi_k"] & \\
GD(j) \arrow[rr, "GD(f)"] & & GD(k) & & D(j) \arrow[rr, "D(f)"] & & D(k)
\end{tikzcd}$$

Of course a similar reasoning shows that left adjoints commute with colimits. This leads to the following theorem.

Theorem 2.8. Let $F : \mathcal{D} \to \mathcal{C}$ be a left adjoint of $G : \mathcal{C} \to \mathcal{D}$. Then G is **continuous**, i.e., preserves limits, and F is **cocontinuous**, i.e., preserves colimits.

Adjoint functors have many powerful properties. We shall finish this chapter with one such property that we will later use in Chapter 7 to prove one property of induction and coinduction.

Lemma 2.9. Consider the following (not necessarily commutative) diagram.

$$\begin{tikzcd}
\mathcal{C}_1 \arrow[r, shift left, "F_1"] & \mathcal{C}_2 \arrow[l, shift left, "G_2"] \arrow[r, shift left, "F_2"] & \mathcal{C}_3 \arrow[l, shift left, "G_1"]
\end{tikzcd}$$

Suppose $F_1 \dashv G_2$ and $F_2 \dashv G_1$. Then $F_2 \circ F_1 \dashv G_2 \circ G_1$.

Proof: Let $c \in \mathcal{C}_1$ and $d \in \mathcal{C}_3$. By first applying the properties of the first adjunction ($F_1 \dashv G_2$) and then the second adjunction, we find that

$$\mathcal{C}_3(F_2F_1c, d) \cong_{\mathbf{Set}} \mathcal{C}_2(F_1c, G_1d) \cong_{\mathbf{Set}} \mathcal{C}_1(c, G_2G_1d).$$

Since the composition of natural bijections is again natural, this isomorphism is natural in c and d. Hence, $F_2 \circ F_1 \dashv G_2 \circ G_1$. $\square$

C3 | Representations of Groups

In this chapter we briefly review some basic notions from the representation theory of groups. In particular, we go over the relation between representations of a group G and representations of a subgroup $H \subset G$. There is a natural way to restrict a representation of G to a representation of H, but there are also ways to turn a representation of H into a representation of G. Of particular interest are the construction of induced and coinduced representations, two ways to turn an H-representation into a G-representation. An important result about these constructions is known as Frobenius reciprocity, Theorem 3.10, which relates (co)induction with restriction. This theorem will serve as a motivation for the more categorical definition needed for groupoids.

Readers familiar with group representations may wish to skip (part of) this chapter.

3.1 Group Representations

Definition 3.1. Let G be a group. A **group representation** is a group homomorphism

$$\rho\colon G \to \mathrm{Aut}(V),$$

where V is a vector space over a field k. We say ρ is n-dimensional if V is an n-dimensional vector space.

Since linear maps are easier to understand, group representations can be used to illustrate the structure of groups. Consider for example the dihedral group

$$D_3 = \langle r, s \mid r^3 = s^2 = (sr)^2 = 1\rangle.$$

At the start of an introductory group theory course one might encounter an alternate, less abstract definition of D_3, namely as the group of symmetries of a triangle, i.e., the group of linear transformations on $\mathbb{R}^2$ that leave a triangle centred at the origin in place. Note that this is an example of a group representation $\rho\colon D_3 \to \mathrm{Aut}(\mathbb{R}^2)$. This representation is depicted in Figure 1.

Definition 3.2. Let $\rho\colon G \to \mathrm{Aut}(V)$ and $\pi\colon G \to \mathrm{Aut}(W)$ be representations of G over the same field k. An **equivariant map** between ρ and π is a linear map $\alpha\colon V \to W$ such that

$$\alpha \circ \rho(g) = \pi(g) \circ \alpha \quad \text{for all } g \in G. \tag{3.1}$$

If α is an isomorphism between V and W then we call ρ and π **isomorphic** or **equivalent**. If V is a subspace of W and α is the inclusion map, then ρ is called a **subrepresentation** of π.

We can construct a 3-dimensional real representation $\pi\colon D_3 \to \mathrm{Aut}(\mathbb{R}^3)$ that has the representation ρ from Figure 1 as a subrepresentation. Let π be given by sending $r \in D_3$ to the rotation of 120 degrees about the z-axis and sending s to the reflection in the yz-plane. One can verify

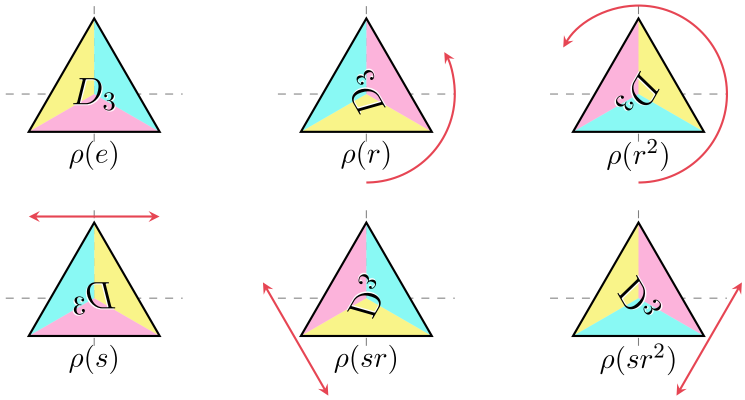

Figure 1: A 2-dimensional real representation $\rho\colon D_3 \to \mathrm{Aut}(\mathbb{R}^2)$.

that ρ is indeed a subrepresentation of π. Notice that reflection in the xy-plane is a non-trivial automorphism of π.

We write $\mathbf{Rep}_G$ for the category whose objects are representations of G and whose morphisms are equivariant maps between them. Note that this category is well-defined, as the composition of equivariant maps is just given by function composition which is associative. Further, for each representation $\rho\colon G \to \mathrm{Aut}(V)$ the identity map $1_V\colon V \to V$ is equivariant and functions as an identity in $\mathbf{Rep}_G$. In the next section we will see how $\mathbf{Rep}_G$ is equivalent to a certain module category, a type of category that is rather well-behaved.

3.2 Representations as $k[G]$-modules

One of the most important results in representation theory is the correspondence between G-representations and modules over the group ring of G. This observation allows us to apply results from module theory to study representations. The group ring is defined as follows.

Definition 3.3. Let G be a group and k a field. The **group ring over k** is the vector space $k[G] = \mathrm{Span}_k(G)$ with multiplication given by linear extension of the group multiplication, i.e., for elements $\sum_{g\in G} \alpha_g g$ and $\sum_{g\in G} \beta_g g$ in $k[G]$ we have

$$\left(\sum_{g\in G} \alpha_g g\right) \cdot \left(\sum_{g\in G} \beta_g g\right) \coloneqq \sum_{g\in G} \left(\sum_{h,k\in G, hk=g} (\alpha_h \beta_k) g\right).$$

Let $e \in G$ be the identity element. Notice that we have the inclusion

$$\begin{aligned} k &\hookrightarrow k[G], \\ \alpha &\mapsto \alpha \cdot e. \end{aligned}$$

In other words, k is a subring of $k[G]$. Any $k[G]$-module is therefore naturally also a k-module, i.e., a k-vector space. We thus find that all $k[G]$-modules are k-vector spaces with a way to multiply by (linear combinations of) elements of G. For this reason, the group ring is sometimes called the **group algebra** as it is an associative algebra and not just a ring. But a vector space together with a way to multiply by elements of G is precisely what a G-representation is. We can restate this observation with the following theorem.

Theorem 3.4. There is a one to one correspondence between G-representations and $k[G]$-

modules, given by

$$\begin{aligned}
\{G\text{-representations}\} &\quad\longleftrightarrow\quad \{k[G]\text{-modules}\} \\
(\rho\colon G \to \operatorname{Aut}(V)) &\quad\longmapsto\quad \begin{pmatrix} V \text{ with } G\text{-action} \\ g \cdot v := \rho(g)v \end{pmatrix} \\
\begin{pmatrix} \rho\colon G \to \operatorname{Aut}(M) \\ \rho(g)(v) := g \cdot v \end{pmatrix} &\quad\longleftarrow\!\shortmid\quad M.
\end{aligned}$$

Suppose M and N are $k[G]$-modules and let $\rho\colon G \to \operatorname{Aut}(M)$ and $\pi\colon G \to \operatorname{Aut}(N)$ be the corresponding representations. Notice that a $k[G]$-linear map $f\colon M \to N$ satisfies

$$f \circ \rho(g) = f(g \cdot -) = g \cdot f(-) = \pi(g) \circ f$$

for all $g \in G$. This shows that f is also an equivariant map. One can also show that any equivariant map between ρ and π is a $k[G]$-linear map between the corresponding $k[G]$-modules. This observation shows that the category of G-representations $\mathbf{Rep}_G$ is isomorphic to the category of $k[G]$-modules $\mathbf{Mod}(k[G])$ via the above correspondence.

Due to this correspondence, representations viewed as group homomorphisms and $k[G]$-modules are sometimes conflated. A 'group representation' may refer to both, depending on the context. From here on out, a 'representation' can refer both to a group homomomorphism and a module over the group ring.

The usage of modules also allows us to easily define various operations on representations by using known operations on modules. For instance, if ρ and π are representations with corresponding modules M and N, we may define $\rho \oplus \pi$ to be the representation corresponding to the module $M \oplus N$. Of course this works for any operation on $k[G]$-modules, not just the direct sum.

Definition 3.5. Let $(\rho_i)_i$ be a collection of representations $\rho_i\colon G \to \operatorname{Aut}(V_i)$. Define the direct sum $\bigoplus_i \rho_i$ and the direct product $\prod_i \rho_i$ to be the representation corresponding to $\bigoplus_i M_i$ and $\prod_i M_i$ respectively. Define the tensor product $\rho_i \otimes \rho_j$ to be the representation corresponding to $M_i \otimes M_j$.

These operations play an important part in studying the structure of module categories and thus that of $\mathbf{Rep}_G$. In Chapter 6, we will generalise some of the results to the category $\mathbf{Rep}_{\mathcal{G}}$ where $\mathcal{G}$ is a groupoid instead of a group.

Aside from inheriting module-related constructions, the correspondence from Theorem 3.4 also allows us to apply certain results from module theory. A particularly important result is the following, named after Heinrich Maschke.

Theorem 3.6 (Maschke)**.** Let G be a finite group and k a field. If $\operatorname{char}(k)$ does not divide $\#G$, then $k[G]$ is semisimple.

See Theorem 2 in [8] for a proof of the corresponding statement about representations.

The semisimplicity of $k[G]$ means that there are finitely many simple modules S_i such that any finitely generated $k[G]$-module M may be written as a direct sum of finitely many S_i in a way that is unique up to reordering. By the correspondence we find that there are finitely many simple representations of G, and all finite-dimensional representations of G are just direct sums of these simple representations. The category $\mathbf{Rep}_G$ is thus completely classified by these simple representations.

3.3 Induced Representations

Whenever G is a large, complicated group it might be hard to directly study its representations. Instead, one can restrict representations $\rho\colon G \to \mathrm{Aut}(V)$ to subgroups $H \subset G$ whose representations are easier to understand. This restriction is simply given by restricting a group homomorphism $\rho\colon G \to \mathrm{Aut}(V)$ to a group homomorphism $\rho|_H\colon H \to \mathrm{Aut}(V)$. From the module-theoretic point of view, we use that a $k[G]$-module M is also a $k[H]$-module by "forgetting" being able to multiply by elements of G not in H.

Further note that any $k[G]$-linear map is in particular a $k[H]$-linear map. We can thus also restrict morphisms between representations. It is not hard to see that this restriction works functorially, leading to the following definition.

Definition 3.7. Let G be a group with subgroup H. Define the **restriction functor** as

$$\begin{aligned} \mathrm{Res}_H^G\colon \mathbf{Mod}(k[G]) &\to \mathbf{Mod}(k[H]), \\ (M \text{ as } k[G]\text{-module}) &\mapsto (M \text{ as } k[H]\text{-module}), \\ (f\colon M \to N \text{ as } k[G]\text{-linear}) &\mapsto (f\colon M \to N \text{ as } k[H]\text{-linear}). \end{aligned}$$

Naturally one might wonder whether there exists a non-trivial functor going the other way round. The existence of such a functor would allow us to construct representations of a potentially complicated group G by starting with representations of a more manageable subgroup H. In the language of module theory, we wish to give each $k[H]$-module M a $k[G]$-module structure in a functorial manner.

One way to achieve this is by using that $k[G]$ itself has a $(k[G], k[H])$-bimodule structure. We can then define the induced representation of M to be the $k[G]$-module $k[G] \otimes_{k[H]} M$, where the right-$k[H]$-module structure on $k[G]$ is used for the tensor product and the left-$k[G]$-module structure on $k[G]$ is used to define a $k[G]$-module structure on the whole of $k[G] \otimes_{k[H]} M$. Given any $k[H]$-linear map $f\colon M \to N$, there is a natural $k[H]$-linear map $\mathrm{id} \times f\colon k[G] \times M \to k[G] \times N$. The universal property of the tensor product states that there exists a unique $k[H]$-linear map $\overline{f}$ between the two induced representations satisfying $\overline{f}(g \otimes m) = \mathrm{id}_{k[G]}(g) \otimes f(m)$. Since $\mathrm{id}_{k[G]}$ is $k[G]$-linear, this induced map $\overline{f}$ must be $k[G]$-linear as well. The diagram below illustrates the situation.

$$\begin{tikzcd} k[G] \times M \arrow[r, "\otimes"] \arrow[d, "\mathrm{id}\times f"'] \arrow[dr, "\mathrm{id}\otimes f"] & k[G] \otimes_{k[H]} M \arrow[d, dashed, "\exists!\overline{f}"] \\ k[G] \times N \arrow[r, "\otimes"] & k[G] \otimes_{k[H]} N \end{tikzcd}$$

One may verify that this construction of induced representations works functorially, leading to the following definition.

Definition 3.8. Let G be a group with subgroup H. Define the **induction functor** by

$$\begin{aligned} \mathrm{Ind}_H^G\colon \mathbf{Mod}(k[H]) &\to \mathbf{Mod}(k[G]), \\ M &\mapsto k[G] \otimes_{k[H]} M, \\ (f\colon M \to N) &\mapsto (\overline{f}\colon \mathrm{Ind}_H^G(M) \to \mathrm{Ind}_H^G(N)) \end{aligned}$$

Alternatively, we can turn a $k[H]$-module M into a $k[G]$-module is by using the $\mathrm{Hom}_{k[H]}(k[G], -)$ functor. Again, we can give this a $k[G]$-module structure by regarding $k[G]$ as a $(k[H], k[G])$-bimodule. The left module structure is used for the $k[H]$-linearity and the right module structure used to give $\mathrm{Hom}_{k[H]}(k[G], M)$ a $k[G]$-module structure. One may verify that a $k[H]$-linear map $f\colon M \to N$ induces a $k[G]$-linear map between the appropriate modules in a functorial manner. Explicitly, $\mathrm{Coind}_H^G(f)$ is given by $\mathrm{Coind}_H^G(f)(\varphi)(v) = f(\varphi(v))$ for $\varphi \in \mathrm{Hom}_{k[H]}(k[G], M)$ and $v \in k[G]$. We shall call this construction coinduction.

Definition 3.9. Let G be a group with subgroup H. Define the **coinduction functor** by

$$\begin{aligned} \mathrm{Coind}_H^G\colon \mathbf{Mod}(k[H]) &\to \mathbf{Mod}(k[G]), \\ M &\mapsto \mathrm{Hom}_{k[H]}(k[G], M), \\ (f\colon M \to N) &\mapsto (\mathrm{Coind}_H^G(f)\colon \mathrm{Coind}_H^G(M) \to \mathrm{Coind}_H^G(N)) \end{aligned}$$

3.4 Frobenius Reciprocity

Induced representations were first studied by Ferdinand Frobenius. In the 1898 paper in which he defined them, he also provided a proof of what is now known as Frobenius Reciprocity [4], albeit using characters of representations. Concretely, it relates the processes of induction and restriction of representations which allows one to study representations of a group by representations of smaller subgroups.

Theorem 3.10 (Frobenius Reciprocity)**.** Let $H \subset G$ be groups. Given a $k[H]$ module M and a $k[G]$ module N, we have a bijection of sets

$$\mathrm{Hom}_{k[G]}\left(\mathrm{Ind}_H^G M, N\right) \cong \mathrm{Hom}_{k[H]}\left(M, \mathrm{Res}_H^G N\right).$$

The classical formulation of Frobenius Reciprocity used the language of character theory and only yields the bijection shown above, but one can show that this bijection is natural in its arguments meaning restriction and induction are adjoint functors. It is further possible to prove that this bijection is in fact a k-linear isomorphism of vector spaces. One application of Frobenius Reciprocity is the following. If G is a large but finite group, then by Theorem **3.6** any complex representation can be written as the direct sum of finitely many simple $\mathbb{C}[G]$-modules. Given any $\mathbb{C}[G]$-module N and a simple $\mathbb{C}[G]$-module S, one might wonder whether S appears as a direct summand in the decomposition of N. As it turns out, this is the case precisely when $\mathrm{Hom}_{k[G]}(N, S) \neq 0$. When G is particularly large it might be computationally expensive to directly calculate $\mathrm{Hom}_{k[G]}(N, S)$. Instead, if we can cleverly choose a subgroup $H \subset G$ such that $N \cong \mathrm{Ind}_H^G M$ for some $k[H]$-module M, then Frobenius Reciprocity implies that we only need to calculate $\mathrm{Hom}_{k[H]}(M, \mathrm{Res}_H^G S)$ to see whether S appears in N. This may significantly reduce the computation cost if H is of large index in G.

C4 | Groupoids as Generalization of Groups

As a motivating example for groupoids we shall recall the construction of the fundamental group of a topological space. Let X be a topological space and pick a base point $x_0 \in X$. Recall that the fundamental group is defined as the set of homotopy classes of loops based at x_0, with the group operation given by path concatenation. It is denoted by $\pi_1(X, x_0)$. If X is a path-connected space, the fundamental group is independent of the choice of base point. Indeed, given another point $x_1 \in X$ we can construct a group isomorphism

$$\begin{aligned} \pi_1(X, x_0) &\xrightarrow{\sim} \pi_1(X, x_1), \\ [\delta] &\mapsto [\gamma^{-1}] \cdot [\delta] \cdot [\gamma], \end{aligned}$$

where γ is any path from x_0 to x_1. The problem is that this isomorphism depends on the homotopy class of γ. Indeed, if $\pi_1(X, x_0)$ is not abelian, there is no natural way of identifying $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ [7, §52].

Groupoids provide a solution to this problem. Instead of two objects being either isomorphic or not, we label all isomorphisms between them. We can use this idea to generalize the idea of a fundamental group to the **fundamental groupoid**. Concretely, the fundamental groupoid of a topological space X is the category $\Pi_1(X)$ with objects $\mathrm{ob}(\Pi_1(X)) = X$ and hom-sets $\Pi_1(X)(x, y) = \{\text{paths from } x \text{ to } y\}/\text{homotopy}$. Composition of morphisms is given by path concatenation. For a fixed basepoint $x_0 \in X$, the hom-set $\Pi_1(X)(x_0, x_0)$ is precisely the fundamental group with basepoint x_0.

There are several benefits to studying the fundamental groupoid over the fundamental group. First, it removes the dependency of choosing an arbitrary basepoint which makes for a more beautiful construction. It also allows us to talk about *the* fundamental groupoid of a space that is not path-connected. Finally it, can also yield stronger theorems. An important example is the Van Kampen Theorem. The version using only the fundamental group cannot be used to calculate the fundamental group of S^1, as it does not admit a non-trival covering of two open sets whose intersection is connected. In contrast, the groupoid version of the Van Kampen theorem does allow one to verify that $\pi_1(S^1, 1) \cong \mathbb{Z}$ [1, §6.7, §9.1].

4.1 Groupoids

With the example of the fundamental groupoid in mind we turn to the abstract study of groupoids. The contents of this section are mainly based on Chapter 3 from *An Introduction to Groups, Groupoids and Their Representations* [5].

Definition 4.1. A **groupoid** is a small category $\mathcal{G}$ in which every morphism is invertible.

Note that this is a strict generalisation of the notion of groups. Any group G can be 'categorified' by considering the category $\mathcal{G}$ consisting of a singular object '$*$' and with hom-set $\mathcal{G}(*, *) = G$. The group axioms (existence of a unit, associativity, and invertibility) are then equivalent to the axioms of $\mathcal{G}$ being a groupoid.

An interesting example of a groupoid is the **groupoid of pairs**. Given a number $n \in \mathbb{Z}_{>0}$, consider the groupoid $\mathcal{G}$ with $\operatorname{ob}\mathcal{G} = \{1, \ldots, n\}$ and with hom-sets $\mathcal{G}(i,j) = \{l_{i,j}\}$ consisting of just one element.

Definition 4.2. Let $\mathcal{G}$ be a groupoid. A **subgroupoid** $\mathcal{H}$ of $\mathcal{G}$, denoted $\mathcal{H} \subset \mathcal{G}$, is a groupoid satisfying $\operatorname{ob}\mathcal{H} \subset \operatorname{ob}\mathcal{G}$ and $\mathcal{H}(x,y) \subset \mathcal{G}(x,y)$ for all $x, y \in \operatorname{ob}\mathcal{H}$. A subgroupoid is called **wide** if $\operatorname{ob}\mathcal{H} = \operatorname{ob}\mathcal{G}$.

Let $\mathcal{H} \subset \mathcal{G}$ be groupoids. Notice that there is a canonical inclusion functor

$$\begin{aligned} I \colon \mathcal{H} &\to \mathcal{G}, \\ x &\mapsto x, \\ (f \colon x \to y) &\mapsto (f \colon x \to y). \end{aligned}$$

In some later constructions we will require our subgroupoid to be wide. This is a very reasonable ask. Indeed, any subgroupoid $\mathcal{H} \subset \mathcal{G}$ can always be extended to a wide subgroupoid $\widetilde{\mathcal{H}} \subset \mathcal{G}$ which is obtained by adding all missing points $x \in \operatorname{ob}(\mathcal{G}) \setminus \operatorname{ob}(\mathcal{H})$ and the required identity morphisms 1_x for $x \in \operatorname{ob}(\mathcal{G}) \setminus \operatorname{ob}(\mathcal{H})$. This extension $\widetilde{\mathcal{H}}$ is called the **natural extension** of $\mathcal{H}$. From now on, we will assume all subgroupoids are wide.

Let $x, y \in \operatorname{ob}\mathcal{G}$. We say x and y are connected if there exists a morphism between them, i.e., if $\mathcal{G}(x,y) \neq \varnothing$. It follows quite quickly from the axioms of a groupoid that this defines an equivalence relation on $\operatorname{ob}\mathcal{G}$. Fix some element $x \in \operatorname{ob}\mathcal{G}$. The subset of $\operatorname{ob}\mathcal{G}$ of all objects connected with x is called the connected component of x. Note that the groupoid is a disjoint union of connected components. A groupoid $\mathcal{G}$ is called connected if it consists of just one connected component.

Seeing as groupoids are just categories with an extra assumption, a morphism between groupoids should just be a functor. Do note that all functors preserve inverses. Indeed, given a functor $T \colon \mathcal{H} \to \mathcal{G}$ between groupoids and a morphism $f \colon x \to y$ in $\mathcal{H}$, we find that $1_{T(x)} = T(1_x) = T(f^{-1} \circ f) = T(f^{-1}) \circ T(f)$ and similarly $1_{T(y)} = T(f) \circ T(f^{-1})$. It follows that $T(f^{-1}) = T(f)^{-1}$ as expected.

Definition 4.3. Let $\mathcal{H}$ and $\mathcal{G}$ be groupoids. A **morphism of groupoids** is a functor $T \colon \mathcal{H} \to \mathcal{G}$.

4.2 Cosets of Subgroupoids

Many constructions applicable to groups can be extended to groupoids in a natural way. In this section, we shall define cosets of a subgroupoid $\mathcal{H} \subset \mathcal{G}$. In this thesis we will only focus on right cosets which will be relevant in later chapters. This section is partly based on Section 6.1 of [5].

Definition 4.4. Let $\mathcal{G}$ be a groupoid and $\mathcal{H} \subset \mathcal{G}$ a wide subgroupoid. Define the following equivalence relation on $\operatorname{mor}(\mathcal{G})$.

$$\alpha_1 \sim \alpha_2 \iff \exists \beta \in \operatorname{mor}(\mathcal{H}) \text{ such that } s(\beta) = t(\alpha_2) \text{ and } \alpha_1 = \beta \circ \alpha_2. \tag{4.1}$$

A **right coset** of $\mathcal{H}$ in $\mathcal{G}$ is an equivalence class of this equivalence relation. We write $[\alpha]$ for the equivalence class of α and $\operatorname{mor}(\mathcal{G})/\sim$ for the set of all equivalence classes.

First we verify that $\sim$ is indeed an equivalence relation. By assumption, $\mathcal{H}$ is wide meaning $1_{t(\alpha)} \in \operatorname{mor}(\mathcal{H})$. Thus, $\alpha \sim \alpha$ as $\alpha = 1_{t(\alpha)} \circ \alpha$. Second, if $\alpha_1 = \beta\alpha_2$ then $\alpha_2 = \beta^{-1}\alpha_1$. Thus, $\sim$ is reflexive. Lastly, if $\alpha_1 = \beta\alpha_2$ and $\alpha_2 = \beta'\alpha_3$ then $\alpha_1 = (\beta\beta')\alpha_3$. Thus, $\sim$ is transitive.

Cosets of groupoids share many of the properties that cosets of groups have. For example, the coset of $\alpha \in \mathrm{mor}(\mathcal{G})$ may be computed by

$$[\alpha] = \{\beta \circ \alpha : \beta \in \mathrm{mor}(\mathcal{H}) \text{ and } s(\beta) = t(\alpha)\}.$$

Further, if $\mathcal{H}$ is connected then $\mathrm{mor}(\mathcal{H})$ is itself a coset. Indeed, if we further assume that $\mathcal{G}$ is connected, then all cosets $[\alpha] \in \mathrm{mor}(\mathcal{G})/\sim$ are in bijection with $\mathrm{mor}(\mathcal{H})$. This bijection is given as follows:

$$\begin{aligned} [\alpha] &\xrightarrow{\sim} \mathrm{mor}(\mathcal{H}), \\ \beta \circ \alpha &\longmapsto \beta. \end{aligned}$$

Unlike groups, not all morphisms in a groupoid are composable. One particular consequence is that two morphisms $\alpha_1, \alpha_2 \in \mathrm{mor}(\mathcal{G})$ with $s(\alpha_1) \neq s(\alpha_2)$ can never lie in the same coset. For this reason, it is unhelpfull to consider all cosets when studying the local interaction between a subgroupoid and the larger groupoid. Rather, we might want to restrict the relation to consider just local morphisms. This will be needed later in Chapter 7.

Definition 4.5. Let $x \in \mathrm{ob}\,\mathcal{G}$. Write

$$\mathcal{G}_x := \{\alpha \in \mathrm{mor}(\mathcal{G}) : s(\alpha) = x\}.$$

Let $\sim_x$ be the restriction of the equivalence relation from equation (4.1) to $\mathcal{G}_x \subset \mathrm{mor}(\mathcal{G})$. A **right coset of $\mathcal{H}$ at x** is an equivalence class $\sim_x$. Write $[\alpha]_x$ for the equivalence class of $\alpha \in \mathcal{G}_x$ and $\mathcal{G}_x/\sim_x$ for the set of all equivalence classes.

There is a connection between this equivalence relation and the category $(x \downarrow I)$, cf. Definition 2.2. In $(x \downarrow I)$, the objects are pairs $\langle \alpha, t(\alpha) \rangle$ for $\alpha \in \mathcal{G}_x$ and the morphisms between objects $\langle \alpha_1, t(\alpha_1) \rangle$ and $\langle \alpha_2, t(\alpha_2) \rangle$ are maps $h \in \mathcal{H}(t(\alpha_2), t(\alpha_1))$ satisfying $\alpha_1 = h \circ \alpha_2$. Two elements $\alpha_1, \alpha_2 \in \mathrm{mor}(\mathcal{G})$ thus lie in the same coset in $\mathcal{G}_x/\sim_x$ precisely when they lie in the same connected component of $(x \downarrow I)$.

C5 | Groupoid Representations

5.1 Groupoid Representations

Definition 5.1. Let $\mathcal{G}$ be a groupoid. A **representation** of $\mathcal{G}$ is a functor $T\colon \mathcal{G} \to \mathrm{Vec}$. A finite-dimensional representation is a functor $T\colon \mathcal{G} \to \mathrm{FinVec}$.

First notice that this definition is a generalization of group representations in the following sense. Suppose G is a group and $\mathcal{G}$ its categorification. Then, a group representation $\rho\colon G \to \mathrm{Aut}(V)$ for a vector space V can also be viewed as a functor $T\colon \mathcal{G} \to \mathrm{Vec}$ with $T(*) = V$ as the functoriality of T precisely corresponds to the compatibility of ρ with the group structure.

Let $f\colon x \to y$ be a morphism in $\mathcal{G}$. Since $\mathcal{G}$ is a groupoid, f is invertible. By using functoriality of T we find that $1_{T(y)} = T(1_y) = T(f \circ f^{-1}) = T(f) \circ T(f^{-1})$ and similarly $1_{T(x)} = T(f^{-1}) \circ T(f)$. Hence, $T(f)$ is invertible with inverse $T(f^{-1})$, showing $T(x) \cong T(y)$. We thus find that T maps all objects of the same connected component of $\mathcal{G}$ to isomorphic vector spaces. In particular, if $\mathcal{G}$ is connected then all objects are mapped to isomorphic vector spaces.

Further note that if $\mathcal{G}$ consists of multiple connected components, the image of a representation T in Vec consists of exactly as many connected components. As these connected components do not interact with one another, the representations of a non-connected groupoid are just given by the the representations of the connected component.

An important representation is the the **trivial representation**. It is given by the functor $\mathbb{1}\colon \mathcal{G} \to \mathrm{Vec}$ sending all objects to k and all morphisms in $\mathcal{G}$ to id_k. Verifying that this is indeed a representation is trivial. This representation will play an important role in the monoidal structure of $\mathbf{Rep}_{\mathcal{G}}$ as will be shown in Chapter 6.

Definition 5.2. Let $S, T\colon \mathcal{G} \to \mathrm{Vec}$ be representations of a groupoid $\mathcal{G}$. A **morphism of representations** is a natural transformation $\alpha\colon S \Rightarrow T$. We call S and T **equivalent** if there exists a natural isomorphism $S \Rightarrow T$.

One particular instance of a morphism between representations is an inclusion.

Definition 5.3. A subrepresentation of a representation $T\colon \mathcal{G} \to \mathrm{Vec}$ is a representation $S\colon \mathcal{G} \to \mathrm{Vec}$ such that for all $x \in \mathrm{ob}\,\mathcal{G}$ we have that $S(x) \subset T(x)$ and for all $f\colon x \to y$ in $\mathcal{G}$ we have $S(f) = T(f)|_{S(x)}$.

Equivalently, there exists a natural transformation $\iota\colon S \Rightarrow T$ whose components are all inclusions. The last condition in the definition above expresses the commutativity of the diagram below.

$$\begin{tikzcd} S(x) \arrow[r, hook, "\iota_x"] \arrow[d, "S(f)"] & T(x) \arrow[d, "T(f)"] \\ S(y) \arrow[r, hook, "\iota_y"] & T(y) \end{tikzcd}$$

We can lift various vector space operations to operations on $\mathcal{G}$-representations by using pointwise definitions and verifying that the resulting $\mathcal{G}$-representation is well-defined. Consider for instance the direct sum of vector spaces. If T and S are representations of $\mathcal{G}$, then we can define their direct sum pointwise by

$$\begin{aligned}(T \oplus S)(x) &:= T(x) \oplus S(x),\\ (T \oplus S)(f\colon x \to y) &:= T(f) \oplus S(f).\end{aligned}$$

Since direct sums work functorially on linear maps, we find that the pointwise construction extends to a functor $T \oplus S\colon \mathcal{G} \to \mathrm{Vec}$, i.e., a $\mathcal{G}$-representation. On vector spaces there is a natural inclusion $\iota_x\colon T(x) \hookrightarrow T(x) \oplus S(x)$. One can verify that this inclusion is natural in x, yielding a natural transformation $\iota\colon T \Rightarrow T \oplus S$. This is expressed by the following commutative diagram.

$$\begin{tikzcd} T(x) \arrow[r, "\iota_x"] \arrow[d, "T(f)"] & T(x) \oplus S(x) \arrow[d, "T(f)\oplus S(f)"] \\ T(y) \arrow[r, "\iota_y"] & T(y) \oplus S(y) \end{tikzcd}$$

Similarly, we can define the direct product of representations $T \times S$ pointwise. These operations are in fact equal, which will be described in detail in Chapter 6 where we prove that $\mathbf{Rep}_{\mathcal{G}}$ is an abelian category with this definition of the direct sum and the direct product.

One more important construction is the tensor product of representations. Similar to the direct sum and direct product, the tensor product is defined pointwise. Proving this is a well-defined operation is a bit more involved however.

Definition 5.4. Let S and T be two representations of a groupoid $\mathcal{G}$. We define the **tensor product representation**, denoted $S \otimes T$, as the representation given by

$$\begin{aligned} S \otimes T\colon \mathcal{G} &\to \mathrm{Vec},\\ x &\mapsto S(x) \otimes T(x),\\ (f\colon x \to y) &\mapsto S(f) \otimes T(f).\end{aligned}$$

Here, $S(x) \otimes T(x)$ is the standard tensor product of vector spaces. The linear map $S(f) \otimes T(f)$ is the unique linear map satisfying $(S(f) \otimes T(f))(v \otimes w) = S(f)(v) \otimes T(f)(w)$ for all $v \in S(x)$ and $w \in T(x)$.

We shall quickly verify the validity of this construction. Suppose we have the following diagram in $\mathcal{G}$.

$$\begin{tikzcd} x \arrow[r, "f"'] \arrow[rr, bend left, "h"] & y \arrow[r, "g"'] & z \end{tikzcd}$$

Applying the definition of $S \otimes T$ yields the diagram below, where the diagonal arrows are given by composition of the obvious morphisms and the dashed arrows are obtained by applying the universal property of the tensor product of vector spaces.

$$\begin{tikzcd}
S(x)\times T(x) \arrow[r, "\otimes"] \arrow[d, "S(f)\times T(f)"'] \arrow[dr] \arrow[dd, bend right=60, "S(h)\times T(h)"'] & S(x)\otimes T(x) \arrow[d, dashed, "\exists! S(f)\otimes T(f)"] \arrow[dd, dashed, bend left=60, "\exists! S(h)\otimes T(h)"] \\
S(y)\times T(y) \arrow[r, "\otimes"] \arrow[d, "S(g)\times T(g)"'] \arrow[dr] & S(y)\otimes T(y) \arrow[d, dashed, "\exists! S(g)\otimes T(g)"] \\
S(z)\times T(z) \arrow[r, "\otimes"] & S(z)\otimes T(z)
\end{tikzcd}$$

Commutativity of the left triangle follows from the functoriality of S and T. Commutativity of the whole diagram follows from commutativity of the left-most triangle and the universal property of tensor products. This diagram shows that the construction of the product representation respects composition. One can further verify that

$$\begin{aligned}(S\otimes T)(1_x) &= S(1_x)\otimes T(1_x)\\ &= 1_{S(x)}\otimes 1_{T(x)}\\ &= 1_{S(x)\otimes T(x)},\end{aligned}$$

meaning $S\otimes T$ is indeed a functor from $\mathcal{G}$ to Vec.

5.2 Restricted Representations

Defining restriction of representations of groupoids turns out to be rather straightforward. Indeed, given groupoids $\mathcal{H}\subset\mathcal{G}$ and the inclusion functor $I\colon\mathcal{H}\to\mathcal{G}$, then any representation $S\in\mathrm{ob}(\mathbf{Rep}_{\mathcal{G}})$ naturally restricts to a representation $S\circ I\in\mathrm{ob}(\mathbf{Rep}_{\mathcal{H}})$. One can verify that precomposition with I extends to a functor between the categories of representations. This observation leads to the following definition.

Definition 5.5. Let $I\colon\mathcal{H}\to\mathcal{G}$ be the inclusion functor. Define the **restriction functor** as

$$\begin{aligned}\mathrm{Res}^{\mathcal{G}}_{\mathcal{H}}\colon \mathbf{Rep}_{\mathcal{G}} &\to \mathbf{Rep}_{\mathcal{H}},\\ S &\mapsto S\circ I,\\ (\alpha\colon S_1\Rightarrow S_2) &\mapsto (\alpha I\colon S_1\circ I\Rightarrow S_2\circ I).\end{aligned}$$

In Chapter 7, we will define induction and coinduction functors which are respectively the left and right adjoint of this restriction functor.

C6 | The Category of $\mathcal{G}$-Representations

In this chapter we shall investigate the structure of the category of $\mathcal{G}$-representations. In the case of finite groups, the category of G-representations is equivalent to the category of modules over the ring $k[G]$. This immediately gives $\mathbf{Rep}_{\mathcal{G}}$ a lot of structure. For the case of arbitrary groupoids, we cannot simply use existing results from module theory however. Most definitions in this chapter are taken from [6].

6.1 $\mathbf{Rep}_{\mathcal{G}}$ is Abelian

A natural first question to ask is whether $\mathbf{Rep}_{\mathcal{G}}$ is **abelian**. Intuitively, abelian categories are categories which share some of the properties of $\mathbf{Ab}$, the category of abelian groups. Examples include $\mathbf{Ab}$ itself, or more generally the category of modules over any ring. Generally, any functor category whose codomain is abelian is itself abelian [6, p.199]. In our case we will make use of the abelian structure on Vec to give $\mathbf{Rep}_{\mathcal{G}}$ an abelian structure.

Theorem 6.1. $\mathbf{Rep}_{\mathcal{G}}$ is an **abelian** category, meaning all of the following hold.
- All hom-sets are abelian groups.
- There exists a **zero object** which is both terminal and initial in $\mathbf{Rep}_{\mathcal{G}}$.
- All finite direct products and direct sums exist.
- Every morphism in has both a kernel and a cokernel.
- All monomorphisms and epimorphisms are normal.

All properties listed above could be verified by lifting the corresponding properties on $\mathbf{Vec}$. Verifying that pointwise definitions obey the required axioms will be a running theme throughout this chapter. As a warm up, consider the direct sum of representations $S \oplus T$ which we defined in Section **5.1** for fixed representations. We shall now verify that $S \oplus T$ obeys the universal property of direct sums. Let $i\colon S \Rightarrow S \oplus T$ and $j\colon T \Rightarrow S \oplus T$ be the natural transformations given by the pointwise inclusions $i_x\colon Tx \hookrightarrow Sx \oplus Tx$ and $j_x\colon Sx \hookrightarrow Sx \oplus Tx$. Suppose we are given another representation U with natural transformations $p\colon S \Rightarrow U$ and $q\colon T \Rightarrow U$. Let $x \in \operatorname{ob} \mathcal{G}$. Since $Tx \oplus Sx$ is the coproduct in Vec, there exists a unique $u_x\colon Tx \oplus Sx \to U$ such that $u_x \circ j_x = p_x$ and $u_x \circ i_x = q_x$. Define $u\colon S \oplus T \Rightarrow U$ as the collection of these morphisms u_x. First we verify that u is natural. Let $f\colon x \to y$ be a morphism in $\mathcal{G}$. By applying the definition of direct sums of linear maps and the properties of coproducts in Vec, we find that

$$\begin{aligned}
u_y \circ (Sf \oplus Tf) &= (u_y \circ j_y \circ Sf) \oplus (u_y \circ i_y \circ Tf) \\
&= (p_y \circ Sf) \oplus (q_y \circ Tf) \\
&= (Uf \circ p_x) \oplus (Uf \circ q_x) \\
&= (Uf \circ u_x \circ j_x) \oplus (Uf \circ u_x \circ i_x) \\
&= Uf \circ u_x \circ (j_x \oplus i_x) \\
&= U_f \circ u_x.
\end{aligned}$$

Hence, u is natural. The uniqueness of u follows immediately from the fact that all components u_x are unique. Hence, $S \oplus T$ satisfies the universal property of coproducts. Indeed, this construction of direct sums can be extended even further.

Theorem 6.2. The direct sum of representations extends to a bifunctor

$$- \oplus -\colon \mathbf{Rep}_{\mathcal{G}} \times \mathbf{Rep}_{\mathcal{G}} \to \mathbf{Rep}_{\mathcal{G}}.$$

Proof: Consider a morphism $(\alpha, \beta)\colon (S, T) \Rightarrow (S', T')$ in $\mathbf{Rep}^2_{\mathcal{G}}$. Define $\alpha \oplus \beta\colon S \oplus T \Rightarrow S' \oplus T'$ by $(\alpha \oplus \beta)_x = \alpha_x \oplus \beta_x$ on points. Since α and β are natural, we find that

$$\begin{aligned}(\alpha_y \oplus \beta_y) \circ (Sf \oplus Tf) &= (\alpha_y \circ Sf) \oplus (\beta_y \circ Tf)\\ &= (S'f \circ \alpha_x) \oplus (T'f \circ \beta_x)\\ &= (S'f \oplus T'f) \circ (\alpha_x \oplus \beta_x).\end{aligned}$$

Hence, $\alpha \oplus \beta$ is natural. If we are further given a morphism $(\gamma, \delta)\colon (S', T') \Rightarrow (S'', T'')$, then for any $x \in \operatorname{ob} \mathcal{G}$ we have

$$\begin{aligned}((\gamma \oplus \delta) \circ (\alpha \oplus \beta))_x &= (\gamma_x \oplus \delta_x) \circ (\alpha_x \oplus \beta_x)\\ &= (\gamma \circ \alpha)_x \oplus (\delta \circ \beta)_x.\end{aligned}$$

We also find that

$$\begin{aligned}(1_T \oplus 1_S)_x &= (1_T)_x \oplus (1_S)_x\\ &= 1_{Tx} \oplus 1_{Sx}\\ &= 1_{(T \oplus S)(x)}.\end{aligned}$$

Thus, the direct sum extends to a bifunctor. □

6.2 Closed Symmetric Monoidal Structure of $\mathbf{Rep}_{\mathcal{G}}$

In Section 5.1 we defined the tensor product $S \otimes T$ of fixed representations S and T. Just as with the direct sum of representations, this construction can be extended to a bifunctor on the whole of $\mathbf{Rep}_{\mathcal{G}}$. Indeed, this bifunctor gives rise to a monoidal structure. Our goal in this section will be proving that $\mathbf{Rep}_{\mathcal{G}}$ is a symmetric closed monoidal category. Proving this will require some work. We shall start by properly defining the tensor product functor.

Proposition 6.3. The tensor product of representations from Definition 5.4 extends to a bifunctor

$$- \otimes -\colon \mathbf{Rep}_{\mathcal{G}} \times \mathbf{Rep}_{\mathcal{G}} \to \mathbf{Rep}_{\mathcal{G}}.$$

First we shall specify how natural transformations are acted upon by the tensor product. Let $\alpha \times \beta\colon (S_1 \times T_1) \Rightarrow (S_2 \times T_2)$ be a natural transformation in $\mathbf{Rep}_{\mathcal{G}} \times \mathbf{Rep}_{\mathcal{G}}$. Our goal will be defining a natural transformation $\alpha \otimes \beta\colon (S_1 \otimes T_1) \Rightarrow (S_2 \otimes T_2)$. Let $f\colon x \to y$ in $\mathcal{G}$. Define $(\alpha \otimes \beta)_x$, which we will also denote by $\alpha_x \otimes \beta_x$, as the unique morphism for which $(\alpha_x \circ \beta_x)(v \otimes w) = \alpha_x(v) \otimes \beta_x(w)$ for all $(v, w) \in S_1(x) \times T_1(x)$. Diagrammatically, this morphism is given as follows.

$$\begin{tikzcd}
S_1x \times T_1x \arrow[r, "\otimes"] \arrow[d, "\alpha_x \times \beta_x"'] \arrow[dr] & S_1x \otimes T_1x \arrow[d, dashed, "\exists! \alpha_x \otimes \beta_x"] \\
S_2x \times T_2x \arrow[r, "\otimes"] & S_2x \otimes T_2x
\end{tikzcd}$$

By assumption, $\alpha \times \beta$ is a natural transformation and thus the diagram below commutes.

$$\begin{tikzcd}
S_1x \times T_1x \arrow[r, "\alpha_x \times \beta_x"] \arrow[d, "S_1f \times T_1f"'] & S_2x \times T_2x \arrow[d, "S_2f \times T_2f"] \\
S_1y \times T_1y \arrow[r, "\alpha_x \times \beta_y"] & S_2y \times T_2y
\end{tikzcd}$$

By combining this diagram with earlier observations, we obtain the following diagram.

$$\begin{tikzcd}
S_1x \times T_1x \arrow[r, "\alpha_x \times \beta_x"] \arrow[d, "\otimes"] \arrow[dr] \arrow[ddd, bend right=60, "S_1f \times T_1f"'] & S_2x \times T_2x \arrow[d, "\otimes"'] \arrow[ddd, bend left=60, "S_2f \times T_2f"] \\
S_1x \otimes T_1x \arrow[r, dashed, "\alpha_x \otimes \beta_x"] \arrow[d, "S_1f \otimes T_1f"'] \arrow[dr, dashed, "\exists!"] & S_2x \otimes T_2x \arrow[d, "S_2f \otimes T_2f"] \\
S_1y \otimes T_1y \arrow[r, dashed, "\alpha_y \otimes \beta_y"] & S_2y \otimes T_2y \\
S_1y \times T_1y \arrow[r, "\alpha_y \times \beta_y"] \arrow[u, "\otimes"] \arrow[ur] & S_2y \times T_2y \arrow[u, "\otimes"']
\end{tikzcd}$$

The observation we just made implies commutativity of the outer-most square. Commutativity of the top, bottom, left, and right squares follows from the definition of the tensor product. We therefore only need to check whether the middle square commutes. Define $\varphi\colon S_1x \times T_1x \to S_2y \otimes T_2y$ to be the morphism

$$\varphi := (S_2f \otimes T_2f) \circ (\alpha_x \otimes \beta_x) \circ \otimes.$$

By applying, in order, commutativity of the top, the right, the outer, the bottom, and the left square of the diagram, we find that

$$\begin{aligned}
\varphi :=&(S_2f \otimes T_2f) \circ (\alpha_x \otimes \beta_x) \circ \otimes \\
=&(S_2f \otimes T_2f) \circ \otimes \circ (\alpha_x \times \beta_x) \\
=&\otimes \circ (S_2f \times T_2f) \circ (\alpha_x \times \beta_x) \\
=&\otimes \circ (\alpha_y \times \beta_y) \circ (S_1f \times T_1f) \\
=&(\alpha_y \otimes \beta_y) \circ \otimes \circ (S_1f \times T_1f) \\
=&(\alpha_y \otimes \beta_y) \circ (S_1f \otimes T_1f) \circ \otimes.
\end{aligned}$$

Now, by the universal property of the tensor product we know that there exists a unique morphism $u\colon S_1x \otimes T_1x \to S_2y \otimes T_2y$ satisfying $\varphi = u \circ \otimes$. It immediately follows that

$$(S_2f \otimes T_2f) \circ (\alpha_x \otimes \beta_x) = (\alpha_y \otimes \beta_y) \circ (S_1f \otimes T_1f)$$

as both are candidates for this unique function u. We may thus conclude that

$$\alpha \otimes \beta\colon S_1 \otimes T_1 \Rightarrow S_2 \otimes T_2$$

is a natural transformation.

Functoriality of the tensor product follows quickly from the construction. If $S \times T \in \mathbf{Rep}_\mathcal{G} \times \mathbf{Rep}_\mathcal{G}$ has the identity $1_S \times 1_T$, then by construction $1_S \otimes 1_T$ is pointwise defined by the diagram below.

$$\begin{tikzcd}
Sx \times Tx \arrow[r, "\otimes"] \arrow[d, "1_{Sx} \times 1_{Tx}"] & Sx \otimes Tx \arrow[d, dotted, "1_{Sx} \otimes 1_{Tx}"] \\
Sx \times Tx \arrow[r, "\otimes"] & Sx \otimes Tx
\end{tikzcd}$$

We can see that $1_{Sx} \otimes 1_{Tx}$ must necessarily be the identity on $Sx \otimes Tx$ for the diagram to commute. Therefore, $1_S \otimes 1_T$ is the identity on $S \otimes T$ as desired.

Similarly, if we are given natural transformations $(\alpha^1 \times \beta^1)\colon (S_1 \times T_1) \Rightarrow (S_2 \times T_2)$ and $(\alpha^2 \times \beta^2)\colon (S_2 \times T_2) \Rightarrow (S_3 \times T_3)$ and we write $\alpha \times \beta$ for their composition, then we obtain the diagram below.

$$\begin{tikzcd}
S_1x \times T_1x \arrow[r, "\otimes"] \arrow[d, "\alpha^1_x \times \beta^1_x"'] \arrow[dr] \arrow[dd, bend right=60, "\alpha_x \times \beta_x"'] & S_1x \otimes T_1x \arrow[d, dashed, "\exists! \alpha^1_x \otimes \beta^1_x"] \arrow[dd, dashed, bend left=60, "\exists! \alpha_x \otimes \beta_x"] \\
S_2x \times T_2x \arrow[r, "\otimes"] \arrow[d, "\alpha^2_x \times \beta^2_x"'] \arrow[dr] & S_2x \otimes T_2x \arrow[d, dashed, "\exists! \alpha^2_x \otimes \beta^2_x"] \\
S_3x \times T_3x \arrow[r, "\otimes"] & S_3x \otimes T_3x
\end{tikzcd}$$

By combining the commutativity of the left and middle squares with the universal property of the tensor product we find that there must exist a unique morphism $S_1x \otimes T_1x \to S_3x \otimes T_3x$, which implies equality of $\alpha_x \otimes \beta_x$ and $(\alpha^2_x \otimes \beta^2_x) \circ (\alpha^1_x \otimes \beta^1_x)$.

We conclude our proof that the tensor product of representations extends to a bifunctor. $\square$

Definition 6.4. A **monoidal category** is a category $\mathcal{C}$ with the following data.

- A bifunctor $- \otimes -\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the **tensor product**;
- An object $\mathbb{1} \in \mathcal{C}$ called the **tensor unit**;
- A natural isomorphism $\alpha\colon ((-) \otimes (-)) \otimes (-) \xrightarrow{\sim} (-) \otimes ((-) \otimes (-))$ with components $\alpha_{c,d,e}\colon (c \otimes d) \otimes e \xrightarrow{\sim} c \otimes (d \otimes e)$, called the **associator**;
- A natural isomorphism $\lambda\colon (\mathbb{1} \otimes (-)) \xrightarrow{\sim} (-)$ with components $\lambda_c\colon \mathbb{1} \otimes c \xrightarrow{\sim} c$, called the **left unitor**;
- A natural isomorphism $\rho\colon ((-) \otimes \mathbb{1}) \xrightarrow{\sim} (-)$ with components $\rho_c\colon c \otimes \mathbb{1} \xrightarrow{\sim} c$, called the **right unitor**;

such that the following two identities hold.

- The triangle identity.

$$\begin{tikzcd}
(T \otimes \mathbb{1}) \otimes S \arrow[rr, "\alpha_{T,\mathbb{1},S}"] \arrow[dr, "\rho_T \otimes 1_S"'] & & T \otimes (\mathbb{1} \otimes S) \arrow[dl, "1_T \otimes \lambda_S"] \\
& T \otimes S &
\end{tikzcd}$$

- The pentagon identity.

$$\begin{tikzcd}
& (T \otimes U) \otimes (V \otimes W) \arrow[dr, "\alpha_{T,U,V \otimes W}"] & \\
((T \otimes U) \otimes V) \otimes W \arrow[ur, "\alpha_{T \otimes U,V,W}"] \arrow[d, "\alpha_{T,U,V} \otimes 1_W"'] & & T \otimes (U \otimes (V \otimes W)) \\
(T \otimes (U \otimes V)) \otimes W \arrow[r, "\alpha_{T,U \otimes V,W}"'] & T \otimes ((U \otimes V) \otimes W) \arrow[ur, "1_T \otimes \alpha_{U,V,W}"'] &
\end{tikzcd}$$

The prototypical example of a monoidal category is **Vec** with the usual tensor product and tensor unit k. Given vector spaces $V, W, U \in \mathrm{ob}(\mathbf{Vec})$, the associator is given by

$$\begin{aligned} \alpha_{V,W,U}\colon (V \otimes W) \otimes U &\xrightarrow{\sim} V \otimes (W \otimes U), \\ (v \otimes w) \otimes u &\mapsto v \otimes (w \otimes u). \end{aligned} \tag{6.1}$$

The left unitor is given by

$$\begin{aligned} \lambda_V \colon k \otimes V &\xrightarrow{\sim} V, \\ c \otimes v &\longmapsto cv \end{aligned} \tag{6.2}$$

and the right unitor by

$$\begin{aligned} \rho_V \colon V \otimes k &\xrightarrow{\sim} V, \\ v \otimes c &\longmapsto cv. \end{aligned} \tag{6.3}$$

Verifying that the triangle and pentagon identities hold is a straightforward calculation. By using these associators and unitors, we can prove that the category of $\mathcal{G}$-representations must also be a monoidal category.

Theorem 6.5. $\mathbf{Rep}_{\mathcal{G}}$ is a monoidal category with the tensor product from Proposition **6.3** and the trivial representation as tensor unit.

Proof. Let $x \in \operatorname{ob}\mathcal{G}$ and $T, U, V \in \operatorname{ob}(\mathbf{Rep}_{\mathcal{G}})$. Let $\alpha^{(x)}_{T,U,V}$ be the component of the associator in **Vec** of the vector spaces Tx, Ux, and Vx, cf. (6.1). Similarly, let $\lambda^{(x)}_T$ and $\rho^{(x)}_T$ be the components of the left and right unitors in **Vec** of the vector space Tx, cf. (6.2), and (6.3) respectively. These pointwise definitions constitute the components of natural isomorphisms

$$\begin{aligned} \alpha_{T,U,V} \colon &\ (T \otimes U) \otimes V \Rightarrow T \otimes (U \otimes V); \\ \lambda_T \colon &\ \mathbb{1} \otimes T \Rightarrow T; \\ \rho_T \colon &\ T \otimes \mathbb{1} \Rightarrow T. \end{aligned}$$

These transformations are themselves components of the associator, the left unitor, and the right unitor which give $\mathbf{Rep}_{\mathcal{G}}$ the structure of a monoidal category.

Verification of naturality is a straightforward calculation. For the sake of exposition we shall show that the left unitor is a natural isomorphism. Recall that the left unitor of a vector space V is given by

$$\begin{aligned} \lambda_V \colon k \otimes V &\to V; \\ c \otimes v &\mapsto cv. \end{aligned}$$

Let $T \in \operatorname{ob}(\mathbf{Rep}_{\mathcal{G}})$ and let $f \colon x \to y$ be a morphism in $\mathcal{G}$. Consider the diagram below.

$$\begin{tikzcd} k \otimes T(x) \arrow[r, "\lambda_{T(x)}"] \arrow[d, "\mathbb{1}(f)\otimes T(f)"'] & T(x) \arrow[d, "T(f)"] \\ k \otimes T(y) \arrow[r, "\lambda_{T(y)}"] & T(y) \end{tikzcd}$$

Commutativity can be verified by evaluating in $c \otimes v \in k \otimes T(x)$. We have $T(f)(\lambda_{T(x)}(c \otimes v)) = T(f)(cv)$ and $\lambda_{T(y)}((\mathbb{1}(f) \otimes T(f))(c \otimes v)) = \lambda_{T(y)}(c \otimes T(f)(v)) = cT(f)(v) = T(f)(cv)$. We thus find that $\lambda_T \colon \mathbb{1} \otimes T \Rightarrow T$ is a natural isomorphism. Now suppose $\alpha \colon S \Rightarrow T$ is a natural transformation in $\mathbf{Rep}_{\mathcal{G}}$. Consider the diagram below.

$$\begin{tikzcd} \mathbb{1} \otimes S \arrow[r, Rightarrow, "\lambda_S"] \arrow[d, Rightarrow, "1_{\mathbb{1}}\otimes\alpha"'] & S \arrow[d, Rightarrow, "\alpha"] & & k \otimes S(x) \arrow[r, "\lambda_{S(x)}"] \arrow[d, "1_k\otimes\alpha_x"'] & S(x) \arrow[d, "\alpha_x"] \\ \mathbb{1} \otimes T \arrow[r, Rightarrow, "\lambda_T"] & T \arrow[r, phantom, "\overset{\text{Evaluating in } x}{\rightsquigarrow}"] & & k \otimes T(x) \arrow[r, "\lambda_{T(x)}"] & T(x) \end{tikzcd}$$

The diagram on the left commutes if and only if all components the natural transformations commute, i.e., if the diagram on right commutes for arbitrary $x \in \operatorname{ob}\mathcal{G}$. This can be shown via a straightforward calculation. Let $c \otimes v \in k \otimes S(x)$. Then,

$$\begin{aligned}\alpha_x(\lambda_{S(x)}(c \otimes v)) &= \alpha_x(cv)\\ &= c\alpha_x(v)\\ &= \lambda_{T(x)}(c \otimes \alpha_x(v))\\ &= \lambda_{T(x)}((1_k \otimes \alpha_x)(c \otimes v)).\end{aligned}$$

We may thus conclude that $\lambda\colon \mathbb{1} \otimes (-) \Rightarrow (-)$ is a natural isomorphism. Verifying that $\alpha_{T,U,V}$, ρ_T, α, and ρ are natural can be done in the same way.

One can verify that both the triangle identity and the pentagon identity hold by evaluating in an object $x \in \operatorname{ob}\mathcal{G}$. Then, commutativity of the resulting diagrams may be verified by a straightforward calculation. We may conclude that $\mathbf{Rep}_{\mathcal{G}}$ is a monoidal category. □

The tensor product in $\mathbf{Rep}_{\mathcal{G}}$ inherits some more structure from the tensor product of vector spaces. In $\mathbf{Vec}$, there is always an isomorphism $V \otimes W \xrightarrow{\sim} W \otimes V$ given by $v \otimes w \mapsto w \otimes v$. Such a map is called a braiding, provided it obeys certain coherence conditions.

Theorem 6.6. $\mathbf{Rep}_{\mathcal{G}}$ is a symmetric monoidal category, meaning there is a natural isomorphism

$$\sigma_{S,T}\colon S \otimes T \Rightarrow T \otimes S$$

for all $S, T \in \operatorname{ob}(\mathbf{Rep}_{\mathcal{G}})$ called the braiding map such that the conditions are met.

- For all $S, T \in \operatorname{ob}(\mathbf{Rep}_{\mathcal{G}})$, we have

$$B_{S,T} \circ B_{T,S} = 1_{S \otimes T}.$$

- For all $S, T, U \in \operatorname{ob}(\mathbf{Rep}_{\mathcal{G}})$, the following diagram commutes.

$$\begin{tikzcd}
(S \otimes T) \otimes U \arrow[r, "\alpha_{S,T,U}"] \arrow[d, "B_{S,T} \otimes 1_U"] & S \otimes (T \otimes U) \arrow[r, "B_{S,T\otimes U}"] & (T \otimes U) \otimes S \arrow[d, "\alpha_{T,U,S}"] \\
(T \otimes S) \otimes U \arrow[r, "\alpha_{T,S,U}"] & T \otimes (S \otimes U) \arrow[r, "1_T \otimes B_{S,U}"] & T \otimes (U \otimes S)
\end{tikzcd}$$

Again, this braiding map is defined pointwise. Let $\sigma^{(x)}_{S,T}\colon Sx \otimes Tx \to Tx \otimes Sx$ be the braiding map in $\mathbf{Vec}$. Define $\sigma_{S,T}$ to be the natural isomorphism whose components are $\sigma^{(x)}_{S,T}$. Verifying that $\sigma_{S,T}$ is a natural transformation and that the two conditions given above hold is straightforward.

Definition 6.7. Let $S, T \in \operatorname{ob}(\mathbf{Rep}_{\mathcal{G}})$. Define the hom representation, denoted $[S,T]$, by

$$\begin{aligned}[S,T](x) &\coloneqq \mathbf{Vec}(S(x), T(x));\\ [S,T](f\colon x \to y) &\coloneqq \varphi \mapsto T(f) \circ \varphi \circ S(f^{-1}).\end{aligned}$$

Define the internal hom functor by

$$\begin{aligned}[-,-]\colon \mathbf{Rep}_{\mathcal{G}}^{\mathrm{op}} \times \mathbf{Rep}_{\mathcal{G}} &\to \mathbf{Rep}_{\mathcal{G}};\\ (S,T) &\mapsto [S,T];\\ ((\alpha,\beta)\colon (S,T) \Rightarrow (S',T')) &\mapsto ([\alpha,\beta]\colon [S,T] \Rightarrow [S',T']),\end{aligned}$$

where $[\alpha,\beta]$ is the natural transformation with components

$$\begin{aligned}[\alpha,\beta]_x\colon \mathbf{Vec}(Sx,Tx) &\to \mathbf{Vec}(S'x,T'x);\\ \varphi &\mapsto \beta_x \circ \varphi \circ \alpha_x\end{aligned}$$

for $x \in \operatorname{ob}\mathcal{G}$.

First let us verify that $[\alpha, \beta]$ is indeed natural. Let $f\colon x \to y$ be a morphism in $\mathcal{G}$. We must prove the commutativity of the diagram below.

$$\begin{tikzcd}
\mathbf{Vec}(Sx, Tx) \arrow[r, "{[\alpha,\beta]_x}"] \arrow[d, "{[S,T](f)}"] & \mathbf{Vec}(S'x, T'x) \arrow[d, "{[S',T'](f)}"] \\
\mathbf{Vec}(Sy, Ty) \arrow[r, "{[\alpha,\beta]_y}"] & \mathbf{Vec}(S'y, T'y)
\end{tikzcd}$$

Picking $\varphi \in \mathbf{Vec}(Sx, Tx)$ and applying naturality of α and β yields

$$\begin{aligned}
[S', T'](f)([\alpha, \beta]_x(\varphi)) &= T'(f) \circ \beta_x \circ \varphi \circ \alpha_x \circ S'f^{-1} \\
&= \beta_y \circ T(f) \circ \varphi \circ Sf^{-1} \circ \alpha_y \\
&= [\alpha, \beta]_y([S', T'](f)(\varphi))
\end{aligned}$$

as desired. Functoriality of $[-,-]$ follows immediately from the observation that postcomposition works covariantly and precomposition works contravariantly.

Note that the definition of $[S, T](f)$ makes explicit use of the invertibility of f. Given $\varphi\colon Sx \to Tx$, the map $[S, T](f)(\varphi)$ is the unique map for which the following diagram commutes.

$$\begin{tikzcd}
Sx \arrow[r, "\varphi"] \arrow[d, "Sf"] & Tx \arrow[d, "Tf"] \\
Sy \arrow[r, "{[S,T](f)(\varphi)}"] & Ty
\end{tikzcd}$$

To illustrate the necessity of invertibility, consider the category $\mathcal{C}$ consisting of objects $\operatorname{ob}\mathcal{C} = \{x, y\}$, and just one non-identity morphism $f\colon x \to y$. Now that $\mathcal{C}$ is not a groupoid as f is not invertible, a representation can send f to a non-invertible linear map. Let S and T both be the trivial representation except that $S(f) = 0$ instead of $S(f) = 1_k$. The situation is pictured below.

$$\begin{tikzcd}
k \arrow[loop above, "1_k"] \arrow[d, "0"] & & x \arrow[loop above, "1_x"] \arrow[d, "f"] & & k \arrow[loop above, "1_k"] \arrow[d, "1_k"] \\
k \arrow[loop below, "1_k"] & \arrow[l, squiggly, "\text{Applying } S"'] & y \arrow[loop below, "1_y"] & \arrow[r, squiggly, "\text{Applying } T"] & k \arrow[loop below, "1_k"]
\end{tikzcd}$$

Note that there does not exist a morphism $[S, T](f)(\varphi)$ for which the diagram below commutes.

$$\begin{tikzcd}
k \arrow[r, "1_k"] \arrow[d, "0"] & k \arrow[d, "1_k"] \\
k \arrow[r, "{[T,S](f)(1_k)}"] & k
\end{tikzcd}$$

Theorem 6.8. The category $\mathbf{Rep}_{\mathcal{G}}$ is monoidally closed, that is, the functor $(-) \otimes T$ has a right adjoint $[T, -]$ for all fixed $T \in \operatorname{ob}(\mathbf{Rep}_{\mathcal{G}})$.

Proof. In order to prove that the given functors are adjoint we shall provide a family of bijections

$$\Phi_{S,U}\colon \mathbf{Rep}_{\mathcal{G}}(S \otimes T, U) \xrightarrow{\sim} \mathbf{Rep}_{\mathcal{G}}(S, [T, U])$$

which is natural in $S, U \in \operatorname{ob}(\mathbf{Rep}_{\mathcal{G}})$. Fix some $\alpha \in \mathbf{Rep}_{\mathcal{G}}(S \otimes T, U)$. Define $\Phi_{S,U}(\alpha) \in \mathbf{Rep}_{\mathcal{G}}(S, [T, U])$ by

$$\begin{aligned}\Phi_{S,U}(\alpha)_x \colon Sx &\to \mathbf{Vec}(Tx, Ux),\\ v &\mapsto (w \mapsto \alpha_x(v \otimes w))\end{aligned}$$

for $x \in \operatorname{ob}\mathcal{G}$. First we shall verify that $\Phi_{S,U}(\alpha)$ is natural. Let $f\colon x \to y$ be a morphism in $\mathcal{G}$ and let $v \in Sx$. Then,

$$\begin{aligned}([T,U](f) \circ \Phi_{S,U}(\alpha)_x)(v) &= ([T,U](f))(w \mapsto \alpha_x(v \otimes w))\\ &= U(f) \circ (w \mapsto \alpha_x(v \otimes w)) \circ T(f^{-1})\\ &= u \mapsto U(f)(\alpha_x(v \otimes T(f^{-1}(u))))\\ &= u \mapsto \alpha_y((S(f) \otimes T(f))(v \otimes T(f^{-1})(u))\\ &= u \mapsto \alpha_y(S(f)(v) \otimes u)\\ &= \Phi_{S,U}(\alpha)_y(S(f)(v))\\ &= (\Phi_{S,U}(\alpha)_y \circ S(f))(v)\end{aligned}$$

as desired. Next we will define an inverse of $\Phi_{S,U}$. Let $\beta \in \mathbf{Rep}_{\mathcal{G}}(S, [T, U])$ and define $\Psi_{S,U}(\beta) \in \mathbf{Rep}_{\mathcal{G}}(S \otimes T, U)$ by

$$\begin{aligned}\Psi_{S,U}(\beta)_x \colon S(x) \otimes T(x) &\to U(x),\\ (v \otimes w) &\mapsto (\beta_x(v))(w).\end{aligned}$$

First we check that $\Psi_{S,U}(\beta)$ is natural. Let $f\colon x \to y$ be a morphism in $\mathcal{G}$ and let $v \otimes w \in Sx \otimes Tx$. Then,

$$\begin{aligned}(\Psi_{S,U}(\beta)_y \circ (S(f) \otimes T(f)))(v \otimes w) &= \Psi_{S,U}(\beta)_y(S(f)(v) \otimes T(f)(w))\\ &= (\beta_y(S(f)(v))(T(f)(w)))\\ &= (U(f) \circ \beta_x(v) \circ T(f^{-1}))(T(f)(w))\\ &= U(f)((\beta_x(v))(w))\\ &= (U(f) \circ \Psi_{S,U}(\beta)_x)(v \otimes w),\end{aligned}$$

meaning $\Psi_{S,U}(\beta)$ is a natural transformation. We will now verify that $\Phi_{S,U}$ and $\Psi_{S,U}$ are indeed inverses. Fix $\alpha \in \mathbf{Rep}_{\mathcal{G}}(S \otimes T, U)$ and $\beta \in \mathbf{Rep}_{\mathcal{G}}(S, [T, U])$. Then,

$$\begin{aligned}\Psi_{S,U}(\Phi_{S,U}(\alpha))_x &= (v \otimes w) \mapsto (\Phi_{S,U}(\alpha)_x(v))(w)\\ &= (v \otimes w) \mapsto (w' \mapsto \alpha_x(v \otimes w'))(w)\\ &= (v \otimes w) \mapsto \alpha_x(v \otimes w)\\ &= \alpha_x,\\ \Phi_{S,U}(\Psi_{S,U}(\beta))_x &= v \mapsto (w \mapsto \Psi_{S,U}(\beta)_x(v \otimes w))\\ &= v \mapsto (w \mapsto (\beta_x(v)(w)))\\ &= \beta_x\end{aligned}$$

as desired. Thus, $\Phi_{S,U}$ and $\Psi_{S,U}$ are indeed inverses. Finally, we must show that Φ is natural in S, U. Let $(\alpha, \alpha')\colon (S, S') \Rightarrow (U, U')$ in $\mathbf{Rep}_{\mathcal{G}}^2$. Consider the diagram below.

$$\begin{tikzcd} \mathbf{Rep}_{\mathcal{G}}(S \otimes T, S') \arrow[r, "\Phi_{S,S'}"] \arrow[d, "\mathbf{Rep}_{\mathcal{G}}(\alpha \otimes T, \alpha')"] & \mathbf{Rep}_{\mathcal{G}}(S, [T, S']) \arrow[d, "\mathbf{Rep}_{\mathcal{G}}(\alpha, [T, \alpha'])"] \\ \mathbf{Rep}_{\mathcal{G}}(U \otimes T, U') \arrow[r, "\Phi_{U,U'}"] & \mathbf{Rep}_{\mathcal{G}}(U, [T, U']) \end{tikzcd}$$

We shall prove the commutativity of the diagram by evaluating in $\beta\colon S\otimes T \Rightarrow S'$ and then considering the component at $x \in \operatorname{ob}\mathcal{G}$. We have

$$\begin{aligned}
(\mathbf{Rep}_{\mathcal{G}}(\alpha,[T,\alpha'])(\Phi_{S,S'}(\beta)))_x &= [T,\alpha']_x \circ \Psi_{S,S'}(\beta)_x \circ \alpha_x \\
&= (\alpha'_x \circ -) \circ (u \mapsto (w \mapsto \beta_x(u\otimes w))) \circ \alpha_x \\
&= v \mapsto (\alpha'_x \circ -) \circ (w \mapsto \beta_x(\alpha_x(v)\otimes w)) \\
&= v \mapsto (w \mapsto \alpha'_x(\beta_x(\alpha_x(v)\otimes w))) \\
&= v \mapsto (w \mapsto \alpha'_x(\beta_x(\alpha_x(v)\otimes w))) \\
&= v \mapsto (w \mapsto (\alpha'_x \circ \beta_x \circ (\alpha\otimes T)_x)(v\otimes w)) \\
&= v \mapsto (w \mapsto \mathbf{Rep}_{\mathcal{G}}(\alpha\otimes T,\alpha')(\beta)_x(v\otimes w)) \\
&= \Phi_{U,U'}(\mathbf{Rep}_{\mathcal{G}}(\alpha\otimes T,\alpha')(\beta))_x.
\end{aligned}$$

Since this holds for arbitrary x and β we find that Φ is natural. We may conclude that $(-)\otimes T$ is a left adjoint of $[T,-]$. Thus, $\mathbf{Rep}_{\mathcal{G}}$ is monoidally closed. □

Definition 6.9. Let $\mathcal{C}$ be a symmetric monoidal category with tensor unit $\mathbb{1}$, associator α, left unitor λ, and right unitor ρ. An object $T \in \operatorname{ob}\mathcal{C}$ is called **dualizable** if there exists an object T^*, a morphism $\mathrm{ev}_T\colon T^*\otimes T \to \mathbb{1}$, and a morphism $i_T\colon \mathbb{1} \to T\otimes T^*$ such that the following diagrams commute.

$$\begin{tikzcd}
T^*\otimes(T\otimes T^*) \arrow[d, "\alpha^{-1}_{T^*,T,T^*}"] & T^*\otimes\mathbb{1} \arrow[l, "1_{T^*}\otimes i_T"'] \arrow[d, "\lambda^{-1}_{T^*}\circ\rho_{T^*}"] \\
(T^*\otimes T)\otimes T^* \arrow[r, "\mathrm{ev}_T\otimes 1_{T^*}"] & \mathbb{1}\otimes T^*
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
(T\otimes T^*)\otimes T \arrow[d, "\alpha_{T,T^*,T}"] & \mathbb{1}\otimes T \arrow[l, "i_T\otimes 1_T"'] \arrow[d, "\rho^{-1}_T\circ\lambda_T"] \\
T\otimes(T^*\otimes T) \arrow[r, "1_T\otimes\mathrm{ev}_T"] & T\otimes\mathbb{1}
\end{tikzcd}$$

The object T^* is called the **dual object** and the maps ev_T and i_T are called the **evaluation map** and **coevaluation map** respectively.

In **Vec**, the dualizable objects are precisely the finite dimensional vector spaces. Given a finite-dimensional vector space V, the evaluation map is given by $\varphi\otimes v \mapsto \varphi(v)$. The coevaluation map is given by $1 \mapsto \sum_i v_i \otimes v_i^*$, where v_i is a basis of V and v_i^* its dual. One can prove that this map does not depend on the choice of basis. Do note that the definition of the coevaluation map requires V to be finite dimensional. We claim that this dualizability can be extended from **Vec** to $\mathbf{Rep}_{\mathcal{G}}$. For this, we must first define dual representations which we can do by lifting the notion of duals of vector spaces.

Definition 6.10. Let $T \in \operatorname{ob}(\mathbf{Rep}_{\mathcal{G}})$. Define the **dual representation** T^* as the representations with $T^*(x) = (T(x))^*$ and $T^*(f) = (T(f^{-1}))^*$. where f is a morphism in $\mathcal{G}$.

We are now ready to prove that $\mathbf{Rep}_{\mathcal{G}}$ contains dualizable objects.

Theorem 6.11. All finite-dimensional representations $T \in \operatorname{ob}(\mathbf{Rep}_{\mathcal{G}})$ are dualizable.

Proof. Let T be a finite-dimensional vector space. Define the evaluation map ev_T as the natural transformation whose components are given by

$$\begin{aligned}
\mathrm{ev}_T^{(x)}\colon T^*(x)\otimes T(x) &\to k; \\
f\otimes v &\mapsto f(v)
\end{aligned}$$

for $x \in \operatorname{ob}\mathcal{G}$. Define the coevaluation map as the natural transformation whose components are given by

$$\begin{aligned} i_T^{(x)} \colon k &\to T(x) \otimes T^*(x); \\ \lambda &\mapsto \sum_i \lambda v_i \otimes v_i^*. \end{aligned}$$

First we shall verify naturality, i.e., commutativity of the diagrams

$$\begin{tikzcd} (Tx)^* \otimes Tx \arrow[r, "\mathrm{ev}_T^{(x)}"] \arrow[d, "T^*f \otimes Tf"'] & k \arrow[d, "1_k"] \\ (Ty)^* \otimes Ty \arrow[r, "\mathrm{ev}_T^{(y)}"] & k \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} k \arrow[r, "i_T^{(x)}"] \arrow[d, "1_k"'] & Tx \otimes (Tx)^* \arrow[d, "Tf \otimes T^*f"] \\ k \arrow[r, "i_T^{(y)}"] & Ty \otimes (Ty)^* \end{tikzcd}$$

where $f \colon x \to y$ is a morphism in $\mathcal{G}$. Let $\varphi \otimes v \in (Tx)^* \otimes Tx$. Then,

$$\begin{aligned} \mathrm{ev}_T^{(y)}((T^*f \otimes Tf)(\varphi \otimes v)) &= \mathrm{ev}_T^{(y)}\left((Tf^{-1})^*(\varphi) \otimes Tf(v)\right) \\ &= (\varphi \circ Tf^{-1})(Tf(v)) \\ &= \varphi(v) \\ &= 1_k(\mathrm{ev}_T^{(x)}(\varphi \otimes v)) \end{aligned}$$

as desired. Next suppose v_i is a basis of Tx with dual basis v_i^*. Let $\lambda \in k$. Then,

$$\begin{aligned} (Tf \otimes T^*f)\left(i_T^{(x)}(\lambda)\right) &= (Tf \otimes T^*f)\left(\lambda \sum_i v_i \otimes v_i^*\right) \\ &= \lambda \sum_i Tf(v_i) \otimes T^*f(v_i^*) \\ &= \lambda \sum_i Tf(v_i) \otimes (v_i^* \circ Tf^{-1}) \\ &= \lambda \sum_i v_i \otimes v_i^* \\ &= i_T^{(y)}(1_k(\lambda)). \end{aligned}$$

We may conclude that both ev_T and i_T are natural transformations. Now we verify the commutativity of the diagrams from Definition 6.9 by considering the component at x, yielding the diagrams

$$\begin{tikzcd} T^*x \otimes (Tx \otimes T^*x) \arrow[d, "(\alpha^{(x)}_{T^*,T,T^*})^{-1}"] & T^*x \otimes k \arrow[l, "1_{T^*x} \otimes i_T^{(x)}"'] \arrow[d, "\left(\lambda^{(x)}_{T^*}\right)^{-1} \circ \rho^{(x)}_{T^*}"] \\ (T^*x \otimes Tx) \otimes T^*x \arrow[r, "\mathrm{ev}_T^{(x)} \otimes 1_{T^*x}"] & k \otimes T^*x \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} (Tx \otimes T^*x) \otimes Tx \arrow[d, "\alpha^{(x)}_{T,T^*,T}"] & k \otimes Tx \arrow[l, "i_T^{(x)} \otimes 1_{Tx}"'] \arrow[d, "(\rho_T^{(x)})^{-1} \circ \lambda_T^{(x)}"] \\ Tx \otimes (T^*x \otimes Tx) \arrow[r, "1_T \otimes \mathrm{ev}_T"] & Tx \otimes k \end{tikzcd}$$

Commutativity of these diagrams can be verified by a simple calculation. First consider the diagram on the left. Let v_j^* be one of the dual basis elements. Then,

$$\begin{aligned}
&\left(\mathrm{ev}_T^{(x)} \otimes 1_{T^*x}\right)\left(\left(\alpha_{T^*,T,T^*}^{(x)}\right)^{-1}\left(\left(1_{T^*x} \otimes i_T^{(x)}\right)(v_j^* \otimes 1)\right)\right) \\
&= \left(\mathrm{ev}_T^{(x)} \otimes 1_{T^*x}\right)\left(\left(\alpha_{T^*,T,T^*}^{(x)}\right)^{-1}\left(v_j^* \otimes \sum_i (v_i \otimes v_i^*)\right)\right) \\
&= \left(\mathrm{ev}_T^{(x)} \otimes 1_{T^*x}\right)\left(\sum_i (v_j^* \otimes v_i) \otimes v_i^*\right) \\
&= \sum_i (v_j^*(v_i)) \otimes v_i \\
&= 1 \otimes v_j \\
&= \left(\lambda_{T^*}^{(x)}\right)^{-1}(\rho_{T^*}^{(x)}(v_j^* \otimes 1)).
\end{aligned}$$

Since this holds for all dual basis vectors v_j^* and all elements of $T^*x \otimes k$ can be written as $\varphi \otimes 1$ for some $\varphi \in T^*x$, we find that the diagram on the left commutes. Commutativity of the right diagram may be verified in a similar way. We may conclude that all finite-dimensional representations are dualizable. $\square$

C7 | Induction and Coinduction as Adjoint Functors

In this chapter we formally define induction and coinduction as adjoints to the restriction functor (Definition 5.5). These adjoints can be given by universal properties using Kan extensions. In Section 7.2 we provide an explicit construction of induced and coinduced representations. By using this explicit construction we can give conditions for the induced restriction to be finite-dimensional. We finish this chapter by investigating the relation between the coinduced representation and the monoidal structure on $\mathbf{Rep}_{\mathcal{G}}$ we described in Chapter 6.

Throughout this chapter, $\mathcal{G}$ will be a fixed groupoid with subgroupoid $\mathcal{H}$. We assume $\mathcal{H}$ is wide, which may be achieved by taking the natural extension of $\mathcal{H}$, cf. Section 4.1.

Theorem 7.1. The restriction functor $\mathrm{Res}^{\mathcal{G}}_{\mathcal{H}}$ admits both a left and right adjoint.

Proof. The existence of the right adjoint is given by Corollary 3 in Section X.3 of [6]. The existence of a left adjoint is given by a dual argument. □

Definition 7.2. Define the **induction functor**

$$\mathrm{Ind}^{\mathcal{G}}_{\mathcal{H}} \colon \mathbf{Rep}_{\mathcal{H}} \to \mathbf{Rep}_{\mathcal{G}}$$

and the **coinduction functor**

$$\mathrm{Coind}^{\mathcal{G}}_{\mathcal{H}} \colon \mathbf{Rep}_{\mathcal{H}} \to \mathbf{Rep}_{\mathcal{G}}$$

as the left and right adjoint of $\mathrm{Res}^{\mathcal{G}}_{\mathcal{H}}$ respectively.

By applying the properties of adjoints we rediscover many known properties of induced representations. For example, the cocontinuity of left adjoints implies that

$$\mathrm{Ind}^{\mathcal{G}}_{\mathcal{H}}(T \oplus S) = \mathrm{Ind}^{\mathcal{G}}_{\mathcal{H}}(T) \oplus \mathrm{Ind}^{\mathcal{G}}_{\mathcal{H}}(S)$$

for $T, S \in \mathrm{ob}(\mathbf{Rep}_{\mathcal{H}})$. Similarly, continuity of right adjoints implies

$$\mathrm{Coind}^{\mathcal{G}}_{\mathcal{H}}(T \times S) = \mathrm{Coind}^{\mathcal{G}}_{\mathcal{H}}(T) \times \mathrm{Coind}^{\mathcal{G}}_{\mathcal{H}}(S).$$

Indeed, as mentioned in Chapter 6 both $\mathbf{Rep}_{\mathcal{H}}$ and $\mathbf{Rep}_{\mathcal{G}}$ are abelian. This implies that the direct product and the direct sum coincide, meaning induction and coinduction both preserve the direct sum and the direct product.

Another consequence of adjointness is the following.

Theorem 7.3 (Induction in stages)**.** Given groupoids $\mathcal{K} \subset \mathcal{H} \subset \mathcal{G}$ and a representation $T \in \mathrm{ob}(\mathbf{Rep}_{\mathcal{H}})$, we have

$$\mathrm{Ind}^{\mathcal{G}}_{\mathcal{K}} T = \mathrm{Ind}^{\mathcal{G}}_{\mathcal{H}} \left(\mathrm{Ind}^{\mathcal{H}}_{\mathcal{K}} T\right) \quad \text{and} \quad \mathrm{Coind}^{\mathcal{G}}_{\mathcal{K}} T = \mathrm{Coind}^{\mathcal{G}}_{\mathcal{H}} \left(\mathrm{Coind}^{\mathcal{H}}_{\mathcal{K}} T\right)$$

Proof. Let $J\colon \mathcal{K} \to \mathcal{H}$ be the inclusion of $\mathcal{K}$ in $\mathcal{H}$. Notice that the inclusion of $\mathcal{K}$ in $\mathcal{G}$ is equal to $I \circ J$, meaning $\operatorname{Res}^{\mathcal{H}}_{\mathcal{K}} \circ \operatorname{Res}^{\mathcal{G}}_{\mathcal{H}} = \operatorname{Res}^{\mathcal{G}}_{\mathcal{K}}$ by definition of the restriction functor. But by Lemma 2.9, we find that $\operatorname{Ind}^{\mathcal{G}}_{\mathcal{H}} \circ \operatorname{Ind}^{\mathcal{H}}_{\mathcal{K}}$ and $\operatorname{Coind}^{\mathcal{G}}_{\mathcal{H}} \circ \operatorname{Coind}^{\mathcal{H}}_{\mathcal{K}}$ are respectively the left and right adjoint of $\operatorname{Res}^{\mathcal{H}}_{\mathcal{K}} \circ \operatorname{Res}^{\mathcal{G}}_{\mathcal{H}} = \operatorname{Res}^{\mathcal{G}}_{\mathcal{K}}$. We thus find that $\operatorname{Ind}^{\mathcal{G}}_{\mathcal{H}} \circ \operatorname{Ind}^{\mathcal{H}}_{\mathcal{K}} = \operatorname{Ind}^{\mathcal{G}}_{\mathcal{K}}$ and $\operatorname{Coind}^{\mathcal{G}}_{\mathcal{H}} \circ \operatorname{Coind}^{\mathcal{H}}_{\mathcal{K}} = \operatorname{Coind}^{\mathcal{G}}_{\mathcal{K}}$. □

7.1 Kan Extensions

Let us go over the existence of the left and right adjoint of the restriction functor in some more detail. In essence we want to extend the domain of a functor $T\colon \mathcal{H} \to \mathbf{Vec}$ to $\mathcal{G}$ in some universal way. In 1960 Daniel Kan gave an explicit construction for such extensions, now known as Kan extensions. In this section we will mostly focus on the right Kan extension. The left Kan extension may be obtained by reversing some of the arrows. The definitions and theorems in this section are taken from Section X.3 in [6].

Definition 7.4. Let $I\colon \mathcal{H} \to \mathcal{G}$ be the inclusion functor and let $T\colon \mathcal{H} \to \mathbf{Vec}$ be a representation. A **right Kan extension of T along I** is a functor $R\colon \mathcal{G} \to \mathbf{Vec}$ and a natural transformation $\varepsilon\colon RI \Rightarrow T$ such that for all pairs $S\colon \mathcal{H} \to \mathbf{Vec}$ and $\alpha\colon SI \Rightarrow T$, there exists a unique $\sigma\colon S \Rightarrow R$ such that $\alpha = \varepsilon \circ \sigma I$.

Viewed as a diagram we have the following.

$$\begin{tikzcd}
\mathcal{H} \arrow[rr, "T"] \arrow[dr, "I"'] \arrow[dr, phantom, shift left=6, "\Uparrow \epsilon"] & & \mathbf{Vec} \\
& \mathcal{G} \arrow[ur, "R"] \arrow[ur, bend right=60, "S"'] \arrow[ur, phantom, shift right=5, "\Leftarrow \exists! \sigma"] &
\end{tikzcd}$$

A direct result of the universal property above is that right Kan extensions are unique up to unique natural isomorphism. For this reason we speak of *the* right Kan extension. We will denote the right Kan extension of T along I by $\operatorname{Ran}_I(T)$. This object can be explicitly calculated using limits.

Theorem 7.5. Write $R = \operatorname{Ran}_I(T)$. For $x \in \operatorname{ob} \mathcal{G}$, the object Rx can be computed by

$$Rx = \varprojlim \Big((x \downarrow I) \xrightarrow{Q} \mathcal{H} \xrightarrow{T} \mathbf{Vec} \Big). \tag{7.1}$$

Further, if $(Rx, \{\varphi_f\}_f)$ and $(Rx', \{\psi_f\}_f)$ are the limiting cones of Rx and Rx' in the limit above and $g\colon x \to x'$ is a morphism in $\mathcal{G}$, then Rg is the unique morphism satisfying $\psi_{f'} \circ Rg = \varphi_{f'g}$ for all $f' \in (x' \downarrow I)$. The existence and uniqueness of Rg is implied by the universal property of limits.

Proof. This is Theorem 1 in Section X.3 of [6]. □

Diagrammatically, Rg is given by the diagram below for all $\langle h\colon f \to f' \rangle \in (x' \downarrow I)$.

$$\begin{tikzcd}
& Rx \arrow[d, dashed, "\exists! Rg"] \arrow[ddl, bend right, "\varphi_{fg}"'] \arrow[ddr, bend left, "\varphi_{f'g}"] & \\
& Ry \arrow[dl, "\psi_f"'] \arrow[dr, "\psi_{f'}"] & \\
TQ(g) \arrow[rr, "TQ(h)"] & & TQ(g')
\end{tikzcd}$$

Generally one may not assume the limits above actually exist. In our case however, all such limits exist as $\mathbf{Vec}$ is complete. Dually, *the* left Kan extension can be given by a colimit which exists by the cocompleteness of **Vec**.

Theorem 7.6. The functor $\mathrm{Res}^{\mathcal{G}}_{\mathcal{H}}$ has a right adjoint that sends all objects $T \in \mathrm{ob}(\mathbf{Rep}_{\mathcal{H}})$ to $\mathrm{Ran}_I(T)$.

Proof. Section X.3 of [6]. ☐

We can explicitly describe the action of this right adjoint on natural transformations. Let $S, T \in \mathrm{ob}(\mathbf{Rep}_{\mathcal{H}})$ and let $\alpha\colon S \Rightarrow T$. Write $R_S = \mathrm{Ran}_I(S)$ and $R_T = \mathrm{Ran}_I(T)$. Consider for all $\langle h\colon f \to f'\rangle \in (x \downarrow I)$ the diagram below.

$$
\begin{tikzcd}
 & R_T(x) \arrow[ddl, bend right, "\psi_f"'] \arrow[ddr, bend left, "\psi_{f'}"] & \\
 & R_S(x) \arrow[u, dashed, "\exists! R\alpha_x"] \arrow[dl, "\varphi_f"'] \arrow[dr, "\varphi_{f'}"] & \\
SQ(f) \arrow[rr, "SQ(h)"] \arrow[d, "\alpha_{Q(f)}"] & & SQ(f') \arrow[d, "\alpha_{Q(f')}"] \\
TQ(f) \arrow[rr, "TQ(h)"] & & TQ(f')
\end{tikzcd}
$$

The bottom square commutes by naturality of α, and the middle and outer triangle commute by definition of $R_S(x)$ and $R_T(x)$. By the universal property, there exists a unique $R\alpha_x\colon R_S(x) \to R_T(x)$ making the whole diagram commute. One can prove that $R\alpha_x$ is natural in x by letting $f\colon x \to y$ be a morphism in $\mathcal{G}$ and proving that both $R\alpha_y \circ R_S(f)$ and $R_T(f) \circ R\alpha_x$ are candidates for an appropriate universal arrow $R_S(x) \to R_T(y)$.

7.2 Explicit Construction of the Limiting Object

In order to actually calculate a coinduced representation we can give an explicit construction of the limiting objects in Theorem 7.5. Let $T \in \mathrm{ob}\,\mathbf{Rep}_{\mathcal{H}}$. Define

$$
Rx := \left\{ (v_f)_f \in \prod_{\langle f, y\rangle \in (x\downarrow I)} T(y) \;\middle|\; T(h)(v_f) = v_{f'} \text{ for all } \langle h\colon f \to f'\rangle \in (x \downarrow I). \right\} \tag{7.2}
$$

and, for $\langle f, y\rangle \in (x \downarrow I)$, define $\varphi_f\colon R(x) \to T(y)$ by projection to the f-th coordinate. In order to avoid clutter we will no longer explicitly write out for which morphisms h, f, and f' the condition $T(h)(v_f) = v_{f'}$ must hold. We claim that $(Rx, \{\varphi_f\}_f)$ is a universal cone of the limit in (7.1). Let $(N, \{\psi_f\}_f)$ be another cone, meaning $TQ(h) \circ \psi_f = \psi_{f'}$.

$$
\begin{tikzcd}
 & N \arrow[ddl, bend right, "\psi_f"'] \arrow[ddr, bend left, "\psi_{f'}"] \arrow[d, dashed, "\exists! u"] & \\
 & Rx \arrow[dl, "\varphi_f"'] \arrow[dr, "\varphi_{f'}"] & \\
TQ(f) \arrow[rr, "TQ(h)"] & & TQ(f')
\end{tikzcd}
$$

Define $u\colon N \to Rx$ as $u = (u_f)_{f\in(x\downarrow I)}$ with $u_f(n) = \psi_f(n)$. Note that $\varphi_f \circ u = \psi_f$ holds by construction. By assumption, $TQ(h) \circ \psi_f = \psi_{f'}$ from which it follows that $T(h)(u(n))_f = (u(n))_{f'}$. In other words, $u(n)$ obeys the condition in (7.2). Hence, u is well-defined. Now let $n \in N$. Writing out the condition that $\psi_f = \varphi_f \circ u$ yields $\varphi_f(u(n)) = \varphi_f((u_g(n))_g) = u_f(n) = \psi_f(n)$. In other words, the commutativity forces $u_f(n) = \psi_f(n)$. Clearly there exists exactly one function u whose components are given by $u_f = \psi_f$, meaning $(Rx, \{\varphi_f\}_f)$ satisfies the universal property. We may thus conclude that $\mathrm{Ran}_I(T)(x)$ is given by the expression in (7.2).

Let us now verify that our new definition of coinduction coincides with the regular module-theoretic description coinduction when applied to a group representation. Suppose $H \subset G$ are groups, and let $\mathcal{H}$ and $\mathcal{G}$ be their categorification. Let N be a representation of H viewed as $k[H]$-module. Recall the construction of the coinduction of a group representation given in **3.9**, namely $\mathrm{Coind}_H^G N = \mathrm{Hom}_{k[H]}(k[G], N)$.

Let T be the categorified version of the representation corresponding to N. By using that H and G are groups we can rewrite (7.2) as

$$R(*) = \left\{ (v_f)_f \in \prod_{f \in G} N \;\middle|\; T(h)(v_f) = v_{hf} \text{ for all } h \in H \text{ and } f \in G \right\}. \tag{7.3}$$

We can already see that $R(*)$ and $\mathrm{Hom}_{k[H]}(k[G], N)$ are really the same object. An element of both $R(*)$ and an element of $\mathrm{Hom}_{k[H]}(k[G], N)$ can be described as assiging to each element $g \in G$ some vector in N in a manner compatible with the structure of H. We can make this more explicit by considering the map

$$\begin{aligned} \varphi\colon R(*) &\to \mathrm{Hom}_{k[H]}(k[G], N), \\ (v_f)_f &\mapsto \left(\sum_{f \in G} \alpha_f f \mapsto \sum_{f \in G} \alpha_f v_f \right). \end{aligned}$$

Clearly φ is k-linear. Note that φ is well-defined as the condition for $\varphi((v_f)_f)$ to be $k[H]$-linear corresponds precisely to the condition in (7.3) that $T(h)(v_f) = v_{hf}$. Indeed, for $h \in H$ and $g \in G$ we have

$$\begin{aligned} \varphi((v_f)_f)(h \cdot g) &= v_{hg} \\ &= T(h)(v_g) \\ &= h \cdot v_g \\ &= h \cdot \varphi((v_f)_f)(g). \end{aligned}$$

It follows that $\varphi((v_f)_f)$ is $k[H]$-linear. We can further see that $\varphi((v_f)_f) \equiv 0$ precisely when all v_f are zero. Hence, $\ker \varphi = 0$. Lastly, any $k[H]$-linear map $\psi\colon k[G] \to N$ defines an element $v = (\psi(f))_{f \in G}$. The $k[H]$-linearity of ψ implies that $h \cdot \psi(f) = \psi(hf)$ and hence $T(h)(v_f) = v_{hf}$ for all $f \in G$. We conclude that $v \in R(*)$ and thus that φ is surjective.

We will now check that the $k[G]$-multiplication on $\mathrm{Hom}_{k[H]}(k[G], N)$ matches the Rg multiplication on $R(*)$. Let $x, g \in G$ and let $(v_f)_f \in R(*)$. Recall that Rg is defined as the unique map for which $\varphi_f \circ Rg = \varphi_{fg}$ for all $f \in (* \downarrow I)$. In particular we find that $Rg((v_f)_f) = (v_{fg})_f$. If we let $x \in G$, then

$$\varphi(Rg((v_f)_f))(x) = \varphi((v_{fg})_f)(x) = v_{xg}.$$

On the other hand, the $k[G]$-multiplication is by defined as $(g \cdot \varphi((v_f)_f))(x) = \varphi((v_f)_f)(x \cdot g)$ for $x \in k[G]$. By specifically picking $x \in G$, we obtain

$$(g \cdot \varphi((v_f)_f))(x) = \varphi((v_f)_f)(x \cdot g) = v_{xg}.$$

Hence, the Rg action coincides with the the g-multiplication in $\mathrm{Hom}_{k[H]}(k[G], N)$. We may conclude that coinduction of a group representation as a right Kan extension as in Definition **7.2** precisely matches the module-theoretic description given in Definition **3.9**.

One can similarly show that the left Kan extension of a group representation precisely matches the module-theoretic construction of the induced representation given in Definition **3.8**.

7.3 Dimension of Coinduction

As stated before, the existence of the induced and coinduced representation depends on the existence of particular limits like those in (7.1). When working in the category of vector spaces, all such limits exist as the category is both complete and cocomplete. In practice however, one might wish to consider only finite-dimensional representations, i.e., functors whose codomain is **FinVec**. This category is neither complete nor cocomplete, as both $\prod_{i=1}^{\infty} k$ and $\bigoplus_{i=1}^{\infty} k$ are infinite-dimensional. In this section we shall provide conditions under which induction and coinduction of finite-dimensional representations are themselves finite-dimensional.

Recall that for groups we can calculate the dimension of the coinduced representation as follows. Let $H \subset G$ be groups such that $n = [G : H] < \infty$. Let $g_1, \ldots, g_n$ be a set of representatives of G/H. For any $\mathbb{C}[H]$-module V, we have a vector space isomorphism

$$\begin{aligned} \operatorname{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G], V) &\xrightarrow{\sim} \prod_{i=1}^{n} g_i V, \\ f &\mapsto (f(g_i))_{i=1}^{n}. \end{aligned}$$

It immediately follows that

$$\dim \operatorname{Coind}_H^G V = [G : H] \cdot \dim V. \tag{7.4}$$

In particular, we find that the coinduction of V is finite-dimensional if and only if H is of finite index in G and V is finite dimensional. One can show that the exact same condition holds for induction. The goal of this section will be proving a similar statement for the groupoid case, using Definition 4.5 of a coset at x.

Theorem 7.7. Let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$ and $T \colon \mathcal{H} \to \mathbf{Vec}$ a representation. Assume $Tx \neq 0$ for all $x \in \operatorname{ob}(\mathcal{H})$. Then $\operatorname{Coind}_{\mathcal{H}}^{\mathcal{G}} T$ is finite-dimensional if and only if T is finite-dimensional and for all $x \in \operatorname{ob}(\mathcal{G})$ with $Tx \neq 0$, there are only finitely many cosets at x.

One can show that the exact same statement holds for induction with a very similar proof. Here we only focus on coinduction.

Proof. First note that $\operatorname{Coind}_{\mathcal{H}}^{\mathcal{G}} T$ is finite-dimensional precisely when it is finite-dimensional on all connected components of $\mathcal{G}$. We may thus assume $\mathcal{G}$ is connected. Let $\mathcal{S}_x$ be a set of representatives of $\mathcal{G}_x / \sim_x$. We shall calculate the dimension by providing an isomorphism to the explicit construction of the limiting object found in Section 7.2. Consider the map

$$\begin{aligned} \Phi \colon \prod_{f_\alpha \in \mathcal{S}_x} T(t(f_\alpha)) &\to \left\{ (w_f)_f \in \prod_{\langle f, y\rangle \in (x \downarrow I)} T(y) \;\middle|\; T(h)(w_f) = w_{f'} \right\}; \\ (v_{f_\alpha})_{f_\alpha \in \mathcal{S}_x} &\mapsto (w_f)_{f \in (x \downarrow I)}; \end{aligned} \tag{7.5}$$

where we define w_f as follows. Since $\mathcal{G}_x$ is a disjoint union of cosets at x, each $f \in (x \downarrow I)$ can be uniquely written as product $f = h \circ f_\alpha$ where $f_\alpha \in \mathcal{S}_x$ and $h \in \mathcal{H}$. Define $w_f := T(h)(v_{f_\alpha})$. Recall that the condition in (7.5) is implicitly understood to hold for all $\langle h \colon f \to f' \rangle \in (x \downarrow I)$. First we shall verify that Φ is well-defined, i.e., that $(w_f)_{f \in (x \downarrow I)}$ satisfies $T(h)(w_f) = w_{f'}$.

Fix some $\langle h \colon f \to f' \rangle \in (x \downarrow I)$, meaning $hf = f'$. This implies f and f' lie in the same coset. Write $f = h_1 f_\alpha$ and $f' = h_2 f_\alpha$ for $h_1, h_2 \in \operatorname{mor}(\mathcal{H})$ and the same $f_\alpha \in \mathcal{S}_x$. It follows that $h h_1 f_\alpha = h_2 f_\alpha$ and thus $h h_1 = h_2$ as all morphisms are invertible. Then, w_f and w_f' are by definition given by $w_f = T(h_1)(v_{f_\alpha})$ and $w_{f'} = T(h_2)(v_{f_\alpha})$ respectively. This implies

$$\begin{aligned} T(h)(w_f) &= T(h)(T(h_1)(v_{f_\alpha})) \\ &= T(hh_1)(v_{f_\alpha}) \\ &= T(h_2)(v_{f_\alpha}) \\ &= w_{f'} \end{aligned}$$

as desired.

Linearity of Φ follows immediately from the definition. Indeed, each component w_f of the image is defined by applying the linear operator $T(h)$ to the input component at f_α for which $f = hf_\alpha$ which is clearly linear in the input.

Suppose $\Phi((v_{f_\alpha})_{f_\alpha \in \mathcal{S}_x}) = 0$. Then in particular, $w_{f_\alpha} := T(1_x)(v_{f_\alpha}) = v_{f_\alpha} = 0$ for all $f_\alpha \in \mathcal{S}_x$. This implies $(v_{f_\alpha})_{f_\alpha \in \mathcal{S}_x} = 0$. We may conclude that $\ker \Phi = 0$.

Finally we shall prove that Φ is surjective. Let $(u_f)_f$ be any element in the codomain of Φ and pick $v_{f_\alpha} := u_{f_\alpha}$ and write $(w_f)_f := \Phi((v_f)_f)$. Given any $f \in (x \downarrow I)$, we have $f = hf_\alpha$ for some unique $f_\alpha \in \mathcal{S}_x$ and $h \in \mathcal{H}$. Notice that $\langle h\colon f_\alpha \to f\rangle \in (x \downarrow I)$, so by assumption we have $T(h)(u_{f_\alpha}) = u_f$. However, by definition of Φ we have $w_f = T(h)(v_{f_\alpha}) = T(h)(u_{f_\alpha}) = u_f$. Hence, Φ is surjective.

Now that we have proven that Φ is an isomorphism, we find that

$$\dim \operatorname{Coind}_{\mathcal{H}}^{\mathcal{G}} T(x) = \sum_{f_\alpha \in \mathcal{S}_x} \dim T(t(f_\alpha)) \tag{7.6}$$

whenever this sum exists. In particular we find that $\operatorname{Coind}_{\mathcal{H}}^{\mathcal{G}} T$ is finite-dimensional if and only if T is finite-dimensional and $\#(\mathcal{G}_x/\sim_x) < \infty$. □

Note that the sum in (7.6) can generally not be rewritten as a product similar to (7.4). This is because $\mathcal{H}$ may be a disconnected groupoid in which case T may send different components to vector spaces of differing dimension.

7.4 Relation to the Tensor Product

In this section we denote by $\mathbb{1}_{\mathcal{H}}$, $\lambda^{(\mathcal{H})}$, $\rho^{(\mathcal{H})}$, and $\alpha^{(\mathcal{H})}$ the tensor unit, the left unitor, the right unitor, and the associator in $\mathbf{Rep}_{\mathcal{H}}$. Similarly we write $\mathbb{1}_{\mathcal{G}}$, $\lambda^{(\mathcal{G})}$, $\rho^{(\mathcal{G})}$, and $\alpha^{(\mathcal{G})}$ for the unit object and the respective transformations in $\mathbf{Rep}_{\mathcal{G}}$.

Given a finite group G with subgroup $H \subset G$, two finite-dimensional representations ρ of G and π of H satsify the following formula:

$$\operatorname{Ind}_{\mathcal{H}}^{\mathcal{G}}(\pi \otimes \operatorname{Res}_{\mathcal{H}}^{\mathcal{G}} \rho) \cong (\operatorname{Ind}_{\mathcal{H}}^{\mathcal{G}} \pi) \otimes \rho$$

[8, p.29]. This relation is sometimes called the *projection formula* and it fits into a much broader context of isomorphisms between certain adjoints [3]. By applying the theory developed in the paper *Isomorphisms between left and right adjoints*, we can prove a more general version of the projection formula for groupoid representations. Doing so first requires us to prove that the restriction functor behaves nicely with respect to the tensor product.

Proposition 7.8. $\operatorname{Res}_{\mathcal{H}}^{\mathcal{G}}$ is a **strong symmetric monoidal functor**, meaning there exist natural isomorphisms

$$\eta\colon \mathbb{1}_{\mathcal{H}} \Rightarrow \operatorname{Res}_{\mathcal{H}}^{\mathcal{G}}(\mathbb{1}_{\mathcal{G}})$$

and

$$\mu_{-,-}\colon \operatorname{Res}^{\mathcal{G}}_{\mathcal{H}}(-) \otimes \operatorname{Res}^{\mathcal{G}}_{\mathcal{H}}(-) \Rightarrow \operatorname{Res}^{\mathcal{G}}_{\mathcal{H}}(- \otimes -)$$

such that the following coherence diagrams commute for all $S, T, U \in \operatorname{ob}(\mathbf{Rep}_{\mathcal{G}})$.

- Associativity.

$$\begin{tikzcd}
\operatorname{Res}(S) \otimes (\operatorname{Res}(T)) \otimes \operatorname{Res}(U)) \arrow[r, "\alpha^{\mathcal{H}}"] \arrow[d, "1\otimes\mu_{S,T}"] & (\operatorname{Res}(S) \otimes \operatorname{Res}(T)) \otimes \operatorname{Res}(U) \arrow[d, "\mu_{S,T}\otimes 1"] \\
\operatorname{Res}(S) \otimes (\operatorname{Res}(T \otimes U))) \arrow[d, "1\otimes\mu_{S,T\otimes U}"] & \operatorname{Res}(S \otimes T) \otimes \operatorname{Res}(U) \arrow[d, "\mu_{S\otimes T,U}"] \\
\operatorname{Res}(S \otimes (T \otimes U)) \arrow[r, "\operatorname{Res}(\alpha^{\mathcal{G}})"] & \operatorname{Res}((S \otimes T) \otimes U)
\end{tikzcd}$$

- Unitality.

$$\begin{tikzcd}
\operatorname{Res}(S) \otimes \mathbb{1}_{\mathcal{H}} \arrow[r, "\rho^{\mathcal{H}}"] \arrow[d, "1\otimes\eta"] & \operatorname{Res}(S) \\
\operatorname{Res}(S) \otimes \operatorname{Res}(\mathbb{1}_{\mathcal{G}}) \arrow[r, "\mu"] & \operatorname{Res}(S \otimes \mathbb{1}_{\mathcal{G}}) \arrow[u, "\operatorname{Res}(\rho^{\mathcal{G}})"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbb{1}_{\mathcal{H}} \otimes \operatorname{Res}(S) \arrow[r, "\lambda^{\mathcal{H}}"] \arrow[d, "\eta\otimes 1"] & \operatorname{Res}(S) \\
\operatorname{Res}(\mathbb{1}_{\mathcal{G}}) \otimes \operatorname{Res}(S) \arrow[r, "\mu"] & \operatorname{Res}(S \otimes \mathbb{1}_{\mathcal{G}}) \arrow[u, "\operatorname{Res}(\lambda^{\mathcal{G}})"]
\end{tikzcd}$$

- Symmetry. Let σ denote the symmetry isomorphism in $\mathbf{Rep}_{\mathcal{H}}$.

$$\begin{tikzcd}
\operatorname{Res}(S) \otimes \operatorname{Res}(T) \arrow[r, "\sigma"] \arrow[d, "\mu"] & \operatorname{Res}(T) \otimes \operatorname{Res}(S) \arrow[d, "\mu"'] \\
\operatorname{Res}(S \otimes T) \arrow[r, "\operatorname{Res}(\sigma)"] & \operatorname{Res}(T \otimes S)
\end{tikzcd}$$

Note that some of the subscripts have been removed to avoid clutter.

Proof sketch. As it turns out $\operatorname{Res}^{\mathcal{G}}_{\mathcal{H}}$ is actually a *strict monoidal functor*, meaning η and $\mu_{S,T}$ are identity morphisms. Indeed, given $x \in \operatorname{ob}\mathcal{H}$ we have $\mathbb{1}_{\mathcal{H}}(x) = k$ and $\operatorname{Res}^{\mathcal{G}}_{\mathcal{H}}(\mathbb{1}_{\mathcal{G}})(x) = \mathbb{1}_{\mathcal{G}}(I(x)) = k$. We can simply define $\eta_x := 1_k$. For μ we find something similar. Given $S, T \in \operatorname{ob}(\mathbf{Rep}_{\mathcal{G}})$, we have

$$\begin{aligned}
(\operatorname{Res}^{\mathcal{G}}_{\mathcal{H}}(S) \otimes \operatorname{Res}^{\mathcal{G}}_{\mathcal{H}}(T)) &= (S \circ I) \otimes (T \circ I) \\
&= (S \otimes T) \circ I \\
&= \operatorname{Res}^{\mathcal{G}}_{\mathcal{H}}(S \otimes T).
\end{aligned}$$

Simply take $\mu_{S,T} := 1_{S\otimes T}$.

To finish the proof one must verify that μ and η are natural and that all four coherence diagrams above are satisfied. This can be done via a straightforward calculation. ☐

Theorem 7.9 (Projection Formula)**.** Let $T \in \operatorname{ob}(\mathbf{Rep}_{\mathcal{G}})$ be a finite-dimensional representation. Then for all representations $S \in \operatorname{ob}(\mathbf{Rep}_{\mathcal{H}})$, we have

$$T \otimes \operatorname{Coind}^{\mathcal{G}}_{\mathcal{H}} S \cong \operatorname{Coind}^{\mathcal{G}}_{\mathcal{H}}((\operatorname{Res}^{\mathcal{G}}_{\mathcal{H}} T) \otimes S)$$

and

$$T \otimes \operatorname{Ind}^{\mathcal{G}}_{\mathcal{H}} S \cong \operatorname{Ind}^{\mathcal{G}}_{\mathcal{H}}((\operatorname{Res}^{\mathcal{G}}_{\mathcal{H}} T) \otimes S).$$

Proof. By Theorem 6.11, T is dualizable. The result is a direct corollary of Proposition 1.12 and Proposition 2.11 in [3]. ☐

7.5 The Algebra of a Groupoid

In section 3.2 we saw that representations of a group G may be viewed as modules over the group ring $k[G]$. We can similarly define the groupoid algebra $k[\mathcal{G}]$ and view representations of $\mathcal{G}$ as modules over this algebra. In this section, which is primarily based on Chapter 10 from [5], we briefly review this correspondence.

Definition 7.10. Let $\mathcal{G}$ be a groupoid and k a field. Define the **groupoid algebra over k** $k[\mathcal{G}] = \mathrm{Span}_k(\mathrm{mor}(\mathcal{G}))$ with multiplication induced by linear extension of

$$f \cdot g = \begin{cases} f \circ g & \text{if } t(g) = s(f); \\ 0 & \text{if } t(g) \neq s(f). \end{cases}$$

for $f, g \in \mathrm{mor}(\mathcal{G})$.

This naturally extends Definition 3.3 from groups to groupoids. For a group G with categorification $\mathcal{G}$, the group ring $k[G]$ is exactly equal to $k[\mathcal{G}]$. Recall that the identity element $e \in G$ is also the multiplicative identity of $k[G]$. For groupoids however, we only have local identities $1_x \in \mathcal{G}_x$ for objects $x \in \mathrm{ob}\,\mathcal{G}$. If there are only finitely many objects, we can still find a global unit $e := \sum_{x \in \mathrm{ob}\,\mathcal{G}} 1_x \in k[\mathcal{G}]$, but no such element exists when $\mathcal{G}$ has infinitely many objects.

Similar to the case of groups (Section 3.2) we have a correspondence between representations of $\mathcal{G}$ and $k[\mathcal{G}]$-modules. Specifically, given a representation $T \in \mathrm{ob}(\mathbf{Rep}_{\mathcal{G}})$, the corresponding $k[\mathcal{G}]$-module N is given by $N = \bigoplus_{x \in \mathrm{ob}\,\mathcal{G}} T(x)$ with multiplication given by $g \cdot v_x = T(g)(v_x)$ for any morphism $g\colon x \to y$ in $\mathcal{G}$ and $v_x \in T(x)$.

Many of the definitions and constructions with $k[G]$-modules from Chapter 3 naturally extend to $k[\mathcal{G}]$-modules. In particular, we may define induction and coinduction of some $k[\mathcal{H}]$-module M by

$$\begin{aligned} \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(M) &:= k[\mathcal{G}] \otimes_{k[\mathcal{H}]} M; \\ \mathrm{Coind}_{\mathcal{H}}^{\mathcal{G}}(M) &:= \mathrm{Hom}_{k[\mathcal{H}]}(k[\mathcal{G}], M). \end{aligned}$$

As was the case with groups, this definition coincides with the abstract definition of induction and coinduction given in Chapter 7. The lack of a global unit significantly complicates the study of $k[\mathcal{G}]$-modules. Consider for instance the following proof of Frobenius Reciprocity.

Theorem 7.11 (Frobenius Reciprocity)**.** For N a $k[\mathcal{G}]$-module and M a $k[\mathcal{H}]$-module, it holds that

$$\mathrm{Hom}_{k[G]}(\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} M, N) \cong \mathrm{Hom}_{k[H]}(M, \mathrm{Res}_{\mathcal{H}}^{\mathcal{G}} N).$$

Proof. Give $\mathrm{Hom}_{k[\mathcal{G}]}(k[\mathcal{G}], N)$ a $k[\mathcal{H}]$-module structure by using the $(k[\mathcal{G}], k[\mathcal{H}])$-module structure on $k[\mathcal{G}]$. Consider the map

$$\begin{aligned} \varphi\colon \mathrm{Hom}_{k[\mathcal{G}]}(k[\mathcal{G}], N) &\to \mathrm{Res}_{\mathcal{H}}^{\mathcal{G}} N, \\ f &\mapsto f(1). \end{aligned}$$

Clearly φ is k-linear. Now let $h \in \mathcal{H}$. Then,

$$\begin{aligned} \varphi(h \cdot f) &= (h \cdot f)(1) \\ &= f(1 \cdot h) \\ &= f(h) \\ &= h \cdot \varphi(f). \end{aligned}$$

Hence, φ is $k[\mathcal{H}]$-linear. Further, $\varphi(f) = 0$ implies $f(1) = 0$ and hence $f(g) = g \cdot f(1) = 0$ for all $g \in k[\mathcal{G}]$. Thus, $f \equiv 0$. Lastly, given any $n \in N$ we can consider $f\colon k[\mathcal{G}] \to N$ given by $f(x) = x \cdot n$. Then $\varphi(f) = n$ as desired. We may conclude that φ is an isomorphism of $k[\mathcal{H}]$-modules.

By applying tensor-hom adjunction, we find that

$$\begin{aligned}
\operatorname{Hom}_{k[G]}(\operatorname{Ind}_{\mathcal{H}}^{\mathcal{G}} M, N) &= \operatorname{Hom}_{k[G]}(k[\mathcal{G}] \otimes_{k[\mathcal{H}]} M, N) \\
&\cong \operatorname{Hom}_{k[H]}(M, \operatorname{Hom}_{k[\mathcal{G}]}(k[\mathcal{G}], N)) \\
&\cong \operatorname{Hom}_{k[H]}(M, \operatorname{Res}_{\mathcal{H}}^{\mathcal{G}} N))
\end{aligned}$$

as desired. □

The proof above demonstrates how the existence of a global unit can play a crucial role in the study of $k[\mathcal{G}]$-modules. For this reason we do not study $k[\mathcal{G}]$-modules in detail in this thesis.

C8 | Conclusion

In this thesis, we have studied induction and coinduction of representations of groupoids from a categorical perspective. By viewing induction not as a particular construction but rather as a functor characterized by its adjunction with restriction, we have a more general framework that naturally extends the classical theory of induced representations of groups.

In the first few chapters we reviewed the necessary foundations in category theory and representation theory. We later gave an explicit construction for coinduced representations by using the theory of Kan extensions. This explicit construction allowed us to provide conditions under which a coinduced representation is finite-dimensional. In Chapter 6 we found that many familiar constructions from **Vec** naturally extend to $\mathbf{Rep}_{\mathcal{G}}$. In particular, $\mathbf{Rep}_{\mathcal{G}}$ is an abelian, closed symmetric monoidal category. This additional structure allowed us to later prove a more general version of the projection formula, which relates induction and coinduction with the tensor product.

The results presented in this thesis suggest several directions for future research. One possibility is to generalize some results to representations whose codomain need not be the category of vector spaces, but for instance the category of modules over certain rings. Another possibility is to study the representations and groupoids with additional structure, such as continuous representations of topological groupoids. In this context, the existence of adjoint to restriction is no longer trivial. One could study which additional conditions need to be imposed in order for adjoints to exist. Finally, this thesis only considers the case where we are given a subgroupoid $\mathcal{H}$ contained in a larger groupoid $\mathcal{G}$. More generally one could study the case where we are given any functor between $\mathcal{H}$ and $\mathcal{G}$, not necessarily an inclusion functor.

8.1 Acknowledgements

I would like thank my thesis supervisors for supporting me throughout this project and for helping me during the many meetings we had. I thank Peter Bruin for helping me come up with examples of representations, for being available for weekly meetings for nearly an entire semester, and for his assistance with the preparation of the presentations for the *Bachelor Seminarium*. I thank Henning Basold for his category-theoretic insights such as the use of Kan extensions, for his help with navigating the organization of the computer science bachelor class, and for his feedback on my writing within a very short deadline.

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