Formalizing the Van Kampen Theorem
for Directed Topology

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Abstract

Directed topology is a fairly new field of mathematics with applications in concurrency. It extends the concept of a topological space by adding a notion of directedness in which directed paths play a very important role. There are direction preserving maps between directed spaces called directed maps. A special case of these is a directed path homotopy that transforms one directed path into another. Using these deformations, directed paths are partitioned into equivalence classes and a special category, the fundamental category, can be linked to a directed space. In this thesis we will explain these definitions and present a special theorem: a directed version of the Van Kampen Theorem. This theorem allows the calculation of fundamental categories by combining local knowledge about paths. Our main contribution is the formalization of this material using the Lean proof assistant and we show how we have implemented this.

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1 Introduction

Directed topology, an extension of topology, is a field of mathematics that has been developed in order to abstractly analyze structures where paths with directions are present. The original motivation, given by Fajstrup et al. in 1999, stems from its applications in concurrency, a field researching the simultaneous execution of programs [FRG99].

Given two programs A and B, we can execute those sequentially in two ways: either first A or first B. That relatively simple case can be seen in the figure below on the left. If parallel execution is possible, any path from the bottom left to the top right is a valid execution path on the condition that the path never turns backward. After all, once an instruction has been executed it cannot be undone. This situation is already substantially more complicated than the sequential version.

In the case of parallel execution, two programs could try to write to the same location at the same time, which could lead to unintended behavior. In order to prevent this, we disallow such execution paths. We would obtain a case like the square on the right with the darker rectangle as an obstacle in it. If an execution path were to pass through that space, both programs would write to the shared location at the same time and we consider that invalid. There are now two truly different paths: the one through the top left and the one through the bottom right. Indeed, these two executions write to the shared location in a different order. Of course, depending on the programs there may be no or many different obstacles. In order to make all of these intuitions precise, the field of directed topology has been developed.

1.1 Directed Topology

There are multiple ways to define the notion of directedness in a topological space. The original approach of Fajstrup et al. uses spaces with a partial ordering or a local partial ordering [FRG06]. We use the somewhat more flexible d-spaces defined by Grandis by letting a directed space be a topological space with a distinguished set of paths [Gra03]. These paths are called directed paths. Between directed spaces there are direction preserving maps called directed maps and together they form the category dTop. A directed version of path homotopies can be used to relate directed paths and obtain equivalence classes. These classes are respected by concatenation of paths and with that the fundamental category of a directed space can be defined. It is the directed equivalent of the fundamental groupoid of a topological space [Bro06].

There is also a directed analogue of the fundamental group called the fundamental monoid, but often it cannot distinguish between different forms of directedness. The main theorem this thesis is concerned with is a directed version of the Van Kampen Theorem, originally stated and proven by Grandis [Gra03]. It allows the calculation of the fundamental category of a directed space by combining the fundamental categories of two subspaces. As we will see, the Lebesgue Number Lemma plays an important role in the proof.
1.2 Contributions

Our main contribution has been the formalization of definitions and theorems relating to directed topology, in particular the Van Kampen Theorem. Formalization is the use of a theorem prover, in our case Lean, in order to write definitions, theorems and proofs in a rigorous manner. A computer is then able to verify the correctness of these statements using type theory or another formal method. The major advantage of formalizing a theorem is that anyone can be fully certain about the correctness of a theorem without needing to sift through its proof in detail. An additional advantage is that the computer is able to aid with proofs by automatically filling in steps or suggesting what theorems to apply. That is why some formalization languages are also called proof assistants. Lastly, the digital nature of formalization lends itself to an efficient form of cooperation by means of file sharing.

We wrote the formalization using the works already present in MathLib. MathLib is a library containing the formalization in Lean of a vast amount of mathematical theory and it is the product of many different collaborators. We have uploaded all of our files in a git repository and it contains a guide on how to get it to run locally. The version of Lean we used is 3.50.3. Resources to learn and understand Lean can be found on the Lean community website.

1.3 Thesis overview

In Section 2, we give a brief overview of the mathematical background on topology and category theory needed to understand the main ideas and concepts of this thesis. Section 3 defines the notion of directed spaces and directed maps and we give a few examples. In Section 4 the definitions and some properties of directed homotopies and directed path homotopies are given. We use those to define relations on the set of directed paths between two points. In Section 5 the equivalence classes of paths under these relations are used to define the fundamental category and the fundamental monoid. The Van Kampen Theorem is stated in Section 6 and an application is given. Finally, in Section 7 we reflect on the ideas presented in this thesis and give some suggestions for further research.

Small excerpts from the Lean formalization can be found throughout the thesis. They can be recognized by the monospace font used. The surrounding text explains the ideas of the formalization and references the corresponding file in which the code can be found.

2 Preliminaries

This thesis assumes basic mathematical knowledge. Familiarity with topology and category theory is useful and this chapter will explain some of the necessary background. For further information about these topics, see [Wal14] (topology) or [Lei14] (category theory).

2.1 Topology

The field of topology is concerned with the properties of spaces and shapes “up to deformation”. Think of squares, circles, donuts or spheres. The notion of a topological space is central.

Definition 2.1 (Topological space). A topological space is a pair \((X, \mathcal{T})\), consisting of a set \(X\) together with a set \(\mathcal{T} \subseteq \mathcal{P}(X)\) such that:

\[
(1) \quad \emptyset \in \mathcal{T} \quad \text{and} \quad X \in \mathcal{T}.
\]
(2) If \( U \subseteq T \) is a set of elements of \( T \), then \( \bigcup_{U \in U} U \in T \).

(3) If \( A, B \in T \) then \( A \cap B \in T \).

The elements of \( T \) are called open sets and \( T \) is called the topology on \( X \). If \( T \) is clear from context, the set \( X \) is often referred to as the topological space instead of the pair \((X, T)\).

Informally speaking, if a point \( x \) is an element of some open set \( U \in T \) then all points “close” to \( x \) are also elements of \( U \). A subset \( V \subseteq X \) is called closed if its complement \( X \setminus V \) is open.

A continuous map between two topological spaces is then a map that maps points that are close together to points that are close together:

**Definition 2.2** (Continuous map). A continuous map between two topological spaces \((X, T_X)\) and \((Y, T_Y)\) is a map \( f : X \to Y \) of sets such that for all \( V \in T_Y \) it holds that \( f^{-1}(V) \in T_X \).

Compositions of continuous maps are again continuous. It is also easy to show that the identity map is always continuous.

One special topological space is the unit interval \([0, 1]\), a subspace of the real number line \(\mathbb{R}\). Continuous maps \( \gamma : [0, 1] \to X \) with \( X \) a topological space are called paths in \( X \). If \( \gamma_1 \) and \( \gamma_2 \) are two paths in \( X \) with \( \gamma_1(1) = \gamma_2(0) \), then you can concatenate those two paths by first following \( \gamma_1 \) and then \( \gamma_2 \). This new path is denoted \( \gamma_1 \circ \gamma_2 \) and given by:

\[
(\gamma_1 \circ \gamma_2)(t) = \begin{cases} 
\gamma_1(2t), & t \leq \frac{1}{2}, \\
\gamma_2(2t-1), & \frac{1}{2} < t.
\end{cases}
\]

These paths turn out to be important invariants of topological spaces and information about them can be captured in the so-called fundamental groupoid [Bro06, chapter 6]. Within this thesis we will take a look at the directed variant: the fundamental category.

### 2.2 Category Theory

Category theory is used to capture abstract similarities between seemingly unrelated mathematical fields. It turns out that many different concepts can be classified as categories:

**Definition 2.3** (Category). A category \( C \) consists of a collection of objects, \( \text{Ob}(C) \), together with sets of morphisms \( C(A, B) \) for all \( A, B \in \text{Ob}(C) \), a composition map \( \circ : C(B, C) \times C(A, B) \to C(A, C) \) and an identity \( \text{id}_A \in C(A, A) \) for all \( A \in \text{Ob}(C) \) such that:

1. For any \( A, B \in \text{Ob}(C) \) and \( f \in C(A, B) \) we have \( f \circ \text{id}_A = f = \text{id}_B \circ f \).

2. For any \( A, B, C, D \in \text{Ob}(C) \) and \( f \in C(A, B) \), \( g \in C(B, C) \), \( h \in C(C, D) \) we have \( (h \circ g) \circ f = h \circ (g \circ f) \).

Instead of \( A \in \text{Ob}(C) \), we often write \( A \in C \). Another way to write \( f \in C(A, B) \) is \( f : A \to B \), as morphisms often behave like maps.

**Example 2.4.** The category \( \text{Set} \) has as objects sets and as morphisms between two sets exactly the maps between those two sets. Composition of morphisms is exactly the composition of those maps. Each set has an identity map and composition of maps is associative, so properties 1 and 2 are satisfied.
**Example 2.5.** Top is the category of topological spaces and continuous maps. They are respectively the objects and the morphisms. Identities are continuous and composition is associative, so it is indeed a category.

There are also maps between different categories, called functors.

**Definition 2.6 (Functor).** Let $C$ and $D$ be two categories. A functor $F : C \to D$ consists of a map on objects and a map on morphisms, where it maps a morphism $f : A \to B$ to a morphism $F(f) : F(A) \to F(B)$, such that the following holds:

1. For any $A \in C$ we have $F(\text{id}_A) = \text{id}_{F(A)}$.
2. For any $A, B, C \in C$, $f : A \to B$ and $g : B \to C$ we have $F(g \circ f) = F(g) \circ F(f)$.

In other words, a functor needs to respect identities and composition.

When formulating the Van Kampen Theorem in Section 6.1, we use the term pushout square. This is a specific type of a so-called colimit. There are many ways to define those, but as we only need the pushout square colimit, we will not bother with the generalization.

**Definition 2.7 (Pushout square).** Let $C$ be a category, $A, B, C, D \in C$ four objects and $i : A \to B$, $j : A \to C$, $f : B \to D$ and $g : C \to D$ four morphisms such that $f \circ i = g \circ j$. We can draw them in a commutative square:

$$
\begin{array}{ccc}
A & \xrightarrow{i} & B \\
\downarrow{j} & & \downarrow{f} \\
C & \xrightarrow{g} & D
\end{array}
$$

We say that that square is a pushout square if for any object $X \in C$ and pair of morphisms $p : B \to X$ and $q : C \to X$ with $p \circ i = q \circ j$ there is a unique $r : D \to X$ such that $p = r \circ f$ and $q = r \circ g$.

$D$ is then called the pushout of the square.

**3 Directed Spaces**

In this section, we will look at the basic structure of a directed space. With directed maps as morphisms, the category of directed spaces $d\text{Top}$ is obtained.
3.1 Directed Spaces

A directed space is a topological space with a distinguished set of paths, whose elements are called directed paths. This is analogous to the set of open sets in a topological space. Similarly, the set of directed paths must satisfy some properties. Firstly, constant paths must be directed. Secondly, if there are two directed paths that connect in an end and start point, the concatenation of those two paths should again be directed. Intuitively, if you can walk from point A to point B and from B to C, then you should be able to walk from A to C by following those paths consecutively. Lastly, it should be possible to walk a part of a directed path at a different speed, as long as the path is not walked backward. This can be captured in the property that monotone subparametrizations of directed paths must also be directed paths.

**Definition 3.1 (Directed space).** A directed space is a topological space $X$, together with a set of paths in $X$, denoted $P_X$. That set must satisfy the following three properties:

1. For any point $x \in X$, we have that $0_x \in P_X$, where $0_x$ is the constant path in $x$.
2. For any two paths $\gamma_1, \gamma_2 \in P_X$ with $\gamma_1(1) = \gamma_2(0)$, we have that $\gamma_1 \circ \gamma_2 \in P_X$.
3. For any path $\gamma \in P_X$ and any continuous, monotone map $\varphi : [0, 1] \to [0, 1]$ we have that $\gamma \circ \varphi \in P_X$.

The elements of $P_X$ are called directed paths or dipaths.

We will first consider some examples of directed spaces.

**Example 3.2 (Directed unit interval).** We can equip the unit interval with a rightward direction. This is done by taking $P_{[0, 1]} = \{\varphi : [0, 1] \to [0, 1] \mid \varphi \text{ continuous and monotone}\}$. We will denote this directed space with $I$. More generally, every (pre)ordered space can be given a set of directed paths this way.

**Example 3.3 (Directed unit circle).** One of the ways the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ can be made into a directed space is by only allowing paths that go monotonously counterclockwise. Specifically, we take the set of directed paths $P_{S^1} = \{t \mapsto \exp(i\varphi(t)) \mid \varphi : [0, 1] \to \mathbb{R} \text{ continuous and monotone}\}$. This directed space will be denoted with $S^1_{\varphi}$.

**Example 3.4 (Maximal directed space).** Any topological space $X$ can be made into a directed space by taking $P_X$ as the set of all paths in $X$. We will call this the maximal directedness on $X$. This is also sometimes called the indiscrete or natural directedness.

**Example 3.5 (Minimal directed space).** Any topological space $X$ can be made into a directed space by taking $P_X = \{0_x \mid x \in X\}$. In other words, only the constant paths are allowed as directed paths. We will call this the minimal directedness on $X$. This is also sometimes called the discrete directedness.

**Example 3.6 (Product of directed spaces).** If $(X, P_X)$ and $(Y, P_Y)$ are two directed spaces, then the space $X \times Y$ with the product topology can be made into a directed space by letting $P_{X \times Y} = \{t \mapsto (\gamma_1(t), \gamma_2(t)) \mid \gamma_1 \in P_X \text{ and } \gamma_2 \in P_Y\}$. As we will see in Section 3.2, with this set of directed paths both projection maps will be directed and $(X \times Y, P_{X \times Y})$ becomes a product in the category of directed spaces and directed maps.

**Example 3.7 (Induced directed space).** Let $X$ be a topological space and $(Y, P_Y)$ a directed space. Let a continuous map $f : X \to Y$ be given. If $\gamma : [0, 1] \to X$ is a path in $X$, then $f \circ \gamma : [0, 1] \to Y$ is a path in $Y$. We can create a direction on $X$ by taking $P_X = \{\gamma \in C([0, 1], X) \mid f \circ \gamma \in P_Y\}$. It is not hard to verify that this satisfies all the properties of a directed space. In the special case that $X$ is a subspace of $Y$ and $f$ is the inclusion map, we find that every subspace of a directed space can be given a natural directedness.
We formalized the concept of a directed space by extending the topological_space class. In Lean, a type class is a collection of types and extending a class means adding new types. Other types can be made into an instance of a class by supplying a term for each type. For example, \( \mathbb{R} \), the type of real numbers, is an instance of topological_space. A useful property of extending classes is that you can get back any subclasses. In this specific case, we can use to_topological_space to obtain the underlying topological space of a directed space.

In our formalization, we do not explicitly use a set of paths. Rather, being directed is a property of a path itself, analogous to how being open is a property of a set in the topological_space class. Paths in topological spaces have been implemented in Mathlib in the file topology/path_connected.lean. A path has type \( \text{path } x \ y \), where its starting point is \( x \) and endpoint is \( y \). This way, concatenation of paths can be done without additional hypotheses. The definition of a directed space can be found in directed_space.lean.

```lean
class directed_space (α : Type u) extends topological_space α :=
  (is_dipath : \( \forall \{x y\}, \text{path } x \ y \to \text{Prop} \))
  (is_dipath_constant : \( \forall (x : α), \text{is_dipath } (\text{path.refl } x) \))
  (is_dipath_concat : \( \forall \{x y z\} \{γ_1 : \text{path } x \ y\} \{γ_2 : \text{path } y \ z\}, \text{is_dipath } γ_1 \to \text{is_dipath } γ_2 \to \text{is_dipath } (\text{path.trans } γ_1 \ γ_2) \))
  (is_dipath_reparam : \( \forall \{x y : α\} \{γ : \text{path } x \ y\} \{t_0 t_1 : I\} \{f : \text{path } t_0 \ t_1\}, \text{monotone } f \to \text{is_dipath } γ \to \text{is_dipath } (f.\text{map } γ.\text{continuous_to_fun}) \))
```

The term \( \text{is_dipath} \) determines whether a path is a directed path or not. The three other terms are exactly the three properties of a directed space. path.refl \( x \) is the constant path in a point \( x \) and path.trans is used for the concatenation of paths. Mathlib only has support for reparametrizations of paths (meaning that the endpoints must remain the same), but we want to also allow strict subparametrizations. We do this by interpreting the subparametrization \( f \) as a monotone path in \([0, 1]\). Then the path \( γ \circ f \) can be obtained using path.map, where we interpret \( γ \) as a continuous map.

In constructions.lean, different instances of directed spaces can be found: topological spaces with a preorder (Example 3.2), products of directed spaces (Example 3.6) and induced directedness (Example 3.7). For brevity, we introduce a notation for the set of all paths between \( x \) and \( y \).

**Definition 3.8.** If \( X \) is a directed space with two points \( x, y ∈ X \), we use the shorthand notation \( P_X(x, y) \) for the set \( \{γ ∈ P_X \mid γ(0) = x \text{ and } γ(1) = y\} \).

This can also be seen as a type for our formalization. That is exactly how to interpret the structure dipath, found in dipath.lean:

```lean
variables {X : Type u} [directed_space X]
structure dipath (x y : X) extends path x y :=
  (dipath_to_path : is_dipath to_path)
```

It extends the path structure and depends on two points \( x \) and \( y \) in a directed space \( X \). The term dipath_to_path is has type is_dipath to_path. That means that the underlying path it extends must be a directed path. Due to the axioms of a directed space, we can define dipath.refl and dipath.trans analogous to their path-counterparts. On the other hand, path.symm, the reversal of a path, cannot be converted to a directed variant as it is not guaranteed that the reversal of a directed path is directed.

We introduce a notation for a special kind of subpath of a directed path.
Definition 3.9. Let \( X \) be a directed space and \( \gamma \in P_X \) a directed path. If \( n > 0 \) and \( 1 \leq i \leq n \) we will define \( \gamma_{i,n} \in P_X \) to be the path from \( \gamma(\frac{i-1}{n}) \) to \( \gamma(\frac{i}{n}) \) given by \( \gamma_{i,n}(t) = \gamma(\frac{i+t}{n}) \).

We can now say what it means for a directed path to be covered by a cover of a directed space. This definition will play a big role in proving the Van Kampen Theorem.

Definition 3.10. Let \( X \) be a directed space and \( \mathcal{U} \) a cover of \( X \). Let \( \gamma \in P_X \) be a directed path and \( n > 0 \) an integer. We say that \( \gamma \) is \( n \)-covered (by \( \mathcal{U} \)) if we have for all \( 1 \leq i \leq n \) that \( \text{Im} \gamma_{i,n} \subseteq U \) for some \( U \in \mathcal{U} \).

In path_cover.lean we formalize this definition of \( n \)-covered in the special case that \( \mathcal{U} \) consists of two elements \( X_0 \) and \( X_1 \):

\[
\text{def covered} \ (\gamma : \text{dipath } x_0 \ x_1) \ (hX : X_0 \cup X_1 = \text{univ}) : \text{Prop} := \\
(\text{range } \gamma \subseteq X_0) \lor (\text{range } \gamma \subseteq X_1)
\]

\[
\text{def covered_partwise} \ (hX : X_0 \cup X_1 = \text{set.univ}) : \Pi \{x y : X\}, \text{dipath } x y \to \mathbb{N} \to \text{Prop} \\
| x y \gamma \emptyset := \text{covered } \gamma \ hX \\
| x y \gamma (\text{nat.succ } n) := \\
\quad \text{covered } (\text{split_dipath.first_part_dipath } \gamma \\
\quad \quad (\text{inv_I_pos } (\text{show } 0 < (\text{nat.succ } + 1), \text{by norm_num})))) \ hX \land \\
\quad \text{covered_partwise} \ (\text{split_dipath.second_part_dipath } \gamma \\
\quad \quad (\text{inv_I_lt_one } (\text{show } 1 < (\text{nat.succ } + 1), \text{by norm_num})))) \ n
\]

Here \( \text{covered} \) corresponds with \( \gamma \) being 1-covered: its image is either contained in \( X_0 \) or in \( X_1 \). We use this definition to inductively define \( \text{covered_partwise} \). As it is easier to start at zero in Lean, \( \text{covered_partwise } hX \ gamma \ n \) corresponds with \( \gamma \) being \( (n + 1) \)-covered. In the case that \( n = 0 \), we have that \( \text{covered_partwise} \) simply agrees with \( \text{covered } hX \ gamma \). Otherwise, we use an induction step to define that \( \text{covered_partwise } hX \ gamma \ (\text{nat.succ } n) \) holds if the first part \( \gamma_{1,n+2} \) is covered and the remainder of \( \gamma \) is \( \text{covered_partwise } hX \ gamma \ n \). Note the use of \( n + 2 \) instead of \( n + 1 \) due to the offset between the definitions. The remainder of path_cover.lean contains lemmas about conditions for being \( n \)-covered.

3.2 Directed Maps

As directed spaces are an extension of topological spaces, directed maps will be extensions of continuous maps. They will need to respect the extra directed structure. If a path in the domain space is given, a path in the codomain space can be obtained by composing the continuous map with the path. If the former is directed, so should be the latter.

Definition 3.11 (Directed map). Let \( X \) and \( Y \) be two directed spaces. A directed map \( f : X \to Y \) is a continuous map on the underlying topological spaces that furthermore satisfies: for any \( \gamma \in P_X \), we have that \( f \circ \gamma \in P_Y \).

Example 3.12. The map \( f : I \to S^1_+ \) given by \( t \mapsto e^{it} \) is a directed map. Firstly, it is continuous on the underlying topological spaces. Secondly, if \( \gamma \in P_I \) is a directed path, that is, continuous and monotone, then \( f \circ \gamma \) is given by \( t \mapsto \exp(i\gamma(t)) \) and that path is by definition of \( P_{S^1_+} \) directed.
Any continuous map (on the underlying topological spaces) from a minimally directed space to a directed space is directed. Similarly, any continuous map to a maximally directed space is directed. By construction of the product of directed spaces the continuous projection maps on both coordinates are directed: a directed path in the product space is a pair of directed paths and a projection returns the original directed path. Similarly, if a continuous map \( f : X \to Y \) is used to induce a direction on \( X \) as in Example 3.7, then \( f \) becomes a directed map from \( X \) to \( Y \), where \( X \) has the induced directedness.

In order to formalize the definition of a directed map in Lean, we first define \texttt{directed_map.directed}, which contains exactly the property that a continuous map between two directed spaces maps directed paths to directed paths:

```
def directed {α β : Type*} [directed_space α] [directed_space β] (f : C(α, β)) : Prop :=
  ∀ {x y : α} (γ : path x y), is_dipath γ → is_dipath (γ.map f.continuous_to_fun)
```

A directed map is then an extension of the \texttt{continuous_map} structure with a proof for being directed.

```
structure directed_map (α β : Type*) [directed_space α] [directed_space β] extends continuous_map α β :=
  (directed_to_fun : directed_map.directed to_continuous_map)
```

We use the notation \( D(\alpha, \beta) \) for the type of directed maps between \( \alpha \) and \( \beta \). Directed paths are also instances of directed maps, because it maps a directed path in \( I \) to a monotone subparametrization of itself and we know that that is also directed path. \texttt{dipath.lean} contains definitions on how to convert the \texttt{dipath} type to the \texttt{directed_map} type and the other way around. These are called \texttt{to_directed_map} and \texttt{of_directed_map} respectively.

Directed spaces and directed maps form a category, which we will denote with \( \mathbf{dTop} \). There are two functors \( Min, Max : \mathbf{Top} \to \mathbf{dTop} \), where \( Min \) equips a topological space with the minimal directedness and \( Max \) equips a topological space with the maximal directedness. If \( U : \mathbf{dTop} \to \mathbf{Top} \) is the forgetful functor that sends a directed space to its underlying topological space, we obtain two adjunctions \( Min \vdash U \vdash Max \) \cite{Gra03}. An adjunction is a special relationship between two functors, see \cite[chapter 2]{Lei14}. Within \( \mathbf{dTop} \) we find an instance of pushouts as the following lemma shows.

**Lemma 3.13.** Let \( X \in \mathbf{dTop} \) be a directed space and \( X_1 \) and \( X_2 \) two open subspaces such that \( X = X_1 \cup X_2 \). Take \( X_0 = X_1 \cap X_2 \) as the intersection of \( X_1 \) and \( X_2 \). Let \( i_k : X_0 \to X_k \) and \( j_k : X_k \to X \), with \( k \in \{1, 2\} \) be the inclusion maps. We then get a pushout square in \( \mathbf{dTop} \):

```
X0 \( \xleftarrow{i_1} \) X1
\bigg\downarrow \quad \bigg\downarrow j_1
X2 \( \xleftarrow{i_2} \) X
\downarrow j_2
X
```

**Proof.** Let \( Y \in \mathbf{dTop} \) be another directed space and \( f_1 : X_1 \to Y \) and \( f_2 : X_2 \to Y \) two directed maps such that \( f_1 \circ i_1 = f_2 \circ i_2 \). We will now construct an unique directed map \( f : X \to Y \) such that \( f \circ j_1 = f_1 \) and \( f \circ j_2 = f_2 \).
$f \circ j_2 = f_2$. Note that $f \circ j_k$ as a map is simply the restriction of $f$ to $X_k$. As $X$ is covered by $X_1$ and $X_2$, it follows that $f$ needs to be defined by

$$f(x) = \begin{cases} f_1(x), & x \in X_1, \\ f_2(x), & x \in X_2. \end{cases}$$

This already gives us uniqueness. If $U \subseteq Y$ is open, then $f^{-1}(U) = f_1^{-1}(U) \cup f_2^{-1}(U)$ is open as a union of open subsets and thus $f$ is continuous. Here we use the fact that $f_1$ and $f_2$ are continuous and that an open subset of an open subset is open in the entire space. In order to see that $f$ is directed, we need to use the Lebesgue Number Lemma [Mun75, p. 179-180]. We will use it to cut up a path into pieces and then apply the two maps $f_1$ and $f_2$ independently and concatenate the results together.

Let $\gamma \in P_X$ be any directed path. We have that $[0, 1] = \gamma^{-1}(X) = \gamma_1^{-1}(X_1) \cup \gamma_2^{-1}(X_2)$, so $\gamma^{-1}(X_1)$ and $\gamma^{-1}(X_2)$ form an open cover of $[0, 1]$. By applying the Lebesgue Number Lemma we can find an integer $n > 0$ such that for all $1 \leq i \leq n$ we have that $[\frac{i}{n}, \frac{i+1}{n}] \subseteq \gamma^{-1}(X_{k_i})$ with $k_i$ either 1 or 2, i.e. $\gamma$ is $n$-covered. With a suitable bijective and monotone reparametrization $\varphi : [0, 1] \to [0, 1]$, we have that

$$\gamma \circ \varphi = \gamma_1 \circ (\gamma_2 \circ \ldots (\gamma_{n-1} \circ \gamma_{n,n}))$$

Note that each of the paths $\gamma_{i,n}$ is directed as they are monotone subparametrizations of $\gamma$. We obtain:

$$(f \circ \gamma) \circ \varphi = f \circ (\gamma \circ \varphi) = f \circ (\gamma_1 \circ (\gamma_2 \circ \ldots (\gamma_{n-1} \circ \gamma_{n,n})))) = (f_k \circ \gamma_1 \circ n) \circ ((f_k \circ \gamma_2 \circ n) \circ \ldots ((f_k \circ \gamma_{n-1} \circ n) \circ (f_k \circ \gamma_{n,n}))))$$

The directedness of the maps $f_1$ and $f_2$ tells us that each of the paths $f_k \circ \gamma_{i,n}$ is directed. By the property of concatenation of directed paths we find that $(f \circ \gamma) \circ \varphi$ is directed. As $\varphi$ is a bijective monotone reparametrization, so is its inverse. This gives us that $f \circ \gamma = (f \circ \gamma) \circ \varphi \circ \varphi^{-1}$ is directed, so $f$ is a directed map. From this, it follows that the above square is indeed a pushout square. □

From proof of the lemma, it follows that within the category $\mathbf{Top}$ a topological space $X$ and a covering of it with two open subsets $X_1$ and $X_2$ also form a pushout square. When $X_1$ and $X_2$ are both closed, they still form a pushout in $\mathbf{Top}$ but that is not guaranteed in $\mathbf{dTop}$. The latter is shown by the following example.

**Example 3.14.** Take $X = [0, 1]$ (maximally directed), $X_1 = [0] \cup \left( \bigcup_{i=0}^{\infty} \left[ \frac{1}{2i+1}, \frac{1}{2i+2} \right] \right)$ and $X_2 = \{0\} \cup \left( \bigcup_{i=1}^{\infty} \left[ \frac{1}{2i-1}, \frac{1}{2i} \right] \right)$. We have that $X_1 \cap X_2 = \{ \frac{1}{n} \mid n \in \mathbb{Z}_{> 1} \}$. Let $Y = [0, 1]$ with directed paths given by $P_Y = [0, 1] \cup \{ \gamma : [0, 1] \to (0, 1) \mid \gamma \text{ continuous} \}$. The directed paths in $Y$ are thus paths contained in $(0, 1]$ and the constant path in 0. Note that this collection does indeed satisfy the three properties of a directed space.

The point 0 in $X_1$ is not connected by any paths to any other points, so we claim that there is a directed map $f_1 : X_1 \to Y$ given by $f_1(x) = x$: we have that for any $\gamma \in P_X$ that $\gamma = 0_0$ or $\gamma$ is contained in $[\frac{1}{2i+1}, \frac{1}{2i+2}]$ for some $i \geq 0$ and thus in $(0, 1]$. In both cases, $f_1 \circ \gamma$ is directed in $Y$, making $f_1$ a directed map. Similarly we have a directed map $f_2 : X_2 \to Y$ given by $f_2(x) = x$.

If $i_1 : X_1 \cap X_2 \to X_1$ and $i_2 : X_1 \cap X_2 \to X_2$ are the inclusion maps, then $(f_1 \circ i_1)(x) = x = (f_2 \circ i_2)(x)$ for all $x \in X_1 \cap X_2$, so $f_1 \circ i_1 = f_2 \circ i_2$ holds. If $X$ were the pushout of $X_1$ and $X_2$, there would have to be a unique directed map $f : X \to Y$ such that $f \circ i_1 = f_1$ and $f \circ i_2 = f_2$. Clearly $f$ must be defined as in the proof of Lemma 3.13:

$$f(x) = \begin{cases} f_1(x), & x \in X_1, \\ f_2(x), & x \in X_2. \end{cases} = \begin{cases} x, & x \in X_1, \\ x, & x \in X_2, \end{cases} = x.$$
If $\gamma \in P_X$ is the directed path given by $t \mapsto t$, we would require that $f \circ \gamma \in P_Y$. As $f \circ \gamma$ is neither the constant path $0_0$ nor a path in $(0, 1]$, we find a contradiction to the directedness of $f$. $X$ is therefore not the pushout of $X_1$ and $X_2$, even though $X_1$ and $X_2$ are both closed.

From this example, it also follows that Max does not preserve colimits, as a pushout is an instance of a colimit.

## 4 Directed Homotopies

In this section, we will look at directed homotopies and directed path homotopies. Those two realize the idea of deformation, while respecting the directedness of a directed space.

### 4.1 Homotopies

A directed homotopy is the deformation of one directed map into another.

**Definition 4.1** (Directed homotopy). Let $X$ and $Y$ be two directed spaces. A homotopy between two directed maps $f, g : X \to Y$ is a directed map $H : I \times X \to Y$ such that for all $x \in X$ we have that $H(0, x) = f(x)$ and $H(1, x) = g(x)$.

Note that $I \times X$ has the product directedness. We say that $H$ is a directed homotopy from $f$ to $g$. This order matters, as unlike in the topological case a directed homotopy cannot generally be reversed. In our formalization, we adhere to the method used in defining homotopies between continuous maps in Mathlib, found in `topology/homotopy/basic.lean`. In an analogous manner, the structure extends the `directed_map (I \times X) Y` structure and has two extra properties.

```lean
structure dihomotopy (f₀ f₁ : directed_map X Y) extends D((I × X), Y) :=
  (map_zero_left' : ∀ x, to_fun (0, x) = f₀.to_fun x)
  (map_one_left' : ∀ x, to_fun (1, x) = f₁.to_fun x)
```

As a directed map is always a continuous map on the underlying topological spaces, we can define how to convert a dihomotopy to a homotopy. Conversely, if we are given a homotopy and we know that it is directed, we can obtain a dihomotopy.

If $f : X \to Y$ is a directed map, there is an identity homotopy $H$ from $f$ to $f$, given by $H(t, x) = f(x)$. Also, if $G$ is a directed homotopy from $f$ to $g$ and $H$ a directed homotopy from $g$ to $h$, we obtain a directed homotopy $G \otimes H$ from $f$ to $h$ given by

$$(G \otimes H)(t, x) = \begin{cases} G(2t, x), & t \leq \frac{1}{2}, \\ H(2t - 1, x), & \frac{1}{2} < t. \end{cases}$$

These two constructions are called `refl` and `trans` in `directed_homotopy.lean`. In both cases the coercion of a homotopy into a dihomotopy is used, by supplying the proof that both homotopies are directed. Here we use the fact that Mathlib already contains proofs that the constructed maps are indeed homotopies, i.e. continuous and satisfying the two mapping properties.
4.2 Path Homotopies

**Definition 4.2** (Directed path homotopy). Let $X$ be a directed space and $x, y \in X$ two points. A directed path homotopy between two directed paths $\gamma_1, \gamma_2 \in P_x(x, y)$ is a directed homotopy $H : I \times I \to X$ from $\gamma_1$ to $\gamma_2$ such that additionally for all $t \in [0, 1]$ we have that $H(t, 0) = x$ and $H(t, 1) = y$.

In other words, a path homotopy is a homotopy between two paths that keeps both endpoints fixed. Again we say that $H$ is a directed path homotopy from $\gamma_1$ to $\gamma_2$. Between two paths $\gamma_1$ and $\gamma_2$ in $I$ with the same endpoints exists a path homotopy under the condition that $\gamma_1(t) \leq \gamma_2(t)$ for all $t \in I$ as the following example shows.

**Example 4.3.** Let $t_0, t_1 \in I$ be two points and $\gamma_1, \gamma_2 \in P_I(t_0, t_1)$. If $\gamma_1(t) \leq \gamma_2(t)$ for all $t \in I$, there is a directed path homotopy $H$ from $\gamma_1$ to $\gamma_2$ given by $H(t, s) = (1 - t) \cdot \gamma_1(s) + t \cdot \gamma_2(s)$. It is continuous by continuity of paths, multiplication and addition. It is easy to show that $H(a_0, b_0) \leq H(a_1, b_1)$ if $a_0 \leq a_1$ and $b_0 \leq b_1$. From this, it follows that $H$ is directed, because a directed path in $I \times I$ is exactly a pair of monotone maps $I \to I$ by definition.

Note that $H$ interpolates two paths $\gamma_1$ and $\gamma_2$. The formalized proof of it being a directed map can be found in the file `interpolate.lean`.

Let $x, y, z \in X$ be three points and $\beta_1, \gamma_1 \in P_X(x, y)$ and $\beta_2, \gamma_2 \in P_X(y, z)$. If there are two directed path homotopies $G$ from $\beta_1$ to $\gamma_1$ and $H$ from $\beta_2$ to $\gamma_2$, we can construct a directed path homotopy $G \odot H$ from $\beta_1 \odot \beta_2$ to $\gamma_1 \odot \gamma_2$ given by

$$(G \odot H)(t, s) = \begin{cases} G(t, 2s), & s \leq \frac{1}{2}, \\ H(t, 2s - 1), & \frac{1}{2} < s. \end{cases}$$

Let $X$ be a directed space and $x, y \in X$ two points. If $\gamma_1, \gamma_2 \in P_X(x, y)$ are two directed paths with a path homotopy from $\gamma_1$ to $\gamma_2$, we will write $\gamma_1 \rightsquigarrow \gamma_2$. This defines a relation on the set $P_X(x, y)$, but that relation is not guaranteed to be an equivalence relation, as it is generally not symmetric. This is due to the fact that the reversal of a directed path may not be directed. The following lemma, which we haven’t formalized in Lean, shows this.

**Lemma 4.4.** Let $X$ be a directed space and $x, y \in X$. The relation $\rightsquigarrow$ on $P_X(x, y)$ is reflexive and transitive, but it is not always symmetric.

**Proof.** The fact that the relation is reflexive and transitive can be obtained from Section 4.1. If $\gamma \in P_X(x, y)$, then the directed homotopy given by $H(t, s) = \gamma(s)$ satisfies the additional conditions of a path homotopy, so $\gamma \rightsquigarrow \gamma$. Similarly if $\gamma_1, \gamma_2, \gamma_3 \in P_X(x, y)$ with $G$ a directed path homotopy from $\gamma_1$ to $\gamma_2$ and $H$ a directed path homotopy from $\gamma_2$ to $\gamma_3$, then $G \odot H$ is a directed homotopy from $\gamma_1$ to $\gamma_3$. It additionally holds that

$$(G \odot H)(t, 0) = \begin{cases} G(2t, 0), & t \leq \frac{1}{2}, \\ H(2t - 1, 0), & \frac{1}{2} < t. \end{cases} = \begin{cases} x, & t \leq \frac{1}{2}, \\ x, & \frac{1}{2} < t. \end{cases} = x,$$

and analogously that $(G \odot H)(t, 1) = y$, so it is also a directed path homotopy and we find $\gamma_1 \rightsquigarrow \gamma_3$. A counterexample to symmetry is as follows: let $\gamma_1, \gamma_2 : I \to I$ be the paths given by

$$\gamma_1(t) = \begin{cases} 0, & t \leq \frac{1}{2}, \\ 2t - 1, & \frac{1}{2} < t. \end{cases} \quad \text{and} \quad \gamma_2(t) = \begin{cases} 2t, & t \leq \frac{1}{2}, \\ 1, & \frac{1}{2} < t. \end{cases}$$
We have that $\gamma_1 \leadsto \gamma_2$, because of Example 4.3. On the other hand, if $H$ were a directed path homotopy from $\gamma_2$ to $\gamma_1$, we would require that $H(0, \frac{1}{2}) = \gamma_2(\frac{1}{2}) = 1$ and $H(1, \frac{1}{2}) = \gamma_1(\frac{1}{2}) = 0$, contradicting with directedness. \hfill \Box

In order get an equivalence relation on the set of directed paths between two points, we will take the symmetric transitive closure of this relation.

**Definition 4.5.** Let $X$ be a directed space and $x, y \in X$ two points. We say that two dipaths $\gamma_1, \gamma_2 \in P_X(x, y)$ are equivalent, or $\gamma_1 \simeq \gamma_2$, if there is an integer $n \geq 0$ together with dipaths $\beta_i \in P_X(x, y)$, for each $1 \leq i \leq n$, such that

$$\gamma_1 \leadsto \beta_1 \leadsto \ldots \leadsto \beta_n \leadsto \gamma_2.$$ 

This alternating sequence of arrows is also called a zigzag. As $\gamma_2 \leadsto \gamma_2$ holds for any path $\gamma_2$ by reflexivity, we can always assume that there is an odd number of paths in a zigzag between two paths $\gamma_1$ and $\gamma_2$. By taking $n = 0$, it follows that $\gamma_1 \simeq \gamma_2$ holds if $\gamma_1 \leadsto \gamma_2$. More precisely, $\simeq$ is the smallest equivalence relation on $P_X(x, y)$ such that that property holds [Lei14, p. 129]. As $\simeq$ is an equivalence relation, we can talk about the equivalence classes of paths, denoted by $[\gamma]$. An important property of these equivalence classes is that they are invariant under maps and path reparametrization.

**Lemma 4.6.** Let $X, Y$ be directed spaces and $x, y \in X$. Let $\gamma_1, \gamma_2 \in P_X(x, y)$ and $f : X \to Y$ directed. If $\gamma_1 \simeq \gamma_2$, then $f \circ \gamma_1 \simeq f \circ \gamma_2$.

*Proof.* Let $n > 0$ odd and $\beta_i \in P_X(x, y)$ for $1 \leq i \leq n$ such that $\gamma_1 \leadsto \beta_1 \leadsto \beta_2 \leadsto \ldots \leadsto \beta_n \leadsto \gamma_2$. If $H : I \times I \to X$ is a directed path homotopy from $\gamma_1$ to $\beta_1$, then $f \circ H$ is a directed path homotopy from $f \circ \gamma_1$ to $f \circ \beta_1$. We find that $f \circ \gamma_1 \leadsto f \circ \beta_1$. Repeating this for all other arrows in the zigzag gives us $f \circ \gamma_1 \leadsto f \circ \beta_1 \leadsto f \circ \beta_2 \leadsto \ldots \leadsto f \circ \beta_n \leadsto f \circ \gamma_2$, so $f \circ \gamma_1 \simeq f \circ \gamma_2$. \hfill \Box

**Lemma 4.7.** Let $X$ be a directed space and $x, y \in X$. Let $\gamma \in P_X(x, y)$ and $\varphi, \varphi' : I \to I$ continuous and monotone with $\varphi(0) = \varphi'(0) = 0$ and $\varphi(1) = \varphi'(1) = 1$. Then $\gamma \circ \varphi \simeq \gamma \circ \varphi'$.

*Proof.* As $\gamma$ is a directed map from $I$ to $X$, it is enough by Lemma 4.6 to show that $\varphi \simeq \varphi'$. Let $\beta_1 = \varphi \circ 0_1$ and $\beta_2 = 0_0 \circ \varphi'$. Then, by applying Example 4.3 three times, we obtain the zigzag

$$\varphi \leadsto \beta_1 \leadsto \beta_2 \leadsto \varphi'.$$

This shows that $\varphi \simeq \varphi'$, completing the proof. \hfill \Box

In the next section, we will construct the fundamental category of a directed space. For that we need the following four additional equalities.

**Lemma 4.8.** Let $X, Y$ be directed spaces and $x, y, z, w \in X$. Let $\beta_1, \gamma_1 \in P_X(x, y)$, $\beta_2, \gamma_2 \in P_X(y, z)$ and $\gamma_3 \in P_X(z, w)$ such that $\beta_1 \simeq \gamma_1$ and $\beta_2 \simeq \gamma_2$. Then the following holds:

1. $\beta_1 \circ \beta_2 \simeq \gamma_1 \circ \gamma_2$
2. $0_x \circ \gamma_1 \simeq \gamma_1$
3. $\gamma_1 \circ 0_y \simeq \gamma_1$
4. $(\gamma_1 \circ \gamma_2) \circ \gamma_3 \simeq \gamma_1 \circ (\gamma_2 \circ \gamma_3)$
Proof. Statements 2, 3 and 4 are direct applications of Lemma 4.7 as they are all reparametrizations. We will now show statement 1. Let \( n, m > 0 \) odd and \( p_i, q_j \in P_X(x, y) \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \) such that
\[
\begin{align*}
\beta_1 \mapsto p_1 \iff p_2 \mapsto \ldots \mapsto p_n \iff \gamma_1, \\
\beta_2 \mapsto q_1 \iff q_2 \mapsto \ldots \mapsto q_m \iff \gamma_2.
\end{align*}
\]
Let \( G \) be a directed path homotopy from \( \beta_1 \) to \( p_1 \) and \( H \) be the identity homotopy from \( \beta_2 \) to \( \beta_2 \). Then \( G \circ H \) is a directed path homotopy from \( \beta_1 \circ \beta_2 \) to \( p_1 \circ \beta_2 \). We obtain a zigzag
\[
\beta_1 \circ \beta_2 \mapsto p_1 \circ \beta_2 \iff p_2 \circ \beta_2 \mapsto \ldots \mapsto p_n \circ \beta_2 \iff \gamma_1 \circ \beta_2,
\]
so \( \beta_1 \circ \beta_2 \simeq \gamma_1 \circ \beta_2 \). Analogously we obtain a zigzag
\[
\gamma_1 \circ \beta_2 \mapsto \gamma_1 \circ q_1 \iff \gamma_1 \circ q_2 \mapsto \ldots \mapsto \gamma_1 \circ q_m \iff \gamma_1 \circ \gamma_2.
\]
This results in \( \gamma_1 \circ \beta_2 \simeq \gamma_1 \circ \gamma_2 \) and combining both equivalences gives us \( \beta_1 \circ \beta_2 \simeq \gamma_1 \circ \gamma_2 \). \( \square \)

The definition of a directed path homotopy and the three lemmas above have all been been formalized in \texttt{directed_path_homotopy.lean}. For the path homotopies, we followed the more general approach from MathLib, where we first defined directed homotopies that satisfy some property \( P \). Thereafter we defined \texttt{dihomotopy_rel} as directed homotopies that are fixed on a select subset of points. This is all defined in \texttt{directed_homotopy.lean}. A path homotopy is a homotopy that is fixed on both endpoints, that is, on \( \{0, 1\} \subseteq I \), so we can define a directed path homotopy as

\begin{verbatim}
abbreviation dihomotopy (p₀ p₁ : dipath x₀ x₁) :=
  directed_map.dihomotopy_rel p₀.to_directed_map p₁.to_directed_map {0, 1}
\end{verbatim}

As a directed homotopy is defined between two directed maps, we need to convert both paths \( p_0 \) and \( p_1 \) to directed maps. The construction \( \circ \) is called \texttt{hcomp} and \( \otimes \) is called \texttt{trans}. If \( f, g \in D(I, I) \) are two directed maps with \( f(t) \leq g(t) \) for all \( t \in I \), the definition \texttt{dihomotopy.reparam} constructs a homotopy from \( \gamma \circ f \) to \( \gamma \circ g \). This is done by composing \( \gamma \) and the homotopy obtained from Example 4.3. If \( H \) is an homotopy from \( p \) to \( q \) with \( p, q \in P_X(x, y) \), and \( f : X \to Y \) is a directed map, then the homotopy from \( f \circ p \) to \( f \circ q \) given by \( f \circ H \) is exactly what \texttt{dihomotopy.map} entails.

Now we can define the relations \( \mapsto \) and \( \simeq \). These are called \texttt{pre_dihomotopic} and \texttt{dihomotopic} respectively.

\begin{verbatim}
def pre_dihomotopic : Prop := nonempty (dihomotopy p₀ p₁)
def dihomotopic : Prop := eqv_gen pre_dihomotopic p₀ p₁
\end{verbatim}

The term \texttt{nonempty} means exactly that there exists some dihomotopy, which corresponds with our definition of \( \mapsto \). \texttt{eqv_gen} gives the smallest equivalence relation generated by a relation, which is exactly what we want. The lemmas \texttt{map}, \texttt{reparam} and \texttt{hcomp} in the \texttt{dihomotopic}-namespace now correspond with Lemma 4.6, Lemma 4.7 and the first point of Lemma 4.8 respectively. This gives us enough tools to construct the so called fundamental category.

### 5 Fundamental Structures

In this section, we define two structures that contain information about directed paths up to deformation in a directed space: the fundamental category and the fundamental monoid.
### 5.1 The Fundamental Category

Using the properties found in Section 4.2, we can define a category that captures the information of all paths up to directed deformation in a directed space.

**Definition 5.1 (Fundamental Category).** Let $X$ be a directed space. The fundamental category of $X$, denoted by $\vec{\Pi}(X)$, is a category that consists of:

- **Objects:** points $x \in X$.
- **Morphisms:** $\vec{\Pi}(X)(x, y) = P_X(x, y)/ \simeq$.
- **Composition:** $[\gamma_2] \circ [\gamma_1] = [\gamma_1 \circ \gamma_2]$.
- **Identity:** $\text{id}_x = [0_x]$.

**Remark.** The fact that this category is well defined follows from Lemma 4.8. Due to property 1, composition is well defined. Due to properties 2 and 3, the constant path behaves as an identity and property 4 gives us associativity.

Note that $\vec{\Pi}$ maps objects in $\mathbf{dTop}$ to objects in $\mathbf{Cat}$. It turns out that it can also be defined on morphisms.

**Definition 5.2.** Let $f : X \to Y$ be a directed map. We define $\vec{\Pi}(f) : \vec{\Pi}(X) \to \vec{\Pi}(Y)$ as the functor:

- **On objects:** $\vec{\Pi}(f)(x) = f(x)$.
- **On morphisms:** $\vec{\Pi}(f)([\gamma]) = [f \circ \gamma]$.

It is well behaved on morphisms, because of Lemma 4.6. It is easy to verify that $\vec{\Pi}(f)$ respects composition and identity and it is thus a functor.

In our formalization, we follow the construction of the fundamental groupoid in MathLib found in `algebraic_topology/fundamental_groupoid/basic.lean` closely. The implementation is found in the file `fundamental_category.lean`. The MathLib version has some auxiliary definitions for a reparametrization that show that the two paths $(\gamma_1 \circ \gamma_2) \circ \gamma_3$ and $\gamma_1 \circ (\gamma_2 \circ \gamma_3)$ are equal with relation to $\simeq$ for compatible paths $\gamma_1, \gamma_2$ and $\gamma_3$. In order to use these in our directed world, we just need to show that this reparametrization is monotone. This is enough to then define the fundamental category.

```lean
def fundamental_category (X : Type u) := X

... 

instance : category_theory.category (fundamental_category X) :=
{
  hom := \lambda x y, dipath.dihomotopic.quotient x y,
  id := \lambda x, [ dipath.refl x ],
  comp := \lambda x y z, dipath.dihomotopic.quotient.comp,
  id_comp' := \lambda x y f, quotient.induction_on f
  (\lambda a, show [ (dipath.refl x).trans a ] = [ a ],
    from quotient.sound (eqv_gen.rel _ _ (dipath.dihomotopy.refl_trans a))),
  comp_id' := /- Proof omitted -/,
  assoc' := /- Proof omitted -/,
}
```
The first definition makes sure that objects of the fundamental category are terms of type $X$. We then show that `fundamental_category` is an instance of a category by defining the morphisms (hom), identities (id) and composition (comp). The morphisms between two objects $x$ and $y$ are given by `dipath.dihomotopic.quotient x y`. This is the quotient of `dipath x y` under the dihomotopic relation and its definition can be found in the `directed_path_homotopy.lean`. The identity on $x$ is then the equivalence class (denoted by $⟦⟧$) of the constant path in $x$. The composition of the equivalence classes of two compatible paths is defined in `dipath.dihomotopic.quotient.comp` as the equivalence class of the concatenation of the two paths.

Proof that the fundamental category is indeed a category are given by `id_comp'`, `comp_id'` and `assoc'`. The first one, `id_comp'`, requires us to show that `(dipath.refl x).trans a` and `a` are two dihomotopic paths. For this, we use `dipath.dihomotopy.refl_trans a`, which is an explicit directed path homotopy from the path `(dipath.refl x).trans a` to `a`. Its existence shows that the two paths are pre_dihomotopic and they are thus in the same equivalence class.

The file also contains the definition of the $\vec{\Pi}$-functor from $dTop$ to `category_theory.Cat`. Analogous to the undirected MathLib implementation, we use the notation $d\pi$ for this functor.

### 5.2 The Fundamental Monoid

**Definition 5.3** (Monoid). A monoid is triple $(M, \cdot, e)$ consisting of a set $M$, an associative binary operation $\cdot : M \times M \to M$ and an element $e \in M$ such that for all $g \in M$ it holds that $e \cdot g = g \cdot e = g$.

**Example 5.4.** The triples $(\mathbb{N}, +, 0)$ and $(\mathbb{N}, \cdot, 1)$ are both monoids. As they both do not have an inverse for each element, they are not groups.

**Example 5.5.** If $C$ is a locally small category and $x \in C$ and object, then the triplet $(\text{id}(x), \circ, \text{id}_x)$ is a monoid. This follows directly from the definition of a category. The elements in $\text{id}(x)$ are called endomorphisms of $x$.

This last example can be used to define the fundamental monoid of a point in a directed space.

**Definition 5.6** (Fundamental monoid). Let $X$ be a directed space and $x \in X$ a point. Then the fundamental monoid of $X$ in $x$, denoted by $\vec{\pi}(X, x)$ is given by $(\text{id}(x), \circ, \text{id}_x)$.

**Example 5.7.** Let $x \in I$ be a point. Let $\gamma \in P_I(x, x)$. Then $\gamma$ is monotone and $\gamma(0) = \gamma(1) = x$. It follows that $\gamma(t) = x$ for all $t \in I$, so $\gamma = 0_x$. From this, we can conclude that the only morphism in $\text{id}(x)$ is the identity, so $\vec{\pi}(I, x)$ is the trivial monoid.

**Example 5.8.** We have that $\vec{\pi}(S^1_+, 1) \cong (\mathbb{N}, +, 0)$. In Section 6.3 we will support this claim by calculating a fundamental monoid in a finite version of the directed unit circle.

Whether we give the unit interval the minimal directedness, the rightward directedness or maximal directedness, the fundamental monoid at any point is the trivial monoid. Their fundamental categories, on the other hand, are able to distinguish the differences in directedness. There are respectively zero, one and two morphisms between two different objects. We see that the fundamental monoid loses information that is contained in the fundamental category.
6 The Van Kampen Theorem

In this section, we will state and prove the Van Kampen Theorem. We follow the proof of Grandis and work out some of the details that were omitted. In Section 6.2 we show how we have formalized this proof by comparing the proof to the Lean code. We conclude with an application of the Van Kampen Theorem in Section 6.3.

6.1 The Van Kampen Theorem

Before we state and prove the theorem, we will define the notion of being covered for directed homotopies.

**Definition 6.1.** Let $X$ be a directed space and $\mathcal{U}$ a cover of $X$. Let $H : I \times I \to X$ be a directed homotopy and $n, m > 0$ two integers. We say that $H$ is $(n, m)$-covered (by $\mathcal{U}$) if for all $1 \leq i \leq n$ and $1 \leq j \leq m$ the image of $\left[ \frac{i-1}{n} \cdot \frac{j-1}{m} \right] \times \left[ \frac{i}{n} \cdot \frac{j}{m} \right]$ under $H$ is contained in some $U \in \mathcal{U}$.

Once again, by the Lebesgue Number Lemma, for any homotopy $H$ and open cover $\mathcal{U}$ of $X$, there are $n, m > 0$ such that $H$ is $(n, m)$-covered by $\mathcal{U}$.

**Theorem 6.2 (Van Kampen Theorem).** Let $X$ be a directed space and $X_1$ and $X_2$ two open subspaces such that $X = X_1 \cup X_2$ and let $X_0 = X_1 \cap X_2$. Let $i_k : X_0 \to X_1$ and $j_k : X_k \to X$ be the inclusion maps, $k \in \{1, 2\}$. Then we obtain a pushout square in $\text{Cat}$:

\[
\begin{array}{ccc}
\tilde{\Pi}(X_0) & \xrightarrow{\tilde{\Pi}(i_1)} & \tilde{\Pi}(X_1) \\
\downarrow \tilde{\Pi}(i_2) & & \downarrow \tilde{\Pi}(j_1) \\
\tilde{\Pi}(X_2) & \xrightarrow{\tilde{\Pi}(j_2)} & \tilde{\Pi}(X) \\
\end{array}
\]

**Proof.** As $j_1 \circ i_1 = j_2 \circ i_2$ and $\tilde{\Pi}$ is a functor, the square is commutative. It remains to show it satisfies the property of a pushout square. Let $C$ be any category and $F_1 : \tilde{\Pi}(X_1) \to C$ and $F_2 : \tilde{\Pi}(X_2) \to C$ be two functors such that $F_1 \circ \tilde{\Pi}(i_1) = F_2 \circ \tilde{\Pi}(i_2)$. We will explicitly construct a functor $F$ such that $F \circ \tilde{\Pi}(j_1) = F_1$ and $F \circ \tilde{\Pi}(j_2) = F_2$. The construction will show that this functor is necessarily unique with this property.

**Step 1** The objects of $\tilde{\Pi}(X)$ are exactly the points of $X$. If an object $x \in \tilde{\Pi}(X)$ is also contained in $\tilde{\Pi}(X_1)$, it holds that $F(x) = F(j_1(x)) = (F \circ \tilde{\Pi}(j_1))(x)$. The desired condition $F \circ \tilde{\Pi}(j_1) = F_1$ then requires us to define $F(x) = F_1(x)$. A similar argument gives us that if $x \in \tilde{\Pi}(X_2)$ then $F(x) = F_2(x)$. As $X_1$ and $X_2$ cover $X$, we have that for all $x \in \tilde{\Pi}(X)$

\[
F(x) = \begin{cases} F_1(x), & x \in X_1, \\ F_2(x), & x \in X_2. \end{cases}
\]

By the property that $F_1 \circ \tilde{\Pi}(i_1) = F_2 \circ \tilde{\Pi}(i_2)$ this is well defined, so we know how $F$ must behave on objects.
Step 2) Let \( \gamma : x \to y \) be a morphism in \( \mathcal{P}(X) \). Then there is a \( n > 0 \) such that \( \gamma \) is \( n \)-covered by the open cover \( \{X_1, X_2\} \), with \( \gamma_{i,n} \) covered by \( X_{k_i}, k_i \in \{1, 2\} \). One important thing to note is that \( \gamma_{i,n} \) can be both seen as a path in \( X \) and as a path in \( X_{k_i} \) by restricting its codomain. This matters when we talk about \( [\gamma_{i,n}] \), as it could be a morphism in \( \mathcal{P}(X) \) and in \( \mathcal{P}(X_{k_i}) \). Within this proof will always consider it as a morphism in \( \mathcal{P}(X_{k_i}) \) and use \( [j_{k_i} \circ \gamma_{i,n}] \) for the morphism in \( \mathcal{P}(X) \). Note that we have that \( [\gamma] = [j_{k_i} \circ \gamma_{i,n}] \circ \ldots \circ [j_{k_i} \circ \gamma_{1,n}] \) in \( \mathcal{P}(X) \), as \( \gamma \) is equal to \( \gamma_{1,n} \circ (\gamma_{2,n} \circ \ldots (\gamma_{n-1,n} \circ \gamma_{n,n})) \) up to reparametrization. From context, it should always be clear which one we reference. Because we want \( F \) to be a functor and thus to respect composition, we find that necessarily

\[
F[\gamma] = F([j_{k_i} \circ \gamma_{i,n}] \circ \ldots \circ [j_{k_i} \circ \gamma_{1,n}])
\]

\[
= F(j_{k_i} \circ \gamma_{i,n}) \circ \ldots \circ F[j_{k_i} \circ \gamma_{1,n}]
\]

\[
= F(\mathcal{P}(j_{k_i})(\gamma_{i,n})) \circ \ldots \circ F(\mathcal{P}(j_{k_i})(\gamma_{1,n}))
\]

\[
= (F \circ \mathcal{P}(j_{k_i})(\gamma_{i,n})) \circ \ldots \circ (F \circ \mathcal{P}(j_{k_i})(\gamma_{1,n}))
\]

\[
= F_{k_i}[\gamma_{i,n}] \circ \ldots \circ F_{k_i}[\gamma_{1,n}].
\]

As multiple choices were made, we need to make sure that \( F \) is well defined this way. We do this by defining a map \( F' : P_X \to C \) by

\[
F'(|\gamma|) = F_{k_i}[\gamma_{i,n}] \circ \ldots \circ F_{k_i}[\gamma_{1,n}],
\]

where \( \gamma \) is \( n \)-covered with \( \gamma_{i,n} \) covered by \( X_{k_i} \). Firstly, we will show that this map is well defined. Secondly, we show that \( F' \) respects equivalence classes. From this it follows that \( F \) is well defined, as it is simply \( F' \) descended on equivalence classes.

Step 3) We first need to make sure that \( F' \) does not depend on any choices of \( k_i \). In the case that \( \gamma_{i,n} \) is covered by both \( X_1 \) and \( X_2 \), the value of \( k_i \) can be either 1 or 2. The condition that \( F_1 \circ \mathcal{P}(i_1) = F_2 \circ \mathcal{P}(i_2) \) assures us that both options give us the same value.

Step 4) The second choice we made is that of \( n \). It is possible that \( \gamma \) is also \( m \)-covered for another integer \( m > 0 \), with \( \gamma_{j,m} \) being contained in \( X_{p_j} \). We want to show that

\[
F_{k_i}[\gamma_{i,n}] \circ \ldots \circ F_{k_i}[\gamma_{1,n}] = F_{p_m}[\gamma_{m,m}] \circ \ldots \circ F_{p_1}[\gamma_{1,m}].
\]

If we refine the partition of \( \gamma \) in \( n \) pieces into a partition of \( mn \) pieces, that partition will surely also be partwise covered. Let \( l_i \in \{1, 2\} \) for all \( 1 \leq i \leq mn \) such that \( \gamma_{i,mn} \) is covered by \( X_{l_i} \). We now claim that for all \( 1 \leq i \leq n \) it holds that \( F_{k_i}[\gamma_{i,n}] = F_{l_m}[\gamma_{m,m}] \circ \ldots \circ F_{l_{m(i-1)+1}}[\gamma_{m(i-1)+1,mn}] \). As \( \gamma_{m(i-1)+1,mn} \) with \( 1 \leq k \leq n \) is a part of \( \gamma_{i,n} \), we may assume that \( l_{m(i-1)+1} = k_i \). This is because \( F_1 \) and \( F_2 \) agree on \( X_1 \cap X_2 \). As \( F_{k_i} \) is a functor, the claim now follows because functors respect composition and because \( \gamma_{i,n} \) is exactly the concatenation of all the smaller paths up to reparametrization. By a similar claim for \( F_{p_j}[\gamma_{j,m}] \) we find:

\[
F_{k_i}[\gamma_{i,n}] \circ \ldots \circ F_{k_i}[\gamma_{1,n}] = (F_{l_m}[\gamma_{m,m}] \circ \ldots \circ F_{l_{m(i-1)+1}}[\gamma_{m(i-1)+1,mn}]) \circ \ldots \circ (F_{l_m}[\gamma_{m,m}] \circ \ldots \circ F_{l_1}[\gamma_{1,m}])
\]

\[
= F_{l_m}[\gamma_{m,m}] \circ \ldots \circ F_{l_1}[\gamma_{1,m}]
\]

\[
= (F_{l_m}[\gamma_{m,m}] \circ \ldots \circ F_{l_{m(i-1)+1}}[\gamma_{m(i-1)+1,mn}]) \circ \ldots \circ (F_{l_m}[\gamma_{m,m}] \circ \ldots \circ F_{l_1}[\gamma_{1,m}])
\]

\[
= F_{p_m}[\gamma_{m,m}] \circ \ldots \circ F_{p_1}[\gamma_{1,m}]
\]

We conclude that the definition is independent of the value of \( n \). This makes \( F' \) well defined.
Step 5) Before we verify that \( F' \) is independent of the choice of representative \( \gamma \), we will first show that \( F' \) satisfies two properties:

\[
\forall x \in \tilde{\Pi}(X) : F'(0_x) = \text{id}_{F(x)}. \tag{1}
\]
\[
\forall \gamma \in P_X(x, y), \delta \in P_X(y, z) : F'(\gamma \odot \delta) = F'(\delta) \circ F'(\gamma). \tag{2}
\]

Let \( x \in \tilde{\Pi}(X) \) be given. If \( x \in X_1 \), then \( 0_x \) is already contained in \( X_1 \) and so by definition of \( F' \) we find \( F'(0_x) = F_1[0_x] = \text{id}_{F_1(x)} = \text{id}_{F(x)}. \) Otherwise it holds that \( x \in X_2, \) so \( F'(0_x) = F_2[0_x] = \text{id}_{F_2(x)} = \text{id}_{F(x)}. \) This proves Eq. (1).

Let \( \gamma \in P_X(x, y), \delta \in P_X(y, z) \) be two paths in \( \tilde{\Pi}(X) \). We can then find a \( n \) such that both \( \gamma \) and \( \delta \) are \( n \)-covered, with \( \gamma_{i,n} \) contained in \( X_{k_i} \) and \( \delta_{i,n} \) contained in \( X_{p_i} \). It is then true that \( \gamma \odot \delta \) is \( 2n \)-covered as it holds that

\[
(\gamma \odot \delta)_{i,2n} = \begin{cases} 
\gamma_{i,n}, & i \leq n, \\
\delta_{i-n,n}, & i > n.
\end{cases}
\]

We find:

\[
F'(\delta \odot \gamma) = F_{p_n}[((\delta \odot \gamma)_{2n,2n}] \circ \ldots \circ F_{p_1}[(\delta \odot \gamma)_{n+1,2n}] \circ F_{k_n}[(\delta \odot \gamma)_{n,2n}] \circ \ldots \circ F_{k_1}[(\delta \odot \gamma)_{1,2n}] = 
(F_{p_n}[\delta_{n,n}] \circ \ldots \circ F_{p_1}[\delta_{1,n}]) \circ (F_{k_n}[\gamma_{n,n}] \circ \ldots \circ F_{k_1}[\gamma_{1,n}]) = F'(\delta) \circ F'(\gamma).
\]

This shows that Eq. (2) holds.

Step 6) We will now show that \( F' \) respects equivalence classes. Then it can descend to the quotient and it follows that \( F \) is well defined. If \( [\gamma] = [\delta] \) in \( \tilde{\Pi}(X) \) with \( \delta \) another path from \( x \) to \( y \), we want that

\[
F'(\gamma) = F'(\delta). \tag{3}
\]

Because of the way the equivalence class is defined, it is enough to show this for \( \gamma \) and \( \delta \) such that \( \gamma \sim \delta \). Let in that case a directed path homotopy \( H \) from \( \gamma \) to \( \delta \) be given. We take \( n, m > 0 \) such that \( H \) is \((n, m)\)-covered by \( \{X_1, X_2\} \). Firstly assume that \( n > 1 \). Restricting \( H \) to the rectangle \([0, \frac{1}{n}] \times [0, 1]\) gives us a directed path homotopy \( H_1 \) from \( x \) to the directed path \( \eta \) given by \( \eta(t) = H(\frac{1}{n}, t) \). By restricting \( H \) to the rectangle \([\frac{1}{m}, 1] \times [0, 1]\) we get a directed path homotopy \( H_2 \) from \( \eta \) to \( \delta \). It is clear that \( H_1 \) is \((1, m)\)-covered and that \( H_2 \) is \((n-1, 1)\)-covered. By applying induction, we can conclude that it is enough to show that Eq. (3) holds for \((1, m)\)-covered directed path homotopies, as we would obtain that \( F'(\gamma) = F'(\eta) = F'(\delta) \). Here the second equality follows from induction and the first equality from the base case.

Step 7) We will prove the case where \( H \) is \((1, m)\)-covered by showing a more general statement:

Let \( H \) be any directed homotopy – not necessarily a path homotopy – from one path \( \gamma \in P_X(x, y) \) to another path \( \delta \in P_X(x', y') \) that is \((1, m)\)-covered, \( m > 0 \). Let \( \eta_0 \) be the path given by \( \eta_0(t) = H(t, 0) \) and \( \eta_1 \) be given by \( \eta_1(t) = H(t, 1) \). Then \( F'(\eta_0 \odot \delta) = F'(\gamma \odot \eta_1) \). We do this by induction on \( m \).

In the case that \( m = 1 \), we have a homotopy contained in \( X_1 \) or \( X_2 \). Without loss of generality, we can assume it is contained in \( X_1 \). Let \( \Gamma_1 \) be the directed homotopy given by \( \Gamma_1(t, s) = \eta_0(\min(t,s)) \) from \( 0_x \) to \( \eta_0 \). Let \( \Gamma_2 \) be the directed homotopy given by \( \Gamma_2(t, s) = \eta_1(\max(t,s)) \) from \( \eta_1 \) to \( 0_y \). We then can construct a directed path homotopy from \( (0_x \odot \gamma) \odot \eta_1 \) to \( (\eta_0 \odot \delta) \odot 0_y \) given by \( (\Gamma_1 \odot H) \odot \Gamma_2 \).
It is a directed path homotopy because $\Gamma_1(t, 0) = \eta_0(\min(t, 0)) = \eta_0(0) = x$ and $\Gamma_2(t, 1) = \eta_1(\max(t, 1)) = \eta_1(1) = y'$ for all $t \in I$. As $\eta_0, \eta_1$ and $H$ are all contained in $X_1$, so is this directed path homotopy. We find that $[\gamma \circ \eta_1] = [\eta_0 \circ \delta]$ in $H(X_1)$. This gives us that $F'(\gamma \circ \eta_1) = F'_1[\gamma \circ \eta_1] = F'_1[\eta_0 \circ \delta] = F'(\eta_0 \circ \delta)$. Let now $m > 1$ and assume the statement holds for $(1, m - 1)$-covered homotopies. We can restrict $H$ to $[0, 1] \times \left[0, \frac{m - 1}{m}\right]$ to obtain a $(1, m - 1)$-covered homotopy $H_1$ and we can restrict $H$ to $[0, 1] \times \left[\frac{m - 1}{m}, 1\right]$ to obtain a $(1, 1)$-covered homotopy $H_2$:

Note that $F'(\gamma) = F'(\gamma_2) \circ F'(\gamma_1)$, because $\gamma_1$ is $(m - 1)$-covered, $\gamma_2$ is 1-covered and $\gamma$ is $m$-covered. Similarly it holds that $F'(\delta) = F'(\delta_2) \circ F'(\delta_1)$. We find:

$$F'(\gamma \circ \eta_1) = F'(\eta_1) \circ F'(\gamma) = F'(\eta_1) \circ (F'(\gamma_2) \circ F'(\gamma_1)) = (F'(\eta_1) \circ F'(\gamma_2)) \circ F'(\gamma_1) = (F'(\delta_2) \circ F'(\eta')) \circ F'(\gamma_1) = F'(\delta_2) \circ (F'(\eta') \circ F'(\gamma_1)) = F'(\delta_2) \circ (F'(\delta_1) \circ F'(\eta_0)) = (F'(\delta_2) \circ F'(\delta_1)) \circ F'(\eta_0) = F'(\delta) \circ F'(\eta_0) = F'(\eta_0 \circ \delta).$$

(Case $m = 1$)

(Induction Hypothesis)

This proves the statement. From the statement we find that Eq. (3) holds:

$$F'(\delta) = F'(\delta) \circ \text{id}_x = F'(\delta) \circ F'(0_\chi) = F'(0_\chi \circ \delta) = F'(\gamma \circ 0_\chi) = F'(0_\chi) \circ F'(\gamma) = \text{id}_x \circ F'(\gamma) = F'(\gamma).$$
With the right conditions, a Van Kampen type theorem for the fundamental monoid holds. One such condition can be extracted from the work of Grandis [Gra03, p. 306].

We see that a space $X$ is well, so $(\vec{\pi}_1 X, \vec{\pi}_0)$.

The original Van Kampen Theorem, stated by Egbert van Kampen, was concerned with the fundamental group of a space [VK33]. The fundamental group is the undirected version of the fundamental monoid.

Example 6.3. We take $X$ to be the directed unit circle, together with the horizontal diameter directed leftward. Take $X_1$ to be the top semicircle together with the diameter and $X_2$ to be the bottom semicircle together with the diameter. Expand them both a little bit to make sure that they are open subspaces.

In $X_1$, and therefore also in $X_1 \cap X_2$, the only path from $x$ to $x$ is the constant path, so it follows that $\pi_1(X_1, x) \cong 0$ and $\pi_1(X_1 \cap X_2, x) \cong 0$. Endomorphisms of $x$ in $X_2$ behave like endomorphisms of 1 in $S^1_+$, so $\pi_1(X_2, x) \cong (\mathbb{N}, +, 0)$. If $\pi_1(X, x)$ were the pushout, we would find that $\pi_1(X, x) \cong (\mathbb{N}, +, 0)$. This is false, as $\pi_1(X, x) \cong (\mathbb{N}, +, 0) * (\mathbb{N}, +, 0)$ — the free product of $\mathbb{N}$ with itself. We will support this claim in Section 6.3.

With the right conditions, a Van Kampen type theorem for the fundamental monoid holds. One such condition can be extracted from the work of Bubenik [Bub09]. Let $X_1$ and $X_2$ be two open subspaces.
covering the directed space $X$ with base point $x$, both containing $x$. It is now sufficient that for any endomorphism $[\gamma] : x \rightarrow x$ we can split it as $[\gamma] = [\gamma_n] \circ \ldots \circ [\gamma_1]$ with $[\gamma_i] : x \rightarrow x$ and $\gamma_i$ contained in either $X_1$ or $X_2$.

In Example 6.3 this condition is not fulfilled as one counterclockwise loop along the circle is first contained in $X_2$ and then in $X_1$. There is no way to factor that loop into endomorphisms of $x$ in such a way that each morphism is contained in $\tilde{\Pi}(X_1)$ or $\tilde{\Pi}(X_2)$.

## 6.2 Formalization

In the formalization of Theorem 6.2 we follow the constructive nature of its proof. It can be found in `directed_van_kampen.lean`. We have the following global variables, corresponding with the assumptions of the Van Kampen Theorem:

```lean
variables {X : dTop.{u}} {X_1 X_2 : set X}
variables (hX : X_1 \cup X_2 = set.univ)
variables (X_1_open : is_open X_1) (X_2_open : is_open X_2)
```

The statement $hX$ is exactly that $X_1$ and $X_2$ cover $X$. We do not write it as $X_1 \cup X_2 = X$, as that would not be type correct: the left side has type `set X` and the right side has type `dTop.{u}`. The term `set.univ` does have type `set X` and is defined to be the set containing all elements of $X$.

Like in the proof, we introduce a category $C$ and two functors $F_1 : \tilde{\Pi}(X_1) \rightarrow C$ and $F_2 : \tilde{\Pi}(X_2) \rightarrow C$. Using these we are going to explicitly construct a functor from $\tilde{\Pi}(X)$ to $C$ and show that it is unique. We will use that to prove that we indeed have a pushout square.

```lean
variables {C : category_theory.Cat.{u u}} (F_1 : (dTop.of X_1) \rightarrow C))
(F_2 : (dTop.of X_2) \rightarrow C)) (h_comm : (dTop.m i_1) \Rightarrow F_1 = ((dTop.m i_2) \Rightarrow F_2))
```

Here $i_1$ and $i_2$ are the inclusion maps as in the statement of the Van Kampen Theorem. They are obtained by `dTop.directed_subset_hom`, defined in `dTop.lean`. This defines the inclusion morphism $X_0 \rightarrow X_1$ in `dTop` in the case that $X_0 \subseteq X_1 \subseteq X$. Here $X_0 = X_1 \cap X_2$. We first define the functor on objects (**Step 1**).

```lean
def functor_obj (x : dTop.X) : C :=
or.by_cases (((set.mem_union x X_1 X_2).mp (filter.mem_top.mpr hX x)) (λ hx, F_1.obj (x, hx)) (λ hx, F_2.obj (x, hx)))
```

Here we take an object $x$ from the fundamental category of $X$ and return an object from the codomain $C$. We use `filter.mem_top.mpr hX x` to show that $x \in X_1 \cup X_2$. From this, we use `set.mem_union` to obtain $x \in X_1$ or $x \in X_2$ and we can split by those cases to apply either $F_1$ or $F_2$. We abbreviate `functor_obj hX F_1 F_2` to `F_obj` in our formalization to maintain clarity. After this definition, there are two lemmas that prove that if $x \in X_k$, then $F_k(x) = F(x)$ for $k \in \{1, 2\}$.

In the proof of Theorem 6.2, $F'$ is first defined and it is then shown to be a valid definition. Within our Lean formalization, we have to do these two parts in the reverse order. Once we have shown the construction is well-defined, we can define $F'$ in our formalization. That is why **Step 2** will be completed later.

We use the definitions of `covered` and `covered_partwise`, shown in Section 3, to define the mapping of morphisms inductively (**Step 3**):
In `functor_map_of_covered` we define what to do with a path $\gamma$ that is 1-covered, i.e. map it to $F_1([\gamma])$ or $F_2([\gamma])$ depending on whether $\gamma$ is covered by $X_1$ or $X_2$. `functor_map_aux_part_one` specifies what $F_1([\gamma])$ should be. Again, we use a shorter notation $F_0$ for `functor_map_of_covered hX h_comm`. We can then define `functor_map__of_covered_partwise` for a $n$-covered path inductively. If $n = 1$, we can apply $F_0$ by definition. Otherwise, we are able apply $F_0$ to the first part of the path, which is 1-covered, and then `functor_map_of_covered_partwise` to the second part, which is $(n-1)$-covered. We abbreviate `functor_map_of_covered_partwise hX h_comm` to $F_n$.

Note that $n$ is still an input for the definition, so we need to show that this definition is independent of the choice for $n$. This is captured in the lemma `functor_map_of_covered_partwise_unique` (Step 4).

This lemma makes use of the following lemma that shows that the image remains the same if we refine the partition of $\gamma$, so when we use a $nk$-covering instead of a $n$-covering.

Now we know that the image is independent of $n$, and because a $n > 0$ exists such that $\gamma$ is $n$-covered (shown in `has_subpaths`), we can choose one such $n$ and we obtain the following formalization of $F'$, completing Step 2.

We have now formalized the $F'$ from the proof of the Van Kampen Theorem and we first show that Eq. (1) and Eq. (2) from Theorem 6.2 hold (Step 5).
We arrive at Step 6 and want to show it is invariant under the dihomotopic relation. To do this we need to show the claim from the proof: if we have a directed homotopy $H$ from $f$ to $g$ that is $(1,m)$-covered, then $F'[H(\_,1)] \circ F'[f] = F'[g] \circ F'[H(\_,0)]$ (Step 7).

We can now finally define the behavior on morphisms to obtain a functor by using the universal mapping property of quotients.

```
def functor_map {x y : dπₓ X} (γ : x → y) : F_obj x → F_obj y :=
    quotient.lift_on γ F_map_aux
    (functor_map_aux_of_dihomotopic_dimaps (m : ℕ) :
     Π {f g : D(I, X)} {H : directed_map.dihomotopy f g} (hcov : directed_map.dihomotopy.covered_partwise H hX 0 m),
        F_map_aux (dipath.of_directed_map f) ≫ F_map_aux (H.eval_at_right 1) =
        F_map_aux (H.eval_at_right 0) ≫ F_map_aux (dipath.of_directed_map g) :=
    /- Proof omitted -/
```

By using induction once again, we end up with the lemma showing us that the choice of representative does not matter.

```
lemma functor_map_aux_of_dihomotopic (γ γ' : dipath x y) (h : γ.dihomotopic γ') :
    F_map_aux γ = F_map_aux γ' :=
    /- Proof omitted -/
```

Here $F_{\text{map}}$ is an abbreviation for $\text{functor_map} hX X₁_{\text{open}} X₂_{\text{open}} h_{\text{comm}}$ and $\text{Functor}$ is analogously abbreviated to simply $F$. Finally, we get to Step 8. The remaining lemmas show that $F \circ  \tilde{\Pi}(j_k) = F_k$ for $k = 1$ and $k = 2$, and that $F$ is the unique functor with this property.

```
lemma functor_comp_left : (dπᵐ j₁) ≫ F = F₁ := /- Proof omitted -/
lemma functor_comp_right : (dπᵐ j₂) ≫ F = F₂ := /- Proof omitted -/
```
The Van Kampen Theorem is stated as

\[ \text{theorem directed_van_kampen \{hX}_1: is\_open \ X_1\} \{hX}_2: is\_open \ X_2\} \{hX: X_1 \cup X_2 = \text{set.univ}\} : \\
\text{is\_pushout} (d\pi_m i_1) (d\pi_m i_2) (d\pi_m j_1) (d\pi_m j_2) := \\
/- Proof omitted -/
\]

The type \text{is\_pushout} is defined in Lean in terms of colimits and cocones. Those two category theoretical terms are used in the generalization of the definition of a pushout. In \text{pushout\_alternative.lean} we show that our definition of a pushout, as stated in Definition 2.7, implies \text{is\_pushout}. The theorem \text{directed_van_kampen} now follows easily from the lemmas we proved.

6.3 Applications

Topological spaces often have uncountably many points and so their fundamental categories have uncountably many objects making them hard to reason about. It is possible to reduce a fundamental category to a subset of objects that captures the essence of the directedness. This makes the fundamental category easier to work with and with the right selection of points, a Van Kampen Theorem still holds. This requires some more theory [Bub09].

Another way to keep it simple is by looking at spaces with finitely many points. The first space \(X\) we will look at is the so called discrete or finite unit circle.

**Example 6.4.** The discrete unit circle is a finite version of \(S^1\) and consists of four points \(X = \{N, E, S, W\}\). \(E\) and \(W\) are the right and left point of the unit circle and \(N\) and \(S\) represent the upper and lower arc of the circle. The topology is given by

\[ T_X = \{\emptyset, \{N\}, \{S\}, \{N, S\}, \{N, S, E\}, \{N, S, W\}, \{N, E, S, W\}\} \]

Let now four paths be given by

\[ \gamma_1(t) = \begin{cases} E, & t = 0, \\ N, & 0 < t. \end{cases}, \gamma_2(t) = \begin{cases} N, & t < 1, \\ W, & t = 1. \end{cases}, \gamma_3(t) = \begin{cases} W, & t = 0, \\ S, & 0 < t. \end{cases}, \gamma_4(t) = \begin{cases} S, & t < 1, \\ E, & t = 1. \end{cases} \]

Schematically we can represent the space and these paths as:
We can represent this space and these paths as:

We can now ask the question: what do endomorphisms of

We take two maps

X

We now make

X

X

E

additional paths

γ

N

be similar to the discrete unit circle we just considered. We take

⃗π

claimed that

X

is a pushout of

X

and it holds that

On morphisms, we have that necessarily

F

D

commutative square.

Π

⃗π

that

the same if and only if their sequences of these base morphisms are the same up to identities. We claim

morphisms in

D

by the four morphisms

X

has thus three objects with their three identities and three additional morphisms : 

γ

1

γ

2

γ

3

γ

4

to monotone reparametrization in

E

path in that space also runs counterclockwise.

any concatenation of the four paths. Note that any path in

X

will now look at a finite version of the space considered in Example 6.3. This space will

Example 6.5.

We will now look at a finite version of the space considered in Example 6.3. This space will

Example 6.5. We will now look at a finite version of the space considered in Example 6.3. This space will

be similar to the discrete unit circle we just considered. We take

Y = \{N, E, S, W, M\} with the topology given by

\[ \mathcal{T}_Y = \{ \emptyset, \{N\}, \{M\}, \{S\}, \{N, M\}, \{N, S\}, \{M, S\}, \{N, M, S\}, \{N, M, S, E\}, \{N, M, S, W\}, Y \} \]

In this case

N

M

S

all represent open intervals. Note that the discrete unit circle is a subspace of

Y

in a natural way. We take the four paths

\[ \gamma_1, \ldots, \gamma_4 \]

equal to those in the previous example and take two additional paths

\[ \gamma_5(t) = \begin{cases} E, & t = 0, \\ M, & 0 < t < 1, \end{cases}, \gamma_6(t) = \begin{cases} M, & t < 1, \\ W, & t = 1. \end{cases} \]

We can represent this space and these paths as:
We take the open subspaces $Y_1 = \{N, M, S, E\}$ and $Y_2 = \{N, M, S, W\}$. The non-trivial morphisms in $\vec{\Pi}(Y_1)$ are $[\gamma_1], [\gamma_4], [\gamma_5] \circ [\gamma_4]$ and $[\gamma_1] \circ [\gamma_5]$. In $\vec{\Pi}(Y_2)$ they are $[\gamma_2], [\gamma_3], [\gamma_6], [\gamma_3] \circ [\gamma_2]$ and $[\gamma_3] \circ [\gamma_6]$. $\vec{\Pi}(Y_1 \cap Y_2)$ only consists of three points $N, M$ and $S$ together with their identities. Just like in the previous example, Theorem 6.2 tells us that the morphisms in $\vec{\Pi}(Y)$ are freely generated by the six morphisms $[\gamma_i]$, $1 \leq i \leq 6$.

This time we are interested in the non-trivial endomorphisms of $W$. If we start in point $W$, first the morphism $[\gamma_4] \circ [\gamma_3]$ must be followed to reach point $E$. From there we have a choice to return to $W$: either $[\gamma_2] \circ [\gamma_1]$ or $[\gamma_6] \circ [\gamma_5]$. We find that endomorphisms of $W$ are sequences of two loops $[\gamma_2] \circ [\gamma_1] \circ [\gamma_4] \circ [\gamma_3]$ and $[\gamma_6] \circ [\gamma_5] \circ [\gamma_4] \circ [\gamma_3]$. This structure is isomorphic to $\mathbb{N} \ast \mathbb{N}$, agreeing with Example 6.3.

7 Conclusion and Further Research

In this thesis, we presented important concepts from directed topology. These allowed us to state and prove a directed version of the Van Kampen Theorem. If its simple conditions are satisfied, we calculate the fundamental category of a directed space using the fundamental categories of subspaces. Within this thesis, we showcased how we formalized that theorem and the theory leading up to it using the Lean proof assistant.

Our formalization can be extended to other concepts and theorems from directed topology. For example, Bubenik’s approach of restricting the fundamental category to a (finite) full subcategory is a clear extension of the theory we have formalized in Lean.

At the moment, MathLib does not have a version of the Van Kampen Theorem for groupoids, originally proven by Brown in 1968 [Bro68, Bro06]. As Grandis based his proof of the directed version on Brown’s version, our formalization can conversely be adapted to suit the undirected case. The undirected version could also be shown to be a corollary of the directed version. This is because the fundamental category of a topological space equipped with the maximal directedness coincides with the fundamental groupoid of the topological space.

It might also prove interesting to further investigate sufficient and necessary preconditions for the Van Kampen Theorem, as stated in Section 6.1. As the proof of the Van Kampen Theorem suggests, we want to be able to cover both paths and homotopies, but whether that is truly necessary is something that future research will have to tell.
References


