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Three-player combinatorial games

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Contents

1 Introduction

Combinatorial game theory studies games with perfect information and no chance elements. Examples include Go and Chess. Classically, combinatorial game theory studies games with two players, usually called Left and Right, abbreviated as L and R. These players alternate turns, until a player has no available moves, at which point the other player is declared the winner. A useful tool is taking the *sum* of games. This is a larger game, in which the two subgames have been placed next to each other, and a player can move on exactly one of the subgames on their turn. An example use case of addition of games is the analysis of a Go endgame, which usually consist of regions that have no direct impact on each other. Therefore, A Go endgame viewed as the sum of these different regions. An important question in combinatorial game theory is which player will win a given game. This can be determined with the *outcome classes*, which indicate the winner of a game, assuming that every player plays *optimally*. These outcome classes are used to define *equality* of games. Using *addition* and *equality* of games, the set of two-player games form an abelian group.

When studying three-player games, adding Center, abbreviated as C, as player, assumptions must be made to combat "kingmaking". This occurs when a player cannot win themselves, but can choose which of the other players will win. In this thesis, we will use the player preferences introduced by Li [Li78]. Then, with analogous definitions of *addition* and *equality*, the set of three-player games form a commutative monoid, only missing the existence of additive inverse elements. In this thesis, we try to determine if additive inverses exist. We will conclude that the most intuitively logical candidate for inverses, conjectured by Greene [Gre17], is not an inverse. It remains unclear whether three-player games, with Li's player preferences, form an abelian group.

1.1 Overview

In the next section, we give a brief overview of related works in the field of combinatorial game theory. This forms the context in which this thesis should be read.

Section 3 gives the basic results of two player combinatorial game theory. We show that two player game positions, with a certain definition of addition and equality, form an abelian group. We clarify the results using the *ruleset* Domineering.

We begin Section 4 by attempting to create a structure analogous to that of two-player games when expanding the player count to three players. The ruleset we use in this section is Rhombination. Using player preferences introduced by Li [Li78], the game positions with three players form a commutative monoid, only missing the existence of additive inverse elements in order to be an abelian group. Next, we show, using rotations, that all Rhombination *outcome classes* are non-empty. This thesis also shows that the intuitive candidate for additive inverse elements for three player games, hypothesised by Greene [Gre17], are not inverse elements. It remains unclear whether inverses exist. We end Section 4 by considering alternative assumptions, under which we hypothesise that this candidate for inverse elements does work. However, these alternative assumptions suffer from different drawbacks.

We conclude by giving an idea on how one might be able to decide whether three player games, with our original assumptions, have additive inverse elements.

2 Related work

One of the pioneers of combinatorial game theory was Conway in *On Numbers And Games* in 1976 [Con76]. Here, Conway introduces the concept of *partisan games*, which are games in which the available moves for each player differ. Examples of partisan *rulesets* are DOMINEERING and RHOMBINATION, which we will both use in this thesis. Most of the research in combinatorial game theory has been done for two player games, as these have a rich structure. It has been shown that the so called *short games* form an abelian group [Sie13].

As mentioned in the introduction, a problem one must deal with when studying three-player games is that of "kingmaking". Different authors have chosen for different solutions to this problem. Li has chosen to introduce static player preferences [Li78], while Straffin has decided a player's preferences depend on who has wronged them in the game [Jr.85]. Instead of player preferences, some authors, like Loeb, have chosen to look at coalitions of players that are able to win a game [Loe96]. Cincotti focusses on the partial order structure, and restricts himself to a subset of games, called numbers [Cin05]. All of these approaches have different advantages and disadvantages. Thus research regarding three-player games is divided between all these different approaches. Greene has followed in Li's footsteps, and explored the three player ruleset RHOMBINATION with Li's player preferences [Gre17].

In this thesis, we build on Greene's work by answering the open questions left in her thesis. Namely, we show that the intuitive candidate for additive inverse elements are in fact, not inverses, and we show that all RHOMBINATION *outcome classes* are non-empty.

3 Two-player games

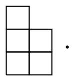
The definitions and theorems in this section are based on the book *Combinatorial Game Theory* by A. Siegel [Sie13]. For this thesis, we define $\mathbb{N} := \mathbb{Z}_{\geq 0}$.

In this section, we will give the relevant definitions and theorems for combinatorial games with two players. We will call our players Left and Right, who take alternating turns. Once a player is unable to make a move, they lose the game, and the other player is declared the winner. This section builds towards Theorem 3.22, in which we conclude that the set of two-player games form an abelian group. In this section we will give a definition of equality for games, which differs from the games being equal as sets. To avoid confusion, we will be using $\cong$ to indicate that two games are set theoretically equal.

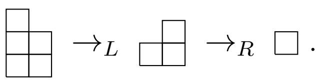
Combinatorial games contain no chance elements and both players have perfect information. As mentioned above, the last player to make a move wins. As such, we do not allow draws.

The ruleset we will be using for our examples with two players is DOMINEERING. In a DOMINEERING game, the players are faced with a grid of available squares. On their turn, a player must place a domino, covering two adjacent empty squares, on this grid. The left player may only do so in a vertical orientation, while the right player may only do so in a horizontal orientation. Once a player cannot make a move, the other player wins.

Example 3.1. The following is a DOMINEERING game:



We let Left make the first move on this game. The following is a possible (not necessarily optimal) play-out of this game, where $\to_\alpha$ means that this is a possible move player α can make:



Left can now no longer make a move, so this would be a win for Right.

We now definite a game recursively.

Definition 3.2. A *game G of birthday $n \in \mathbb{N}_{\geq 0}$* is an ordered pair $(\mathcal{G}^L, \mathcal{G}^R)$, where $\mathcal{G}^L$ and $\mathcal{G}^R$ are sets of games of birthdays between 0 and $n-1$, inclusive on both sides, and such that if $n > 0$, there exists a game $H \in \mathcal{G}^L \cup \mathcal{G}^R$ of birthday $n-1$. A *game* is a game of birthday n for an $n \in \mathbb{N}_{\geq 0}$. We write a game G as $G \cong \{\mathcal{G}^L \mid \mathcal{G}^R\}$. Furthermore, if $\mathcal{G}^L = \{G_1^L, ..., G_m^L\}$ and $\mathcal{G}^R = \{G_1^R, ..., G_k^R\}$ for some $m, k \in \mathbb{N}$, we write $G \cong \{G_1^L, ..., G_m^L \mid G_1^R, ..., G_k^R\}$. For example, the unique game of birthday 0 is $0 := \{\emptyset \mid \emptyset\}$, since there are no games of at most birthday -1. This game can also be written as $0 \cong \{\,\mid\,\}$.

Intuitively, the sets $\mathcal{G}^L$ and $\mathcal{G}^R$ are the sets of moves the Left and Right player, respectively, can make. We call elements of these sets *moves*, *options*, or *children*. In

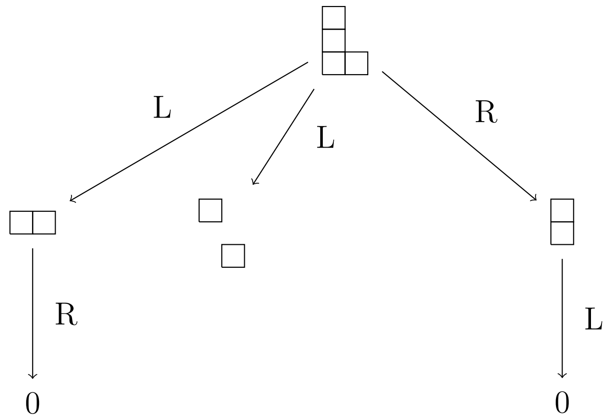


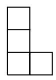
Figure 1: Full game tree of an example DOMINEERING game

most literature, our definition of a game corresponds with that of a *short game*. Since we will only be working with short games, we will simply call them games.

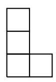
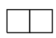
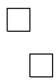
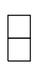
The birthday of a game can be thought of as the height of the game tree, which corresponds to the maximum number of turns a game can last.

An important distinction to be made is between a *game* and a *ruleset*. Intuitively, a game is a position with the possible moves both players can make. A ruleset is a system which governs what the possible games are. Examples of rulesets are GO, CHESS, and DOMINEERING, while a specific GO position is then a game. In order to illustrate the definition of a game, we will give an example of a domineering game.

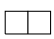
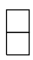
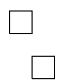
Example 3.3. The following is an example of a domineering game:

.

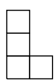
Using the ruleset we introduced, we can determine what the possible moves are for Left and Right and picture this as a game tree as seen in Figure 1. We see that Left has two possible moves, while Right only has one. We can write this game using the notation introduced in Definition 3.2. this gives us:

 $\cong \Big\{$  ,  $\Big|$  $\Big\}$.

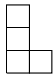
We can go further, as the definition of a game is recursive. Using the fact that:

 $\cong \{\,|\,0\}$,  $\cong \{0\,|\,\}$, and  $\cong 0$,

we find

 $\cong \{\{\,|\,0\}, 0 \,|\, \{0\,|\,\}\}$.

Finally, $0 \cong \{\,|\,\}$ gives us

 $\cong \{\{\,|\,\{\,|\,\}\}, \{\,|\,\}\,|\,\{\{\,|\,\}\,|\,\}\}.$

This is extremely cryptic, so it is usually preferable to visualize games instead.

Definition 3.4. The *player-order function* is the cyclical permutation function (LR), which maps a player to the player that is after them in the player order.

We will now introduce the addition of two games. This is a binary operation defined as follows.

Definition 3.5. Let $G \cong \{\mathcal{G}^L \mid \mathcal{G}^R\}$ and $H \cong \{\mathcal{H}^L \mid \mathcal{H}^R\}$ be games. The *sum* of these games, denoted by $+$, is defined recursively as follows:

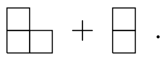
$$G + H \cong \left\{\left(\mathcal{G}^L + H\right) \cup \left(G + \mathcal{H}^L\right) \mid \left(\mathcal{G}^R + H\right) \cup \left(G + \mathcal{H}^R\right)\right\},$$

where for $\alpha \in \{L, R\}$:

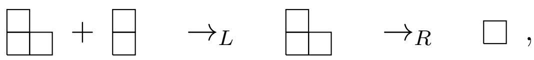
$$\mathcal{G}^\alpha + H := \{X + H : X \in \mathcal{G}^\alpha\}, \qquad G + \mathcal{H}^\alpha := \{G + X : X \in \mathcal{H}^\alpha\}.$$

Intuitively, the sum of two games is a larger game, where the two subgames have been placed next to each other. On their turn, a player may make a move on exactly one of the subgames.

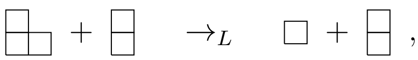
Example 3.6. Consider the following sum of DOMINEERING games:



Assuming that Left makes the first move, the following are two possible play-outs:



in which case Right wins, and



in which case Left wins.

Lemma 3.7. *Let G and H be games. Then $G + H$ is a game.*

Proof. We use induction on the sum of the birthday of G and the birthday of H. Our base case is that both G and H have birthday 0. The only game of birthday 0 is 0 and since 0 has no left or right options, we have $0+0 \cong \{0^L+0\cup 0+0^L \mid 0^R+0\cup 0+0^R\} \cong \{\,|\,\} \cong 0$, which is a game.

Now, let $N \in \mathbb{N}$ be such that $G + H$ is a game for all games G of birthday n' and games H of birthday m' with $n' + m' \leq N$. Let G be a game of birthday n and H be a game of birthday m, such that $n + m = N + 1$. Then for all options $X \in \left(\mathcal{G}^L + H\right) \cup \left(G + \mathcal{H}^L\right) \cup \left(G^R + H\right) \cup \left(G + H^R\right)$ of $G+H$, we can use our induction hypothesis to conclude that X is a game. Let k be the highest birthday of all these games X. Then $G + H$ is a game of birthday $k + 1$. $\square$

Lemma 3.8. *Let G be a game. Then $G + 0 \cong G$.*

Proof. We use induction on the birthday of G. Our base case is that G has birthday zero. So $G \cong 0$. We have shown in the proof of Lemma 3.8 that $0 + 0 = 0$.

Let $N \in \mathbb{N}$ be such that $H + 0 \cong H$ for all games H of birthday at most N. Now, let G be a game of birthday $N + 1$. Using our induction hypothesis, we get $G + 0 \cong \{\mathcal{G}^L + 0 \cup G + 0^L \mid \mathcal{G}^R + 0 \cup G + 0^R\} \cong \{\mathcal{G}^L \cup \emptyset \mid \{\mathcal{G}^R \cup \emptyset\}\} \cong G$. □

Lemma 3.9. *Addition of games is commutative and associative.*

Proof. We will first prove that $G + H \cong H + G$, for all games G and H. We proceed by induction on the sum of the birthday of G and the birthday of H. Our base case is that both G and H have birthday 0, which means that $G \cong H \cong 0$. Since $G \cong H$, it follows that $G + H \cong H + G$.

Now, let $N \in \mathbb{N}$ be such that $G + H \cong H + G$ for all games G of birthday n' and games H of birthday m' with $n' + m' \leq N$. Let $G \cong \{\mathcal{G}^L \mid \mathcal{G}^R\}$ be a game of birthday n and $H \cong \{\mathcal{H}^L \mid \mathcal{H}^R\}$ be a game of birthday m such that $n + m = N + 1$. Then all options $X \in \left(\mathcal{G}^L + H\right) \cup \left(G + \mathcal{H}^L\right) \cup \left(G^R + H\right) \cup \left(G + H^R\right)$ of $G + H$ are games of birthday at most N. Thus

$$\begin{aligned} G + H &\cong \left\{\left(\mathcal{G}^L + H\right) \cup \left(G + \mathcal{H}^L\right) \mid \left(\mathcal{G}^R + H\right) \cup \left(G + \mathcal{H}^R\right)\right\} \\ &\cong \left\{\left(\mathcal{H}^L + G\right) \cup \left(H + \mathcal{G}^L\right) \mid \left(\mathcal{H}^R + G\right) \cup \left(H + \mathcal{G}^R\right)\right\} \cong H + G. \end{aligned}$$

Using induction, we conclude that addition of games is commutative.

We will now prove for all games G, H, J, that $(G + H) + J \cong G + (H + J)$. We proceed by induction on the sum of the birthday of G, the birthday of H, and the birthday of J. Our base case is that G, H, and J have birthday 0, which means that $G \cong H \cong J \cong 0$. Thus $(G + H) + J \cong 0 \cong G + (H + J)$.

Now, let $N \in \mathbb{N}$ be such that $(G + H) + J \cong G + (H + J)$ for all games G of birthday n', games H of birthday m', and games J with birthday k', with $n' + m' + k' \leq N$. Let $G \cong \{\mathcal{G}^L \mid \mathcal{G}^R\}$ be a game of birthday n, $H \cong \{\mathcal{H}^L \mid \mathcal{H}^R\}$ a game of birthday m, and $J \cong \{\mathcal{J}^L \mid \mathcal{J}^R\}$ a game of birthday k, such that $n + m + k = N + 1$. We will show that $(G+H)+J$ and $G+(H+J)$ have the same Left options, which is sufficient to conclude $(G + H) + J \cong G + (H + J)$, due to symmetry. All (Left) options of $(G + H) + J$ have a birthday of at most N. Using our induction hypothesis:

$$\begin{aligned} (G + H) + J &\cong \left\{\left(\left(\mathcal{G}^L + H\right) + J\right) \cup \left(\left(G + \mathcal{H}^L\right) + J\right) \cup \left((G + H) + \mathcal{J}^L\right) \mid \ldots\right\} \\ &\cong \left\{\left(\mathcal{G}^L + (H + J)\right) \cup \left(G + \left(\mathcal{H}^L + J\right)\right) \cup \left(G + \left(H + \mathcal{J}^L\right)\right) \mid \ldots\right\}. \end{aligned}$$

These are the Left options of $G + (H + J)$. Using symmetry, we have $(G + H) + J \cong G + (H + J)$. □

Definition 3.10. Let $\mathbb{G}$ be the set of games. We recursively define the *outcome functions* for both players, where for $\alpha \in \{L, R\}$, this is the function $o_\alpha : \mathbb{G} \to \{L, R\}$, given by

$$o_\alpha(G) = \begin{cases} \alpha \text{ if } o_{\sigma(\alpha)}(H) = \alpha \text{ for some } H \in \mathcal{G}^\alpha \\ \sigma(\alpha) \text{ else} \end{cases}.$$

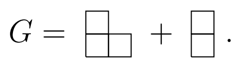
We now define the *outcome function* $o\colon \mathbb{G} \to \{L, R\}^2$, where

$$o(G) = (o_L(G), o_R(G)).$$

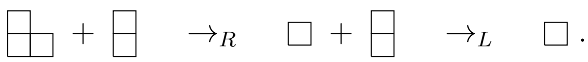
We denote this by $o(G) = o_L(G)o_R(G)$. We use the outcome function to define the *outcome classes*. Let $\alpha, \beta \in \{L, R\}$. Then $\alpha\beta := \{G \in \mathbb{G} : o(G) = \alpha\beta\}$.

If a game G has a valid option that will lead to a win for player α, they will choose that move. If such a move does not exist, which includes the player having no possible move, the other player will win. When a player plays *optimally*, they make moves that follow their outcome function. That is to say, given that player α has an available move, if $o_\alpha(G) = \beta \in \{L, R\}$, player α will move to a child H such that $o_{\sigma(\alpha)}(H) = \beta$. The outcome function tells us which player will win, if both players play optimally. Since the outcome function maps each game to a outcome class, the outcome classes are a partition of set of games.

Example 3.11. Since the game 0 has no moves for either player, we have $0 \in RL$. Now, let



We have seen in Example 3.6 that if Left starts, they have a move with which they can force a win. Thus $o_L(G) = L$. Furthermore, if Right starts on G, the following is the only possible sequence of moves



This leads to a win for Left. Thus $o_R(G) = L$, and $o(G) = LL$.

In the literature, the outcome classes are denoted by $\mathcal{P}, \mathcal{N}, \mathcal{L}$, and $\mathcal{R}$. These capture the characteristics of the outcome class in a single letter. For example, $\mathcal{L} = LL$ is the class of games where $\mathcal{L}$eft can always win, and $\mathcal{N} = LR$ is the class of games where the $\mathcal{N}$ext player to move can win. We have chosen to use a more systematic notation for the sake of consistency with the outcome classes for three players.

Definition 3.12. Let G and H be games. We say that G and H are *equal* if for all games X it holds that $o(G + X) = o(H + X)$. We denote this by $G = H$.

Note that we define what it means for games to be equal and that this differs from being set theoretically equal. Whenever we talk of two games being equal in the remainder of this thesis, we are referring to the game theoretical equality introduced above.

Lemma 3.13. *Equality of games is an equivalence relation.*

Proof. Let G, H and J be games. Then for all games X it holds that $o(G + X) = o(G + X)$. Thus equality is reflexive.

If $o(G + X) = o(H + X)$, then $o(H + X) = o(G + X)$, which gives us symmetry.

If $G = H$ and $H = J$, then for all games X it holds that $o(G+X) = o(H+X) = o(J+X)$. Thus $G = J$ and equality is transitive. $\square$

Since equality of games is reflexive, two games being set theoretically equal implies game theoretical equality. Therefore, Lemma 3.8 still holds game theoretically.

Lemma 3.14. *Let G, H and J be games with $G = H$. Then $G + J = H + J$.*

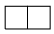
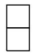
Proof. Let X be a game. Using associativity, we have:

$$o((G+J)+X) = o(G+(J+X)) = o(H+(J+X)) = o((H+J)+X),$$

which gives us that $G + J = H + J$. □

Definition 3.15. Let G be a game. We now recursively define its *negative* $-G := \{-\left(\mathcal{G}^R\right) \mid -\left(\mathcal{G}^L\right)\}$ where for $\alpha \in \{L, R\}$, we define $-\left(\mathcal{G}^\alpha\right) := \{-H : H \in \mathcal{G}^\alpha\}$.

Example 3.16. It holds that $-0 \cong 0$, as there are no available moves for either player. Furthermore

$-$  $\cong -\{\ \mid 0\} \cong \{-0 \mid \ \} \cong \{0 \mid \ \} \cong$  .

In general, it holds that the negative of a Domineering game is the game rotated by 90 degrees. Though we will not prove this rigorously, it is intuitively clear, as the negative of a game swaps the available moves for both players. Since the dominoes the players place on their turn are 90 degrees rotations of each other, it follows that the negative of a game is the game rotated by 90 degrees.

Lemma 3.17. *Let G be a game of birthday n, then $-G$ is also a game of birthday n.*

Proof. The proof of this lemma is an elementary induction proof, which we will not provide. □

Lemma 3.18. *Let G be a game. Then $-(-G) \cong G$.*

Proof. The proof of this lemma is an elementary induction proof, which we will not provide. □

We will now build towards Theorem 3.20, which gives us a one-to-one correspondence between games equal to zero and games in the outcome class RL. In order to prove this theorem, we must first prove the following lemma.

Lemma 3.19. *Let G and H be games, such that $o(G) = o(H) = RL$. Then $o(G+H) = RL$.*

Proof. We will proceed by induction on the sum of the birthday of G and the birthday of H. Our base case is that $G \cong H \cong 0$. It is clear that $o(0+0) = RL$.

Now, let $N \in \mathbb{N}$ be such that $o(G+H) = RL$ for all games G and H of birthday n' and m' respectively, satisfying $n' + m' \leq N$ and $o(G) = o(H) = RL$. Let G and H be games of birthdays n and m respectively, such that $n + m = N + 1$ and $o(G) = o(H) = RL$. We wish to show that $o(G+H) = RL$. This is equivalent to showing that for $\alpha \in \{L, R\}$, the equality $o_\alpha(G+H) = \sigma(\alpha)$ holds. By symmetry, it is sufficient to prove that $o_L(G+H) = R$. If $G + H$ does not have a Left option $o_L(G+H) = R$ is immediately clear. We now assume that $G + H$ has a Left option.

Left, playing optimally on $G + H$, will move to a game $G^L + H$ or $G + H^L$. Since Right can win on both G and H playing second, they can respond with an optimal move on the same subgame. That is, Right can then move to a game $(G^L)^R + H$ with $o_L\left((G^L)^R\right) = R$, or $G + (H^L)^R$ with $o_L\left((H^L)^R\right) = R$. Our induction hypothesis gives us that $o_L\left((G^L)^R + H\right) = R$, and $o_L\left(G + (H^L)^R\right) = R$. Thus Right will always have a winning move, and so $o_L(G + H) = R$ holds. □

Theorem 3.20. *Let G be a game. Then $G = 0$ if and only if $G \in RL$.*

Proof. Suppose $G = 0$. From the definition of equality, we have that for all games X it holds that $o(G + X) = o(X)$. Therefore, we have that $o(G) = o(G + 0) = o(0) = RL$. Thus $G \in RL$.

Now, suppose that $G \in RL$. Let X be a game. We will prove that $o(G+X) = o(X)$. By symmetry, it is sufficient to prove that $o_L(G + X) = o_L(X)$. Suppose that $o_L(X) = R$. Then Lemma 3.19 gives us that $o_L(G+X) = R = o_L(X)$. Now, suppose that $o_L(X) = L$. Then beginning on $G + X$, Left has an option $G + X^L$, such that $o_R\left(X^L\right) = L$. By assumption, it holds that $o_R(G) = L$. We use Lemma 3.19 once again, to conclude that $o_R\left(G + X^L\right) = L$. Since such a Left option exists, it holds that $o_L(G + X) = L = o_L(X)$. □

Theorem 3.21. *Let G be a game. Then $G + (-G) = 0$.*

Proof. Using Theorem 3.20, it is sufficient to prove that $G + (-G) \in RL$. We use induction on the birthday of G. Our base case is that $G \cong 0$, in which case $0+(-0) \in RL$ holds.

Now, let $N \in \mathbb{N}$ be such that $H + (-H) \in RL$ holds for all games H of birthday at most N. Let G be a game of birthday $N + 1$. We will prove that $G + (-G) \in RL$. Due to symmetry, it is sufficient to show that $o_L(G + (-G)) = R$.

Suppose that an optimal move for Left on $G + (-G)$ is to play to a Left option $G^L + (-G)$. Then Right can move to $G^L + (-(G^L))$. Using our induction hypothesis, we have $G^L + (-(G^L)) \in RL$. Thus this leads to a win for Right, if both player play optimally.

The other option for Left is to move to a Left option $G + (-(G^R))$. Using our induction hypothesis, we have $G^R + (-(G^R)) \in RL$, which Right can move to. Thus whatever Left does, Right will be able to win. So $o_L(G + (-G)) = R$.

We conclude, using symmetry and induction, that $G + (-G) \in RL$ for all games G. □

The above theorem tells us that $-G$ is an additive inverse element of G using game theoretical addition and equality. This leads us to our main result.

Theorem 3.22. *The set of games modulo equality is an abelian group.*

Proof. Lemma 3.7 tells us that the sum of games is a game. Addition is commutative and associative by Lemma 3.9. Lemma 3.13 tells us that equality of games is an equivalence relation, thus the set of games modulo equality is well-defined. Furthermore, by Lemma 3.14, addition is consistent with respect to equality, making addition a well-defined operation for the set of games modulo equality. Lemma 3.8 tells us that 0 is a neutral element for addition and through Theorem 3.21 we have the existence of inverse elements. Thus the set of games modulo equality is an abelian group. □

A group structure is an incredibly rich structure and this result tells us that we can analyse combinatorial games using group theory. In the remainder of this thesis, we will try to recreate this structure for combinatorial games with three players.

4 Three-Player games

4.1 The basics

We now introduce a third player to our games, whom we will call Center. We let play proceed cyclically, with Center playing after Left, Right after Center and Left after Right. In this subsection we will be redefining games to include three players, and adapting most of the results from our section on two-player games. Our conclusion will be that three-player games form a commutative monoid, instead of an abelian group, due to the absence of additive inverse elements. For the rest of this thesis, we will be using the ruleset RHOMBINATION for our examples and results. RHOMBINATION is very similar to DOMINEERING, but now a game is a grid of equilateral triangles. On a player's turn, that player must place a rhombus, covering two adjacent empty triangles, in a certain orientation, which can be found in Table 1. RHOMBINATION is introduced by Greene [Gre17].

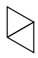
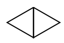
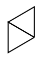
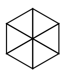
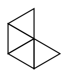
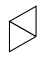
L	**C**	**R**
		

Table 1: Moves for the different players in RHOMBINATION.

Example 4.1. An example play-out of RHOMBINATION games is shown below, where $\to_\alpha$ means that this is a possible move player α can make.

 $\to_{\text{L}}$  $\to_{\text{C}}$  $\to_{\text{R}}$ 0.

Definition 4.2. A *game G of birthday $n \in \mathbb{N}_{\geq 0}$* is an ordered triple $(\mathcal{G}^L, \mathcal{G}^C, \mathcal{G}^R)$, where $\mathcal{G}^L$, $\mathcal{G}^C$, and $\mathcal{G}^R$ are sets of games of birthdays between 0 and $n-1$, inclusive on both sides, and such that if $n > 0$, there exists a game $H \in \mathcal{G}^L \cup \mathcal{G}^C \cup \mathcal{G}^R$ of birthday $n-1$. A *game* is a game of birthday n for an $n \in \mathbb{N}_{\geq 0}$. We write a game G as $G \cong \{\mathcal{G}^L \mid \mathcal{G}^C \mid \mathcal{G}^R\}$. Furthermore, if $\mathcal{G}^L = \{G_1^L, ..., G_m^L\}$, $\mathcal{G}^C = \{G_1^C, ..., G_l^C\}$, and $\mathcal{G}^R = \{G_1^R, ..., G_k^R\}$ for some $m, l, k \in \mathbb{N}$, we write $G \cong \{G_1^L, ..., G_m^L \mid G_1^C, ..., G_l^C \mid G_1^R, ..., G_k^R\}$. For example, the unique game of birthday 0 is $0 := \{\emptyset \mid \emptyset \mid \emptyset\}$, since there are no games of at most birthday -1. This game can also be written as $0 \cong \{\ \mid \mid \ \}$.

When referring to a game for the remainder of this thesis, we are using the definition of a three-player game given above, instead of the definition of a two-player game. The only difference between these definitions is that we have added the possible moves for Center.

We will now formalise our player order with the following definition.

Definition 4.3. The *player-order function* σ is the cyclical permutation function (LCR), which maps a player to the player that is after them in the player order. That is, the player to make a move after player $\alpha \in \{L, C, R\}$, is player $\sigma(\alpha)$.

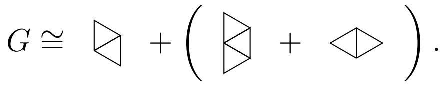
Definition 4.4. Let $G \cong \{\mathcal{G}^L \mid \mathcal{G}^C \mid \mathcal{G}^R\}$ and $H \cong \{\mathcal{H}^L \mid \mathcal{H}^C \mid \mathcal{H}^R\}$ be games. Then the *sum* of these games, denoted by $+$, is defined recursively as follows:

$$G + H \cong \{(\mathcal{G}^L + H) \cup (G + \mathcal{H}^L) \mid (\mathcal{G}^C + H) \cup (G + \mathcal{H}^C) \mid (\mathcal{G}^R + H) \cup (G + \mathcal{H}^R)\},$$

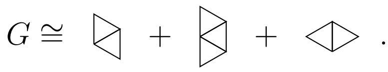
where for $\alpha \in \{L, C, R\}$:

$$\mathcal{G}^\alpha + H := \{X + H : X \in \mathcal{G}^\alpha\}, \qquad G + \mathcal{H}^\alpha := \{G + X : X \in \mathcal{H}^\alpha\}.$$

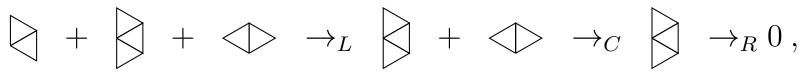
Example 4.5. Consider the following sum of RHOMBINATION games:



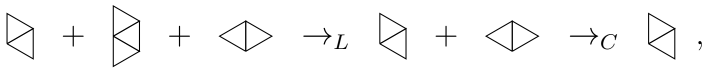
In Lemma 4.8 we will be concluding that addition of three-player games is associative, just like two-player games. Thus



Assuming Left makes the first move, the following are the only possible play-outs:



in which case Right wins, and:



in which case Center wins.

Many proofs in this subsection are almost the same as their analogues from the previous section, with the only difference being that there is a third player. We will, for the most part, not give these proofs. However, we will give the proof of Lemma 4.6 as an example. When there is a major difference, we will give the proof.

Lemma 4.6. *Let G and H be games. Then $G + H$ is a game.*

Proof. We use induction on the sum of the birthday of G and the birthday of H. Our base case is that both G and H have birthday 0. The only game of birthday 0 is 0 and since 0 has no Left, Center, or Right options, we have $0 + 0 \cong \{0^L + 0 \cup 0 + 0^L \mid 0^C + 0 \cup 0 + 0^C \mid 0^R + 0 \cup 0 + 0^R\} \cong \{\,\mid\,\mid\,\} \cong 0$, which is a game.

Now, let $N \in \mathbb{N}$ be such that $G + H$ is a game for all games G of birthday n' and games H of birthday m' with $n' + m' \leq N$. Let G be a game of birthday n and H be a game of birthday m, such that $n + m = N + 1$. Then for all options $X \in \big(\mathcal{G}^L + H\big) \cup \big(G + \mathcal{H}^L\big) \cup \big(\mathcal{G}^C + H\big) \cup \big(G + \mathcal{H}^C\big) \cup \big(G^R + H\big) \cup \big(G + H^R\big)$ of $G + H$, we can use our induction hypothesis to conclude that X is a game. Let k be the highest birthday of all these games X. Then $G + H$ is a game of birthday $k + 1$. $\square$

Lemma 4.7. *Let G be a game. Then $G + 0 \cong G$.*

Proof. The proof for this lemma is analogous to the proof for Lemma 3.8. $\square$

Lemma 4.8. *Addition of games is commutative and associative.*

Proof. The proof for this lemma is analogous to the proof for Lemma 3.9. □

Just like in two-player games, we would like to classify games in *outcome classes*. However, in three-player games we have to deal with "kingmaking". "kingmaking" occurs when a player is unable to win a game themselves, but can choose which of the other players becomes the winner. In Example 4.5, we can see a "kingmaking" situation. If Left starts, they are unable to win themselves, and must choose between letting Center, or Right win. This makes it difficult to classify these kinds of games in outcome classes. In order to combat this problem, many different solutions have been studied, as mentioned in the relevant work section.

In this thesis we will use Li's solution [Li78]. Li proposes to give each player preferences for who they would like to make the last move. A player now wants to have made a move as recently as possible when the game ends. This means that a player α would most like for themselves to have made the last move. If this is not possible they would like player $\sigma(\alpha)$ to win. Their least favourite player to make the last move is $\sigma^{-1}(\alpha)$. For example, if the Left player can choose between letting the Center or Right player win, as in Example 4.5, they will choose to let the Center player win, as $\sigma(\text{L}) = \text{C}$. This gives the player preferences in Table 2.

player	**First preference**	**Second preference**	**Last preference**
Left	Left	Center	Right
Center	Center	Right	Left
Right	Right	Left	Center

Table 2: Player preferences as introduced by Li [Li78].

We solidify these preferences in our definition of the *outcome function* and the *outcome classes*.

Definition 4.9. Let $\mathbb{G}$ be the set of games. We recursively define the *outcome functions* for all players, where for $\alpha \in \{L, C, R\}$, this is the function $o_\alpha \colon \mathbb{G} \to \{L, C, R\}$, given by

$$o_\alpha(G) = \begin{cases} \alpha & \text{if } o_{\sigma(\alpha)}(H) = \alpha \text{ for some } H \in \mathcal{G}^\alpha \\ \sigma(\alpha) & \text{else if } o_{\sigma(\alpha)}(H) = \sigma(\alpha) \text{ for some } H \in \mathcal{G}^\alpha \\ \sigma^{-1}(\alpha) & \text{else} \end{cases}.$$

We now define the *outcome function* $o \colon \mathbb{G} \to \{L, C, R\}^3$, where

$$o(G) = (o_L(G), o_C G(), o_R(G)).$$

We denote this by $o(G) = o_L(G)o_C(G)o_R(G)$. We use the outcome function to define the *outcome classes*. Let $\alpha, \beta, \gamma \in \{L, C, R\}$. Then $\alpha\beta\gamma := \{G \in \mathbb{G} : o(G) = \alpha\beta\gamma\}$.

If a game G has a valid option where player α is able to win, they will choose that move. If such a move does not exist, they will prefer letting player $\sigma(\alpha)$ win, to letting player $\sigma^{-1}(\alpha)$ win. Note that if player α has no available moves, player $\sigma^{-1}(\alpha)$ will immediately win. The outcome function tells us which player will win, if all players follow their established player preferences. Just as in the two-player case, we say that a player is playing *optimally* when they make moves that follow their outcome function.

Definition 4.10. Let G and H be games, we say that G and H are *equal* if for all games X it holds that $o(G + X) = o(H + X)$. We denote this by $G = H$.

Lemma 4.11. *Equality of games is an equivalence relation.*

Proof. The proof for this lemma is analogous to the proof for Lemma 3.13. □

Lemma 4.12. *Let G, H and J be games with $G = H$. Then $G + J = H + J$.*

Proof. The proof for this lemma is analogous to the proof for Lemma 3.14. □

Theorem 4.13. *The set of games, for three players, modulo equality is a commutative monoid.*

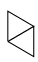
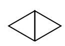
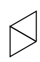
Proof. Lemma 4.6 tells us that the sum of games is a game. Addition is commutative and associative by Lemma 4.8. Lemma 4.11 tells us that equality of games is an equivalence relation, thus the set of games modulo equality is well-defined. Furthermore, by Lemma 4.12, addition is consistent with respect to equality, making addition a well-defined operation for the set of games modulo equality. Lastly, Lemma 4.7 tells us that 0 is a neutral element for addition. □

When dealing with two players, we concluded that the set of game values was an abelian group. The ingredient we are missing, now that we have three players, is the existence of additive inverse elements.

4.2 Rotations

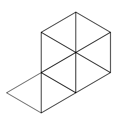
In the change from two to three players, the number of outcome classes has grown from 4 to 27. This leads us to wonder whether these outcome classes are non-empty. Greene has been able to show that, for the rule set Rhombination, 24 of these classes are non-empty [Gre17]. The state of the remaining three outcome classes: RCL, CLR, and LRC was unknown. This section builds towards Theorem 4.22, in which we conclude that all 27 Rhombination outcome classes are non-empty.

For convenience, we define the following Rhombination games:

$1_L \cong \{0 \,|\,|\, \} \cong$  , $1_C \cong \{\,|\, 0 \,|\, \} \cong$  , $1_R \cong \{\,|\,|\, 0\} \cong$  .

These are the games in which a single player has an available move.

Proposition 4.14. *Let*

$G \cong$  .

It holds that $G \in RCL$.

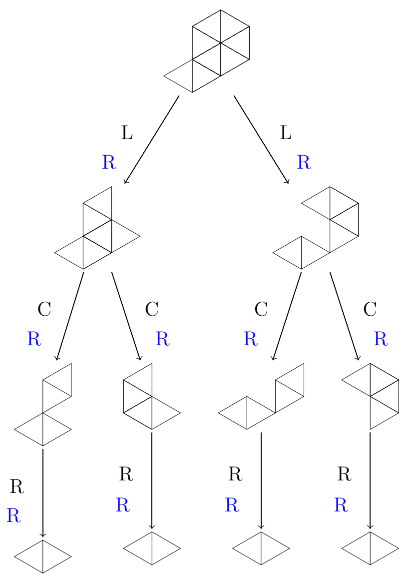


Figure 2: Game tree of G when Left begins. Black letters represent which player makes the move, while the blue letters represent the outcome of the resulting game.

Proof. The game tree when Left starts on G is shown in Figure 2. We can see that, whatever move Left starts with, Right will win. Thus $o_L(G) = R$.

The game tree of G when Center starts can be seen in Figure 3. We see that Center has a move where they will win. Thus $o_C(G) = C$.

The game tree of G when Right starts can be seen in Figure 4. We see that Right can choose whether to let Left or Right win. Recalling our player preferences, Right prefers Left winning, to Center winning, since $\sigma(R) = L$. Thus $o_R(G) = L$.

Putting everything together we have that $o(G) = RCL$. Thus $G \in RCL$ holds. □

Proposition 4.14 gives us that the outcome class RCL is non-empty. Instead of manually finding examples of games in the remaining two outcome classes, we will introduce rotations, with corresponding lemmas, and a theorem which will let us prove these outcome classes are non-empty. Rotations, its associated lemmas, and Theorem 4.19 are based on Greene's work [Gre17].

Definition 4.15. Let G be a game. We recursively define:

$$G^{120} := \left\{ (\mathcal{G}^C)^{120} \mid (\mathcal{G}^R)^{120} \mid (\mathcal{G}^L)^{120} \right\},$$

where for $\alpha \in \{L, C, R\}$: $(\mathcal{G}^\alpha)^{120} := \{(H)^{120} : H \in \mathcal{G}^\alpha\}$.

Example 4.16. Let

$$G \cong 1_L \cong \{0 \mid \mid \}.$$

Then, using the fact that $\emptyset^{120} \cong \emptyset$, we find that $0^{120} \cong \{ \mid \mid \} \cong 0$. This gives us:

$$G^{120} \cong \{ \mid \mid 0\} \cong 1_R,$$

and

$$G^{240} := (G^{120})^{120} \cong \{ \mid 0 \mid \} \cong 1_C.$$

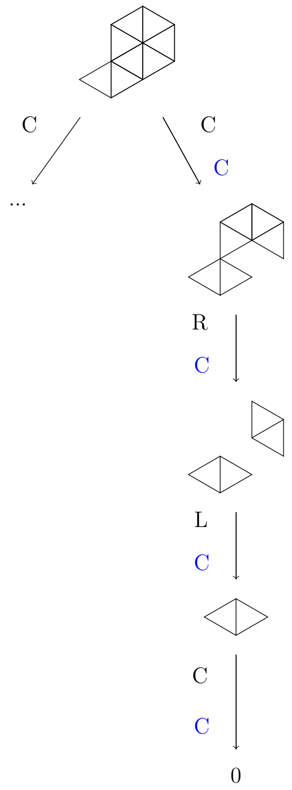


Figure 3: Game tree of G when Center begins. Black letters represent which player makes the move, while the blue letters represent the outcome of the resulting game. Dots represent that there are more children, which are not necessary for our analysis.

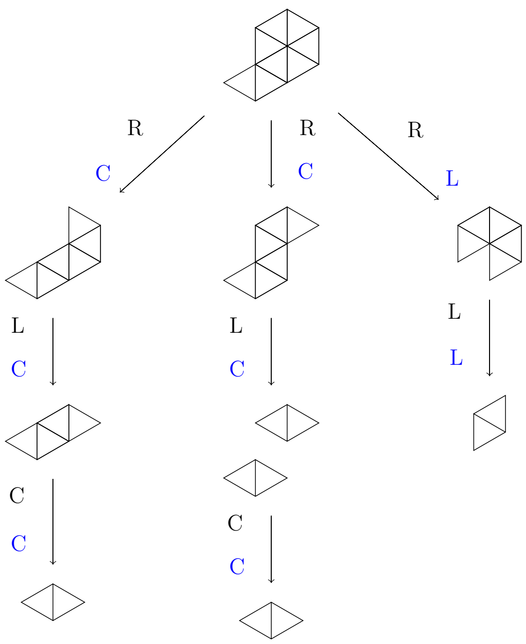


Figure 4: Game tree of G when Right begins. Black letters represent which player makes the move, while the blue letters represent the outcome of the resulting game.

In general, if G is a Rhombination game, it holds that G^{120} is the game G rotated counter clockwise by 120 degrees. Though we will not prove this rigorously, it is intuitively clear, as for each player $\alpha \in \{L, C, R\}$, the moves available to them, playing on G^{120}, are the moves available to player $\sigma(\alpha)$ on G. Since the rhombus player $\sigma(\alpha)$ places on their turn is a 120 degree counter clockwise rotation of the rhombus player α must place, it follows that G^{120} is the game G rotated counter clockwise by 120 degrees.

Lemma 4.17. *Let G be a game of birthday n. Then G^{120} is also a game of birthday n.*

Proof. We use induction on the birthday of G. Our base case is the game 0, for which it holds, as mentioned in Example 4.16, that $0^{120} = 0$, which is of course a game of birthday 0.

Suppose $N \in \mathbb{N}$ is such that H^{120} is a game with the same birthday as H for all games H of birthday at most N. Let G be a game of birthday $N + 1$, then all of its options have a birthday of at most N, which means that the children of G^{120} also all have a birthday of at most N by our induction hypothesis. Furthermore, there must exist one option $H \in \mathcal{G}^L \cup \mathcal{G}^C \cup \mathcal{G}^R$ with a birthday of exactly N. Thus H^{120} also has a birthday of N. We now conclude that G^{120} is a game of birthday $N + 1$. □

Lemma 4.18. *Let G be a game. Then $G^{360} := ((G^{120})^{120})^{120} \cong G$.*

Proof. We use induction on the birthday of G. Our base case is that $G \cong 0$. Since $0^{120} \cong 0$, we have that $0^{360} \cong 0$.

Let $N \in \mathbb{N}$ be such that for all games H of birthday at most N, the identity $H^{360} \cong H$ holds. Let G be a game of birthday $N + 1$. Then for all games $X \in \mathcal{G}^L \cup \mathcal{G}^C \cup \mathcal{G}^R$, it holds that, $X^{360} \cong X$. Thus $(\mathcal{G}^L)^{360} \cong \mathcal{G}^L$, $(\mathcal{G}^C)^{360} \cong \mathcal{G}^C$, and $(\mathcal{G}^R)^{360} \cong \mathcal{G}^R$. Now we can conclude that:

$$G^{360} \cong \left\{(\mathcal{G}^L)^{360} \mid (\mathcal{G}^C)^{360} \mid (\mathcal{G}^R)^{360}\right\} \cong \{\mathcal{G}^L \mid \mathcal{G}^C \mid \mathcal{G}^R\} \cong G.$$

□

Theorem 4.19. *Let G be a game, and $\alpha, \beta, \gamma \in \{L, C, R\}$. Then $G \in \alpha\beta\gamma$ if and only if $G^{120} \in \sigma^{-1}(\beta)\sigma^{-1}(\gamma)\sigma^{-1}(\alpha)$.*

We will first give an example of the theorem.

Example 4.20. It holds that $1_L \in LLC$. Using Theorem 4.19, we conclude that $(1_L)^{120} \in \sigma^{-1}(L)\sigma^{-1}(C)\sigma^{-1}(L) = RLR$. In Example 4.16, we have shown that $(1_L)^{120} = 1_R$, which is consistent, since $1_R \in RLR$

Proof of Theorem 4.19. We assume, without loss of generality, that Left makes the first move on G. We use induction on the birthday of G. The base case is that the birthday of G is 0, which means that $G \cong 0$. It holds that $0 \in RLC$. Furthermore, it holds that $0^{120} = 0 \in RLC = \sigma^{-1}(L)\sigma^{-1}(C)\sigma^{-1}(R)$.

Let $N \in \mathbb{N}$ be such that for all games H with birthday at most N, and outcome classes $\alpha\beta\gamma$, the following holds:

$$H \in \alpha\beta\gamma \text{ if and only if } H^{120} \in \sigma^{-1}(\beta)\sigma^{-1}(\gamma)\sigma^{-1}(\alpha).$$

Let G be a game of birthday $N+1$ and $\alpha\beta\gamma$ an outcome class. Every Left option $G^L \in \mathcal{G}^L$ is a game of birthday at most N. Thus we can use our induction hypothesis on these games. This gives us that for $\delta \in \{L, C, R\}$: $o_C(G^L) = \delta$ if and only if $o_L((G^L)^{120}) = \sigma^{-1}(\delta)$.

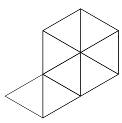
We claim that $o_L(G) = L$ if and only if $o_R(G^{120}) = R$ holds. By definition, it holds that $o_L(G) = L$ if and only if there exists an option G^L with $o_C(G^L) = L$. Using our induction hypothesis, this is true if and only if there exists a Right option $(G^L)^{120}$ of G^{120}, with $o_L((G^L)^{120}) = \sigma^{-1}(L) = R$. Since this is a winning move for Right, such a move exists if and only if $o_R(G^{120}) = R$. Thus $o_L(G) = L$ if and only if $o_R(G^{120}) = R = \sigma^{-1}(L)$.

Now, we claim that $o_L(G) = C$ if and only if $o_R(G^{120}) = L$ holds. By definition, it holds that $o_L(G) = C$ if and only if there does not exists an option G^L with $o_C(G^L) = L$, and there does exist an option G^L with $o_C(G^L) = C$. Using our induction hypothesis, this is true if and only if there does not exists a Right option $(G^L)^{120}$ of G^{120}, with $o_L((G^L)^{120}) = \sigma^{-1}(L) = R$, and there does exist a Right option $(G^L)^{120}$ of G^{120}, with $o_L((G^L)^{120}) = \sigma^{-1}(C) = L$. Now, Right does not have a winning move on G^{120}, but can let Left win, which is their second choice. Thus this is true if and only if $o_R(G^{120}) = L$. Thus $o_L(G) = C$ if and only if $o_R(G^{120}) = L = \sigma^{-1}(C)$.

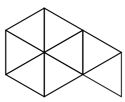
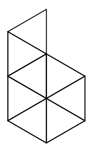
All that remains is to prove that $o_L(G) = R$ if and only if $o_R(G^{120}) = C$ holds. By definition, it holds that $o_L(G) = R$ if and only if for all Left options G^L, it holds that $o_C(G^L) = R$, which includes the case that Left has no options. Using our induction hypothesis, this is true if and only if for all Right options $(G^L)^{120}$, it holds that $o_L((G^L)^{120}) = \sigma^{-1}(R) = C$. Now, all Right's options on G^{120} lead to a win for Center. So this is true if and only if $o_R(G^{120}) = C$. Thus $o_L(G) = R$ if and only if $o_R(G^{120}) = C = \sigma^{-1}(R)$.

We have now proven for all $\delta \in \{L, C, R\}$ that $o_L(G) = \delta$ if and only if $o_R(G^{120}) = \sigma^{-1}(\delta)$. By induction and generality, we conclude that for all games G, and outcome classes $\alpha\beta\gamma$, the following holds: $G \in \alpha\beta\gamma$ if and only if $G^{120} \in \sigma^{-1}(\beta)\sigma^{-1}(\gamma)\sigma^{-1}(\alpha)$. □

Example 4.21. Let

$G \cong$  .

Proposition 4.14 tells us that $G \in RCL$. We now use Theorem 4.19 to conclude that $G^{120} \in \sigma^{-1}(C)\sigma^{-1}(L)\sigma^{-1}(R) = LRC$. Furthermore $G^{240} = (G^{120})^{120} \in \sigma^{-1}(R)\sigma^{-1}(C)\sigma^{-1}(L) = CLR$. G^{120} and G^{240} are picture below for reference:

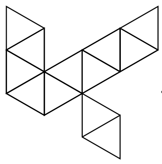
$G^{120} \cong$  , $G^{240} \cong$  .

Theorem 4.22. *All* 27 *outcome classes are non-empty for the rule set* RHOMBINATION.

Proof. Greene has presented examples of RHOMBINATION games in all outcome classes, except for RCL, CLR, and LRC [Gre17]. So all that remains to show is that RCL, CLR, and LRC are non-empty. Proposition 4.14 tells us that RCL is non-empty. Now, Example 4.21, which uses Theorem 4.19, shows that CLR and LRC are also non-empty. □

4.3 Inverses

At the end of Section 4.1, we concluded that three player games form a commutative monoid. In order to form an abelian group, as in the two player case, the existence of additive inverses is required. Greene hypothesises that for all games G, it holds that $G + G^{120} + G^{240} = 0$ [Gre17]. This would mean that $-G = G^{120} + G^{240}$. The intuition behind this is that these rotations give every player the same opportunities. Theorem 4.25 shows that does not hold. For this subsection, we let:

$H \cong$  .

We will show that the outcome classes of H and $H + 1_L + 1_C + 1_R$ are not equal. This proves that $1_L + 1_L^{120} + 1_L^{240} = 1_L + 1_C + 1_R \neq 0$

Lemma 4.23. *It holds that* $o_L(H) = L$.

Proof. The game tree when Center starts on H is shown in Figure 5. We can see that whatever move Center starts with, Left will win. Thus $o_C(G) = L$. □

Lemma 4.24. *It holds that* $o_C(H + 1_L + 1_C + 1_R) \neq L$.

Proof. In Figure 6 a partial game tree of $H + 1_L + 1_C + 1_R$ when Center starts can be seen. The tree shows a specific starting move for Center, which leads to a win for Right. Since Center prefers Right winning to Left winning, we know that regardless of Center's other options, they will not choose to let Left win. Therefore, it holds that $o_C(H + 1_L + 1_C + 1_R) \neq L$. □

Theorem 4.25. *It does not hold for all games* G *that* $G + G^{120} + G^{240} = 0$.

Proof. Suppose it does hold for all games G that $G+G^{120}+G^{240} = 0$. Then, through the definition of equality, we have that for all games X, the equality $o(G+G^{120}+G^{240}+X) = o(X)$ holds. We let $X = H$ and $G = 1_L$. This gives us $o(1_L + 1_C + 1_R + H) = o(H)$. However, Lemma 4.23 and Lemma 4.24 tell us that $o(1_L + 1_C + 1_R + H) \neq o(H)$, which gives us a contradiction. □

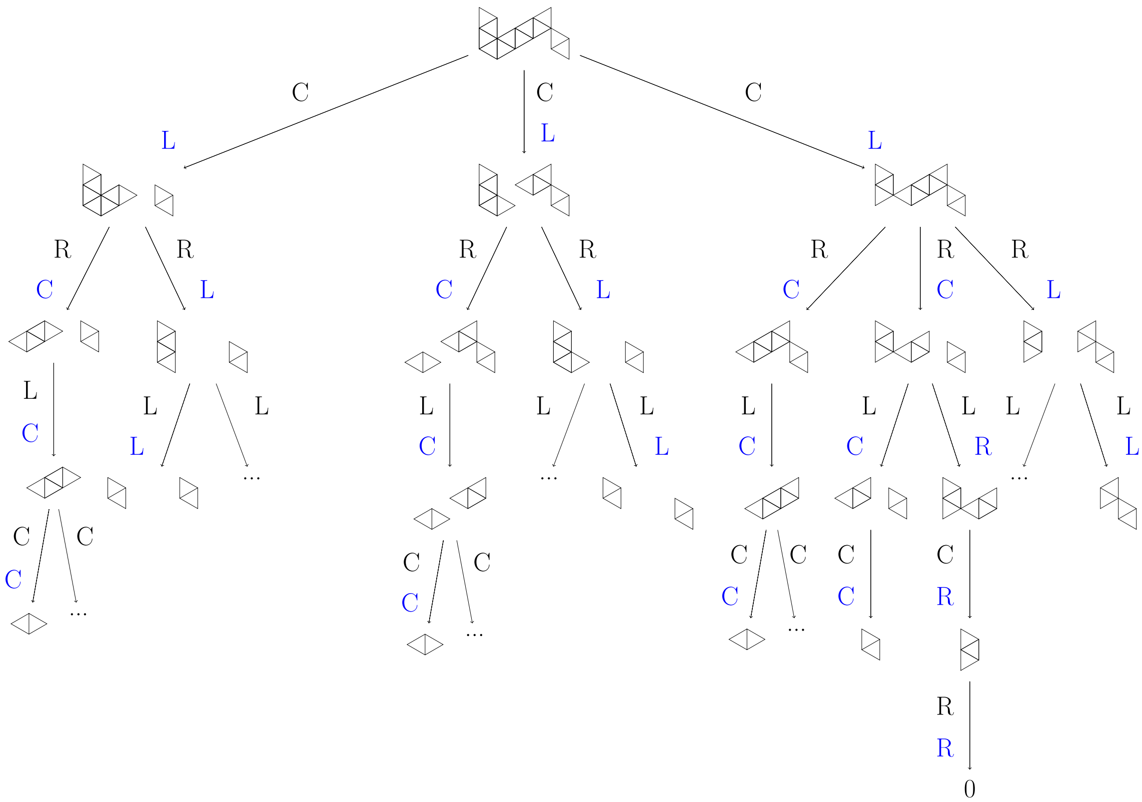


Figure 5: Game tree of H when Center begins. Black letters represent which player makes the move, while the blue letters represent the outcome of the resulting game. Dots represent that there are more children, which are not necessary for our analysis.

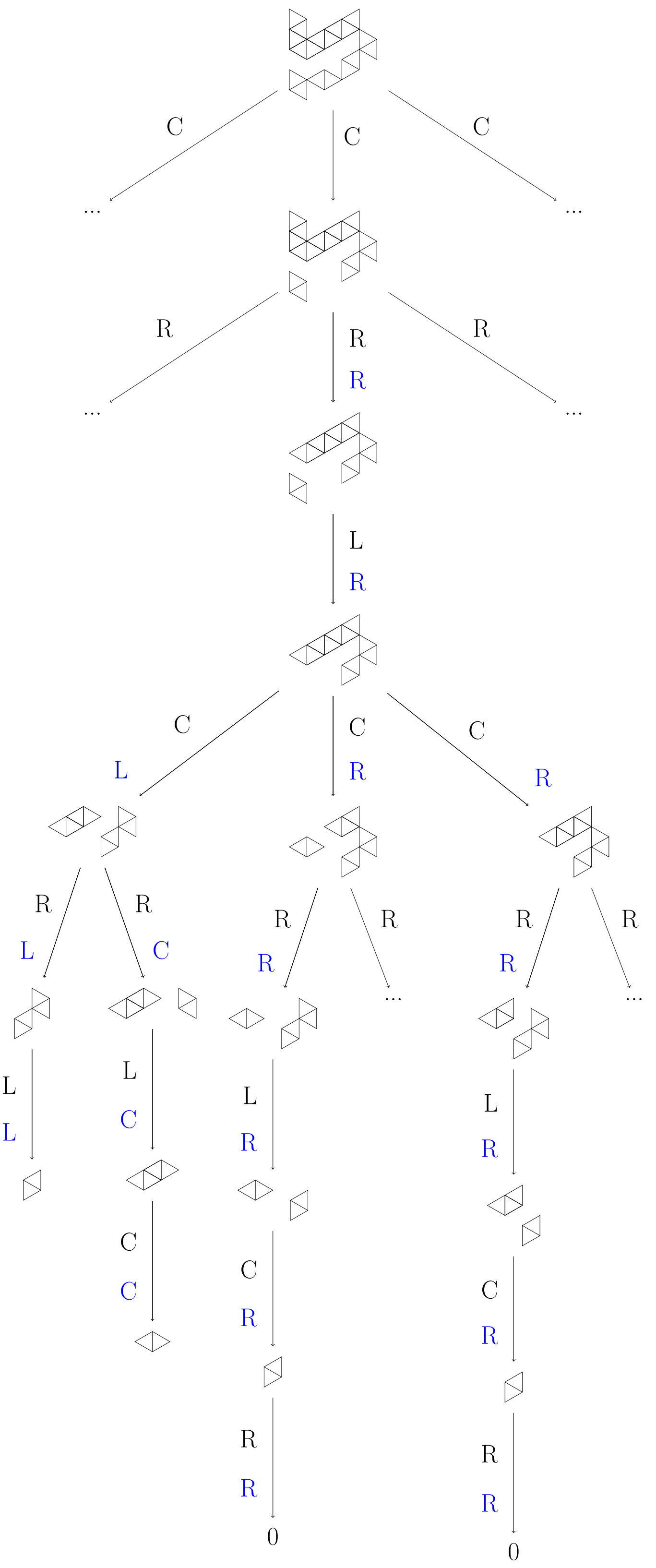


Figure 6: Partial game tree of $H + 1_L + 1_C + 1_R$ when Center begins. Black letters represent which player makes the move, while the blue letters represent the outcome of the resulting game. Dots represent that there are more children, which are not necessary for our analysis.

A possible reason why these are not additive inverses, even though each player gets the same moves, is that it is no longer optimal for a player to ensure they have as many future moves as possible, as in the two-player case. Now, a player, knowing they cannot win, might sabotage their own position in order to help the player after them in the player order.

4.4 Alternative assumptions

In the previous section, we have concluded it does not hold, with Li's player preferences, for all (Rhombination) games G that $G + G^{120} + G^{240} = 0$. We now ask ourselves if we could redefine our outcome classes such that this does hold. Our goal is to eliminate any preferences between the players. Furthermore, we would like to have fewer outcome classes, which will lead to a less strict definition of equality. We hypothesise that this intuitive candidate for additive inverse elements does hold using these new outcome classes, though we have been unable to prove this. However, we will conclude that this approach has several flaws. These make it far from a perfect solution, even if one were to show inverses exist. We will now only differentiate between the case that the starting player can force a win, whatever their opponents do, and the case where they cannot. So when a "kingmaking" situation occurs, the only thing we care about is that the active player cannot force a win.

We now redefine our outcome classes as follows:

Definition 4.26. Let $\mathbb{G}$ be the set of games. We recursively define the functions $f_\alpha \colon \mathbb{G} \to \{L, C, R, O\}$, for $\alpha \in \{L, C, R\}$, given by

$$f_\alpha(G) = \begin{cases} \alpha & \text{if } o_{\sigma(\alpha)}(H) = \alpha \text{ for some } H \in \mathcal{G}^\alpha \\ \sigma^{-1}(\alpha) & \text{else if } \mathcal{G}^\alpha = \emptyset \\ \beta & \text{else if } o_{\sigma(\alpha)}(H) = \beta \text{ for all } H \in \mathcal{G}^\alpha \\ O & \text{else} \end{cases}.$$

Now the *outcome functions* for all players are given by $o_\alpha \colon \mathbb{G} \to \{T, F\}$, for $\alpha \in \{L, C, R\}$, where

$$o_\alpha(G) = \begin{cases} T & \text{if } f_\alpha(G) = \alpha \\ F & \text{else} \end{cases}.$$

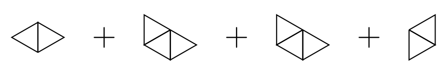
We now define the *outcome function* as $o(G) \colon \mathbb{G} \to \{T, F\}^3$, where

$$o(G) = (o_L(G), o_C(G), o_R(G)).$$

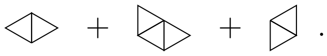
We denote this by $o(G) = o_L(G)o_C(G)o_R(G)$. We use the outcome function to define the *outcome classes*. Let $\alpha, \beta, \gamma \in \{L, C, R\}$. Then $\alpha\beta\gamma := \{G \in \mathbb{G} : o(G) = \alpha\beta\gamma\}$.

The functions f_α tell us which, if any, player can force a win, whatever the other players do. Since we are only interested in whether the active player can force a win, the outcome functions set every other case to F.

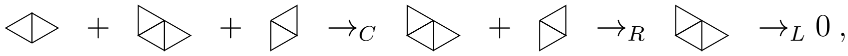
Example 4.27. It holds that $0 \in FFF$. Furthermore $1_L \in TFF$. Now let

$G \cong$  .

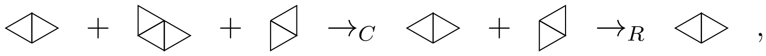
Then, assuming Left makes the first move, they have one distinct move, which is to move to the following game:



On Center's turn, they have two possible moves, which leads to the following play-outs:



in which case Left wins, and:



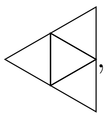
in which case Right wins. Since Center can choose whether Left wins, Left is unable to force a win. Therefore, $o_L(G) = F$.

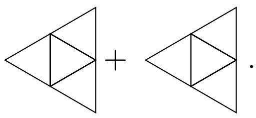
We use the same definition of equality as before, but now using our new outcome classes.

As previously mentioned, we conjecture that, with these outcome classes, the equality $G + G^{120} + G^{240} = 0$ holds for all games G. Our belief stems from the fact that we have eliminated preferences between the players, which seems to have been a key reason the desired equality does not hold with Li's preferences. Furthermore, we have reduced the number of outcome classes. This relaxes the definition of equality.

Even if our conjecture were to hold, there are still drawbacks to these assumptions. First of all, we no longer have a one-to-one correspondence between an outcome class and games that are equal to zero as in Theorem 3.20, this can be seen in Example 4.28. This would be useful, as it is far easier to show a game is in an outcome class then showing it is equal to zero explicitly. The second problem is that a majority of all games, especially the games with a higher birthday, are in the outcome class FFF, as no starting player can force a win. Thus the outcome classes say quite little about the games. Another problem with this approach is that it is no longer possible to determine the outcome class of a game based on the outcome class of its children, as information is lost.

Example 4.28. Let

$X =$ ,

$G =$ .

Then $G \in FFF$, as the second player to move will win. Furthermore, it holds that $X \in TTT$. If $G = 0$ were to hold, then we would have $o(G + X) = o(X)$. However, it holds that $o(G+X) = FFF \neq o(X)$. Therefore $G \neq 0$. This shows that $G \in FFF$ does not imply $G = 0$. Since $0 \in FFF$, it follows that there is no one-to-one correspondence between an outcome class and games equal to 0.

5 Conclusion and future directions

In this thesis, we have tried to create a structure like that of two-player combinatorial games, when adding a third player. In order to do this, we have adopted the player preferences introduced by Li [Li78]. Using these player preferences, we have shown that all 27 outcome classes are non-empty for the rule set RHOMBINATION. Lastly, we have proven that the intuitive candidates for additive inverse elements, hypothesised by Greene [Gre17], are not inverse elements.

Future work could be done in studying the alternative outcome classes, introduced in Section 4.4. An important result would be whether additive inverses exist in this scenario.

Another open question we have not been able to answer is whether three player games form an abelian group. Since the intuitive candidate for additive inverse elements does not work, our hypothesis is that three player games do not form an abelian group. A possible approach to prove our hypothesis is to look at the *group of differences*. The lemmas in this section are based on the work of Bruns and Gubeladze [BG09]. We will work with an arbitrary commutative monoid in the following lemmas. In Lemma 5.3, we construct a map which is bijective if and only if our commutative monoid is an abelian group. This gives an approach to determine whether the set of three-player games modulo equality is an abelian group.

Let M a commutative monoid. We now define an equivalence relation $\sim$ on the Cartesian product $M \times M$ by

$$(a, b) \sim (c, d) \iff \exists k \in M \text{ such that } a + d + k = b + c + k.$$

Lemma 5.1. *It holds that $\sim$ is an equivalence relation on $M \times M$.*

Proof. Let $(a, b) \in M \times M$. Using $k = 0$, we have $a + b = a + b$. This gives us that $(a, b) \sim (a, b)$. Thus $\sim$ is reflexive.

Let $(a, b), (c, d) \in M \times M$ with $(a, b) \sim (c, d)$. There exists a $k \in M$ such that $a + d + k = b + c + k$. Using commutativity of $+$, and symmetry of $=$, we have $c + b + k = d + a + k$. Thus $(c, d) \sim (a, b)$, and $\sim$ is symmetric.

Let $(a, b), (c, d)$, and $(e, f) \in M \times M$, with $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then there exist $k, k' \in M$, such that $a + d + k = b + c + k$ and $c + f + k' = d + e + k'$. Let $x = c + d + k + k'$. Then $a + f + x = b + e + x$. Thus $(a, b) \sim (e, f)$, and $\sim$ is transitive. □

We denote elements of $(M \times M)/\sim$ as $\overline{(a, b)}$, where $a, b \in M$, and $\overline{(a, b)}$ is the equivalence class of (a, b).

Lemma 5.2. *The set $(M \times M)/\sim$, with coordinate wise addition, is an abelian group.*

Proof. We must first show that coordinate wise addition is a well-defined operation on $(M \times M)/\sim$. That is, that it is independent of representative elements. Let $(a, b), (c, d), (e, f) \in M \times M$, such that $(a, b) \sim (c, d)$. It is simple to show that

$\overline{(a+e, b+f)} \sim \overline{(c+e, d+f)}$. Using this fact, we have $\overline{(a,b)} + \overline{(e,f)} = \overline{(a+e, b+f)} = \overline{(c+e, d+f)} = \overline{(c,d)} + \overline{(e,f)}$. Thus coordinate wise addition is well defined on $(M \times M)/\sim$.

Commutativity and associativity follow immediately from commutativity and associativity of addition on M. Furthermore, for all elements $\overline{(a,b)} \in (M \times M)/\sim$, it holds that $\overline{(a,b)} + \overline{(0,0)} = \overline{(a+0, b+0)} = \overline{(a,b)}$. Thus $\overline{(0,0)}$ is a neutral element for coordinate wise addition. Lastly, for all elements $\overline{(a,b)} \in (M \times M)/\sim$, it holds that $\overline{(a,b)} + \overline{(b,a)} = \overline{(a+b, a+b)}$. Using $(a+b, a+b) \sim (0,0)$, we have $\overline{(a+b, a+b)} = \overline{(0,0)}$. Thus $\overline{(b,a)}$ is an additive inverse element of $\overline{(a,b)}$. We now conclude that $(M \times M)/\sim$ is an abelian group. □

Lemma 5.3. *The map $i \colon M \to (M \times M)/\sim$, defined by $i(a) = (\overline{(a,0)})$ is a homomorphism. Furthermore, it is bijective if and only if M is an abelian group.*

Proof. We will first prove that i is a homomorphism. It is easy to see that $i(0) = \overline{(0,0)}$. Let $a, b \in M$. Then $i(a+b) = \overline{(a+b, 0)} = \overline{(a,0)} + \overline{(b,0)} = i(a) + i(b)$. Thus i is a homomorphism.

Suppose that i is bijective. Let i^{-1} be its inverse map. Then for $G \in M$, we have

$$\begin{aligned} G + i^{-1}\left(\overline{(0,G)}\right) &= i^{-1}\left(\overline{(G,0)}\right) + i^{-1}\left(\overline{(0,G)}\right) \\ &= i^{-1}\left(\overline{(G,0)} + \overline{(0,G)}\right) \\ &= i^{-1}(0) = 0. \end{aligned}$$

Thus for every $G \in M$, the element $i^{-1}\left(\overline{(0,G)}\right) \in M$ is G's inverse. Thus M is an abelian group if i is bijective.

Now, suppose that M is an abelian group. In order to show that i is bijective, we construct the map $j \colon (M \times M)/\sim \, \to M$ defined by $j\left(\overline{(a,b)}\right) = a - b$, where $-b$ is the additive inverse of b. Let $(a,b), (c,d) \in M \times M$, such that $(a,b) \sim (c,d)$. To show j is well-defined, we must prove that $j\left(\overline{(a,b)}\right) = j\left(\overline{(c,d)}\right)$. Using $(a,b) \sim (c,d)$, we have that $a - b = c - d$. So $j\left(\overline{(a,b)}\right) = a - b = c - d = j\left(\overline{(c,d)}\right)$. Thus j is well-defined. Let $a \in M$. Then $j(i(a)) = j\left(\overline{(a,0)}\right) = a - 0 = a$. Therefore $j \circ i = Id_M$. Now, let $\overline{(a,b)} \in (M \times M)/\sim$. Then $i\left(j\left(\overline{(a,b)}\right)\right) = i(a-b) = \overline{(a-b, 0)}$. Using $(a-b, 0) \sim (a,b)$, we have that $\overline{(a-b,0)} = \overline{a,b}$. Thus $i \circ j = Id_{(M \times M)/\sim}$. We conclude that j is an inverse map of i, which implies that i is bijective. □

Using the set of three-player games modulo equality as our monoid, we have that this is an abelian group if and only if the map i is bijective. If one were to show that the map i is not bijective, Lemma 5.3 would prove that M is not an abelian group. On the other hand, this gives a possible approach to show M is an abelian group, without having to explicitly find additive inverse elements, though the explicit inverse elements might be required to show that i is bijective.

References

[BG09] Winfried Bruns and Joseph Gubeladze. *Polytopes, rings, and K-theory*, volume 27, page 52. Springer, 2009.

[Cin05] Alessandro Cincotti. Three-player partizan games. *Theoretical Computer Science*, 332(1-3):367–389, 2005.

[Con76] J. Conway. *On Numbers and Games.* Academic Press, 1976.

[Gre17] K. Greene. Exploration of the three-player partizan game of rhombination. Master's thesis, Wake Forest University Graduate School of Arts and Sciences, 2017.

[Jr.85] Philip D. Straffin Jr. Three person winner-take-all games with mccarthy's revenge rule. *The College Mathematics Journal*, 16(5):386–394, 1985.

[Li78] S.-Y.R Li. N-person nim and n-person moore's games. *International Journal of Game Theory*, 7:31–36, 1978.

[Loe96] Daniel E Loeb. Stable winning coalitions. *Games of No Chance*, 29:451–471, 1996.

[Sie13] A. Siegel. *Combinatorial game theory.* American Mathematical Society, 2013.