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Master Thesis

Strategies for Restricted Nim

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MASTER THESIS

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Abstract

Nim is a combinatorial game played by two players. The game consists of one or multiple piles of chips of any size. The players take turns taking as many chips as they like from one pile at a time. If a player is unable to make a move because there are no more chips left, this player will lose the game of Nim.

For any game of Nim it is known whether the starting player or the second player will win the game given optimal play and what strategy has to be used in order to actually win the game. In this thesis we look at different restrictions for this game of Nim. These restrictions alter the outcome of the normal game in such a way that the previous strategies can no longer be used to win.

An example would be disallowing the same number of chips to be taken as were taken in the previous turn, or only allowing up to a certain number of chips to be taken. These new games of restricted Nim require different strategies to win, which can be calculated and determined.

Contents

1 Introduction

Nim is a mathematical game of strategy played by two players [1, p.2–4]. The players take turns removing chips from one or more piles of chips. In normal play, a player wins if he[1] is the last player to remove one or more chips from a pile. The players can remove as many chips as they like but they can only remove chips from one pile at a time.
The strategy for playing Nim with two players is known and the outcome of any game of Nim can be easily decided [4]. The goal of this thesis is to change the rules of Nim by posing restrictions to the players in order to change the necessary strategies to win.
These restrictions limit the options players have compared to regular Nim. For instance, we can restrict the previous move that has been made, so a player is not allowed to remove the same number of chips that has been removed by the other player in the previous turn. We can also let a player choose which move to restrict after their turn, or let players ban moves in advance, before even starting the game of Nim.

In Section 2 the rules of regular and restricted Nim are further explained. An explanation of Sprague-Grundy Values are also given, which can be used to indicate whether a position is winning or losing for the next player. Section 3 gives some related work on solving Nim. Table 1 gives an overview of the rest of the sections in this thesis. The table shows what game type was examined in each section and with how many player and piles this game was played. It also shows in what way the results are represented and gives a short comment on these results. Abbreviations used in the table are explained below.

Table 1: Contents and result types of the sections of this thesis.

Section	Game Type	Players	Piles	Result Type	Comment
4.1	PR	2	1	SG	Repeating Pattern
4.2	CR	2	1	SG	Repeating Pattern
4.3	PR	2	2	W/L/WM	Strategies Three Cases
4.5	PR/Ranking	N	1	WP	Repeating Strategy
4.6	PR	3	1	WP%	Complicated Cases
5	Wythoff/PR	2	2	W/L/WM	Differs from Regular
5.1	Wythoff/PR	2	3	W/L/WM	Differs from Regular
6.1.1	M	2	2	W/L	Similar to Regular
6.1.2	M/SG	2	2	W/L	Similar to Regular
6.1.3	M/PR	2	2	W/L/WM/WR	Almost Opposite from PR
6.2	B	2	1	W/L	Final Ban Important
6.2.1	B/PR	2	1	W/L	Better for Starting Player
6.2.2	B/NE	2	1	W/L	Better for Starting Player
6.3	U	2	2	W%	Better for Second Player
6.4	PB	2	1-5	W/L	Easy for Low Pile Count

If the results field in the table says SG, this means that for the result the Sprague-Grundy values of the examined situations are given. W/L means that it is given which player will win and which player will lose. For WM it is also possible that a player can only win with one certain move, and for WR a player sometimes can only win if one certain move is restricted to them. WP indicates that the winning player is given, this is the case when there are more than two players. Finally $W\%$ means that the winning probability for the

[1]The personal pronoun he refers here, and in future instances, to he or she.

starting player is given.
For the game types PR means that the previous move made is restricted for the next player. CR means that a player gets to choose which move they restrict for the next player. For more than two players the results can be based on a ranking or not, this is explained in Section 4.5. Wythoff can be seen as a different game similar to Nim, this is explained in Section 5. An M indicates Misère play, now the winning condition is reversed compared to regular play. M/SG indicates the Misère version of the Subtraction Game. A B means that players need to ban moves for the other player in advance, before playing Nim. B/NE means that players need to decide on their bans simultaneously, which creates a situation with a Nash Equilibrium. U indicates that the starting pile sizes are unknown to both players and for PB players have the option to put some of the chips that have removed back onto a different pile.

This thesis is a master thesis written for the Leiden Institute of Advanced Computer Science (LIACS) at Leiden University. The supervisors for this thesis are Walter Kosters and Hendrik Jan Hoogeboom.

2 How to Play Nim

Regular Nim is a game played with two players. Nim is played with piles of chips stacked onto each other. There can be any number of piles, and these piles can all have any number of chips. During their turn, players can remove any number of chips from one pile, as long as they remove at least one chip. If there are no more chips left, no move can be made. The player that makes the last move wins the game. Nim is called an impartial game because both players have the same options when making a move, the move a player can make depends only on the state of the game and not on which player is making a move.
When the game is played with only one pile, the starting player will always win by taking the entire pile. If the game is played with more than one pile the game will be more interesting. Figure 1 shows an example of a position in Nim with three piles of chips. The first pile has four chips, the second pile has two chips and the third pile has five chips.

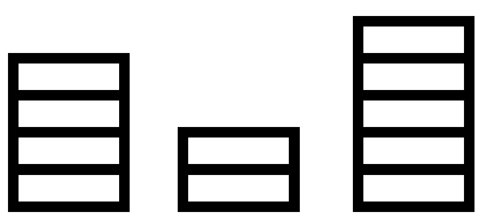

Figure 1: A position of Nim played with three piles, consisting of four, two and five chips.

2.1 Sprague-Grundy Values

The Sprague-Grundy Function can be used to indicate whether a certain situation of a game is a winning position or a losing position for the next player [6]. The Sprague-Grundy

Function is given as follows:

$$g(x) = min\{n \geq 0 : n \neq g(y) \text{ for } y \in F(x)\}$$

Here $g(x)$ indicates the Sprague-Grundy value (SG value) of a position x in a certain game. $F(x)$ indicates all positions that can be reached from position x after making a single move. This means that the SG value of x is given by the smallest number larger or equal to 0 that is not an SG value of one of the positions that can be reached from x in a single move. The smallest number larger or equal to 0 that is not included in a set of numbers is called the Minimum Excluded Value, or mex, of that set of numbers. So the SG value of a position is the mex of the SG values of the positions that can be reached in one move from that position.

If there are no moves that can be made from a certain position, for example if all piles are empty during a game of Nim, this position will get an SG value of 0. If a position has an SG value of 0, this means that the player who is about to make a move will lose the game. If the SG value of a position is greater than 0, the player who will make the next move will always win the game. If a position has an SG value greater than 0, one of its following positions must have an SG value of 0, otherwise the mex of that position would have been 0. This means that the next player can make a move that will put the other player in a position with SG value 0. A position with SG value 0 will have following positions with an SG value greater than 0, otherwise the mex would not have been 0. This means that a position with SG value 0 cannot lead to a position where no more moves are possible, but instead will always lead to a position that has a position with an SG value of 0 as a following position. So the player who goes next in a position with SG value of 0 always loses and the player who goes next in a position with SG value greater than 0 always wins.

2.2 Strategy for Playing Regular Nim

The SG values can be used to devise a winning strategy for Nim. If the game is played with only one pile, every position where the pile is not empty will have an SG value larger than 0, as it is possible to empty the pile and the game with just the empty pile has an SG value of 0.

Now we want to know what a player needs to do to win a game with more than one pile. If we add two games of Nim together the resulting SG value of the new game will be the so called Nim-Sum of the SG values of the two separate components. The Nim-sum of two values is obtained by adding the individual bits of the values without carry, or in other words performing a bitwise XOR operation on the two values [4]. When combining two games each consisting of one pile this means that the SG value of the game with two piles will only be 0 if both individual components had the same SG value. This makes sense as the so-called mirror image strategy can be used by the second player to win. They will always need to make the same move the first player made, but on the other pile, until no more chips are left.

Even for a game with more than two piles it is always possible to move to a position with a Nim-Sum of 0 from a position with a Nim-Sum larger than 0, and not possible to move to a position with Nim-Sum larger than 0 from a position with a Nim-Sum of 0. How this is done is described in [4].

2.3 Restricted Nim

There are many different ways to change the rules of a standard game of Nim by introducing restrictions. One example of a restriction is to give an upper bound for the number of chips that a player is allowed to take in one turn. This version of Nim is called the Subtraction Game $S(1, 2, \ldots, k)$, where k is the maximum number of chips a player can take in one turn. So when playing $S(1, 2, \ldots, k)$, a player can take 1 or 2 or ... or k chips in one turn, they need to take at least one chip and can take a maximum of k chips. Using this restriction will give more substance to games played with only one pile.
It is also possible to restrict the moves a player can make based on the move made by the previous player. For example if one player removes a certain number of chips, the next player cannot remove this same number of chips during his next turn. We can also restrict the number of chips that a player took in their own previous turn, or restrict the moves made over a certain window of previous turns. We will go over multiple different restrictions like this and also look at versions played with more than two players.

3 Related Work

In order to see if a game of Nim is winnable by the starting player the game can be translated to a Quantified Boolean Formula (QBF) [5, p.201–202]. A QSAT Solver can then be used to figure out if the game is indeed winnable by the first player considering optimal play.
A game of Nim can be translated to QBF by defining variables, clauses and a formula, as follows. We can use the following variables, from [7].

L_i: Variables which indicate the height of the piles at time step i.
M_i^A/M_i^B: Variables which indicate the moves the starting player (player A) and the second player (player B) can make at time step i.

And the following clauses:

I: Set of clauses representing the initial height of piles at time step 0.
T_i: Set of clauses representing transition axioms at time step i. These are the same for both players in Nim since they have the same options in identical situations.
G: Set of clauses representing the goal. In the case of Nim, all piles have a height of 0 and player B is the next player to make a move.

We get the following formula:

$$\exists M_0^A \forall M_1^B \exists M_2^A \forall M_3^B \ldots \exists M_{k-2}^A \forall M_{k-1}^B \exists M_k^A \exists L_0 \ldots L_{k+1}$$
$$(I \wedge (T_0^A \wedge \ldots \wedge T_k^A) \wedge (T_0^B \wedge \ldots \wedge T_k^B) \wedge G)$$

So there has to exist a move for player A in time step t, so that for all moves for player B in time step $t+1$ there exists a move for Player A in time step $t+2$, ending with a move in time step k for player A, so that there exists a course of play starting in the initial state and ending in the goal state.
In order to prevent Player B from making any illegal moves we can introduce a variable z which is true iff Player B makes an illegal move. This gives the following formula:

$$\exists M_0^A \forall M_1^B \exists M_2^A \forall M_3^B \ldots \exists M_{k-2}^A \forall M_{k-1}^B \exists M_k^A \exists L_0 \ldots L_{k+1} \exists ilm_1 \ldots ilm_{k-1} \exists z$$
$$(((I \wedge (T_0^A \wedge \ldots \wedge T_k^A) \wedge (T_0^B \wedge \ldots \wedge T_k^B) \wedge G) \vee z) \wedge (z \leftrightarrow ilm_1 \vee \ldots \vee ilm_{k-1}))$$

If this formula is true the player that makes the first move will win the game of Nim given optimal play.

For more information on Nim and combinatorial game theory see [1], [2], [3] and [4].

4 Restricting the Previous Move

In this section we look at the Sprague-Grundy values (SG values) obtained by playing the Subtraction Game with the added restriction that a player cannot take the same number of chips as the other player in the previous turn.

4.1 Version with One Pile

We first look at the version played with one pile. The starting player is allowed to take any number of chips that is allowed by the Subtraction Game, as no previous move has been made, there is no extra restriction. Table 2 shows the SG values when playing the Subtraction Game $S(1,2,3)$ with this restriction, using one pile of up to 11 chips. A player can take either 1, 2 or 3 chips unless restricted. When the restricted number is given as 0, this indicates a starting position, the starting player can take either 1, 2 or 3 chips without a restriction. If the number that is restricted is greater than the number of chips left on the pile, the next player can make a move as if there is no restricted value, as if the restricted number was 0.

Table 2: Sprague-Grundy values when playing Substraction Game S(1,2,3) with the restriction that a player cannot make the same move that was made in the previous turn.

Chips\Restricted	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	1	1	1	1
3	2	2	2	0
4	0	0	0	0
5	3	0	2	1
6	1	1	1	1
7	3	1	2	0
8	0	0	0	0
9	3	0	2	1
10	1	1	1	1
11	3	1	2	0

Any situation in which it is impossible to make another move has an SG value of 0. In Table 2 these are the positions where there are 0 chips left on the pile and the position where there is 1 chip left, but it is restricted to take 1 chip because that was the number

of chips taken in the previous turn.
For every other position the SG value is determined by the SG values of the positions that can be reached in one move. If there are n chips left on the pile and the restricted number is 0, the three positions that can be reached are $n-1(1)$, $n-2(2)$ and $n-3(3)$, where the number in brackets is the restricted number. In the table these are the values that are diagonally to the upper right of the current position. So the values for $n=5$ are the 0 from 4(1), the 2 from 3(2) and the 1 from 2(3). The SG value for 5(0) is now obtained by taking the minimum excluded value (mex) of these three values, which is 3. In order to obtain the SG value for 5(1) we need to do the same thing except now we disregard the value of 4(1), as it is not possible to reach this position when taking 1 chip is restricted. We now take the mex of the 2 from 3(2) and the 1 from 2(3), which gives 0. For 5(2) we need to disregard the value of 3(2), so we get the mex of 0 and 1 which is 2, and for 5(3) we need to disregard the value of 2(3), so we get the mex of 0 and 2 which is 1.
In Table 2 it can be seen that the SG values of piles with 4, 5, 6, and 7 chips retain the same SG values when you add exactly four chips to these piles. The SG values of the pile with 4 chips are the same as the values of the pile with 8 chips, the values of the pile with 5 chips are the same as the values of the pile with 9 chips, etcetera. It can be shown that this pattern will keep on repeating no matter how many chips are on the pile. The SG values for the pile with 12 chips are solely dependent on the SG values of the piles with 11, 10 and 9 chips. Since the SG values of the piles with 11, 10 and 9 chips are exactly the same as the SG values of the piles with 7, 6 and 5 chips respectively, the SG values of the pile with 12 chips has to be exactly the same as the SG values of the pile with 8 chips. Using these SG values of the pile with 12 chips we can determine the SG values of the pile with 13 chips in the same way, and after that for the pile with 14 chips and so on.

Proposition 4.1. If the SG values of three consecutive numbers of chips on a pile are the same as the SG values of three other consecutive numbers of chips on a pile, there is a pattern that will repeat indefinitely.

Proof. Every set of SG values that comes after the first three consecutive numbers of chips will be the same as the sets of SG values that come after the second three consecutive numbers of chips. When playing Subtraction Game $S(1,2,\ldots,k)$, this is the case for every possible value of k. If it is possible to find two sets of k consecutive numbers of chips on a pile with exactly the same SG values, there is a pattern that will repeat indefinitely. The SG values when playing with a pile of n chips is solely dependent of the SG values of the piles with $n-1$ through $n-k$ chips. □

So now we see that if there are two sets of k consecutive numbers of chips on a pile with the same SG values, there is a pattern that will repeat indefinitely. We can also say the following.

Proposition 4.2. There will always be two sets of k consecutive numbers of chips on a pile with the same SG values.

Proof. This can be said because the possible SG values are finite. If you can take a maximum of k chips from the pile, there is a maximum of k positions that can be reached from a certain position. The SG value is obtained by taking the mex of at most k values. This means that the SG value of a position can never be higher than k. So the SG value of every position is a value between 0 and k. This means that there is only a finite number of combinations of SG values k consecutive numbers of chips on a pile can have. Eventually this exact same combination of SG values will be found again, since there are only finite possibilities. □

4.2 Choosing the Restricted Number

We can also play the same version of the Subtraction Game as in the previous section except now allow a player to choose which move will be restricted to the next player. This means that in addition to taking away chips from a pile the player can choose the number of chips the next player is not allowed to take during their next turn. Table 3 shows the SG values when playing the Subtraction Game $S(1, 2, 3)$ with this restriction, using one pile of up to 13 chips. The starting player is allowed to take 1, 2 or 3 chips, so there is no extra restriction aside from the rules of the Subtraction Game.

Table 3: Sprague-Grundy values when playing Substraction Game S(1,2,3) where a player can restrict a move for the next player.

Chips\Restricted	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	1	2	2
3	3	2	3	3
4	4	3	4	0
5	5	0	5	1
6	6	1	4	2
7	6	2	5	3
8	6	3	4	0
9	6	0	5	1
10	6	1	4	2
11	6	2	5	3
12	6	3	4	0
13	6	0	5	1

In this version of the Subtraction Game the SG value of a pile with n chips is still only decided by the SG values of the piles with $n-1$ though $n-k$ chips, as k is the maximum number of chips that a player can remove form the pile in one move. This means that just as with the previous version there will be a pattern of SG values that repeats indefinitely. If there is a pattern of k consecutive numbers of chips on a pile that repeats once, it will keep repeating. The upper limit for the SG value will be k^2, as there are a maximum of k moves that can be made and for every move k numbers can be restricted for the next player, so every position can lead to a maximum of k^2 other positions. Eventually a pattern of k consecutive numbers of chips on a pile will repeat.
Something interesting to note from Table 3 is that the starting player will always be able to win when the pile has at least 1 chip. An SG value of 0 means that the next player will lose and an SG value greater than 0 means the next player will win. When the number of chips that is restricted is 0, there is no number of chips on the pile with an SG value of 0. In Table 3 there are only a few position in which the next player would lose. Here they are: $0, 1(1), 4(3), 5(1), 8(3), 9(1), 12(3)$ and $13(1)$, where the first number indicates the number of chips that are left on the pile and the number in brackets is the number of chips that is restricted.

Proposition 4.3. The numbers of chips on a pile that have a losing situation are always $n(k+1)$ and $n(k+1)+1$ for any positive integers n and k, where k is the maximum number of chips a player can take in one move.

Proof. The only positions in which a player immediately loses are either, when there are no more chips left on the pile or when there is only 1 chip left, but it is restricted to take 1 chip. In both situations no more moves can be made so the next player loses. When the number of chips left is greater than 1 and up to k chips, the next player will always win. This is because he can always take a certain number of chips to make sure that there are either 0 or 1 chips left. Even if one of these two options is restricted, the player can still go for the other option. For example, if there are k chips left, but it is restricted to take k chips, the next player can take $k-1$ chips, leaving only 1 chip, and choose to restrict taking 1 chip for the other player. A losing position occurs when there are $k+1$ chips left on the pile, the next player will not have both the options to leave 0 or 1 chips. The player can take a maximum of k chips so it is impossible to leave 0 chips. When the option to take k chips is restricted, so the player is not allowed to leave 1 chip, the next player will lose. He has to move to a position where both these options are available. If there are $k+2$ chips left there will also be a losing position. A player can now neither decrease the number of chips to 0 or to 1, however they can decrease the number of chips to $k+1$ and then restrict k in order to win. There is no other option to win however, so if taking 1 chip is restricted the next player will lose. So if a player can move to either $k+1$ chips or $k+2$ chips they will win, which is possible for piles with a number of chips greater than $k+2$ up to $k+1+k$ chips.
When the number of chips is $2(k+1)$ a similar situation occurs as when the number of chips was $k+1$. A player can no longer reach one of the losing positions, so if reaching the other losing position is restricted, that means that the next player will lose, since he can only move to a position which is winning for the next player. When the number of chips is $2(k+1)+1$, the same thing happens as when the number of chips was $k+2$, if a player is not allowed to take 1 chip they will lose. This means that $3(k+1)$ will have a losing position because it can only reach $2(k+1)+1$ and not $2(k+1)$. And $3(k+1)+1$ will also have a losing position when taking 1 chip is restricted. Because of this $4(k+1)$ and $4(k+1)+1$ will also have losing position. This pattern will repeat indefinitely, so for any positive integer n, $n(k+1)$ and $n(k+1)+1$ will have losing positions. $\square$

4.3 Two Piles of Chips

Before, we have only considered games of Nim played with a single pile of chips, but we will now look what happens when there are two piles. We look at the version of the Subtraction Game where a player is not allowed to take the same number of chips that the other player had taken in the previous turn. This restriction is applied to both piles, so for instance if a player takes 3 chips from the first pile, the next player is not allowed to take 3 chips from either pile, so neither the first nor the second pile. A player has to take at least 1 chip and there is a maximum number of chips k a player is allowed to take in one turn. We only look at the cases where $k \geq 2$, the value of k has to be at least 1 and if the value is exactly 1, the game will always end after maximally 1 turn as the second player is not allowed to take 1 chip, which was the only option.
We will now look whether a game of this version of Nim with certain pile sizes can be won by the next player to make a move. Every position of this game can be categorised into three outcomes:

- The next player always loses, no matter which move is restricted.
- The next player always wins, no matter which move is restricted.
- The next player can win if he is allowed to make a specific move, if he is not allowed to make this move, he will lose.

Which of the three cases a position has depends on the number of winning moves the next player can make. If the next player has no winning moves, he will always lose, it does not matter which move is restricted because the player did not have any moves at all. If the player can always win he needs to have at least two different winning moves. If one of these two moves is restricted he can still win by going for the other winning move. This is why he is always able to win no matter which move is restricted, he is always able to play a winning move. Only two winning moves are necessary because only one move can be restricted at a time. If the next player only has one winning move, he can only win if this move is not restricted, since none of the other moves will lead to a winning position for the player. So the first case occurs when the next player has no winning moves, the third case when the next player has exactly one winning move and the second case occurs when the next player has two or more winning moves.
We now know that every position of a game of this version of Nim belongs to one out of three outcome groups. We want to find a pattern in the behaviour of these positions when more chips are added to one or both piles. The value of k is important for the outcome of the positions of a game. When analysing the positions we look at three different cases for these values of k:

- k is an even number.
- k is an odd number and $(k+1)/2^m$ is an odd natural number if m is an even number. So $k+1$ becomes an odd natural number after it has been divided by 2 an even number of times.
- k is an odd number and $(k+1)/2^m$ is an odd natural number if m is an odd number. So $k+1$ becomes an odd natural number after it has been divided by 2 an odd number of times.

The reason for dividing the odd values of k will be explained later, we will first look at the even values of k.

4.3.1 Even Values of k.

Table 4 shows the results of this version of Nim played on two piles of a size up to 30 chips. The value of k is 8, so the players can take 1 through 8 chips during their turn, given that the other player did not take the same number of chips in their previous turn. In this table a P indicates that the position is a win for the previous player. This means that the next player to make a move will always lose in this position, no matter which move is restricted. An N indicates that the next player to make a move will always win, no matter which move is restricted. If a position has a number in the table, this means that this is the only move the next player can make to win. If this move is restricted, the next player will always lose. An empty space indicates that the next player can only win by removing 1 chip from a pile, and will lose if taking 1 chip is restricted. We chose to indicate this situation as an empty space because it is the situation that appears the most and indicating it with an empty space improves the readability of the table. Of course, the table is symmetric in the main diagonal.

Table 4: Results of the restricted version of Nim with two piles up to 30 chips and a maximum number of chips that can be taken of 8. Empty spaces indicate a 1.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	P		N	3	4	5	N	7	N	P	N	2	3	N	N	N	7	8	P		N	3	4	N	N	N	N	P		N	3
1																															
2	N									N		N							N									N			
3	3			N						N			N						N			N						N			N
4	4				N					N									N				N					N			
5	5					N				N									N									N			
6	N									N									N									N			
7	7							N		N							N		N									N			
8	N									N								N	8								N	8			
9	P		N	N	N	N	N	N	N	P		N	N	N	N	N	N	8	P		N	N	N	N	N	N	8	P		N	N
10	N																														
11	2		N							N									N									N			
12	3			N						N									N									N			
13	N									N									N									N			
14	N									N									N									N			
15	N									N									N									N			
16	7							N		N									N									N			
17	8								N	8								N	8								N	8			
18	P		N	N	N	N	N	N	8	P		N	N	N	N	N	N	8	P		N	N	N	N	N	N	8	P		N	N
19																															
20	N									N									N									N			
21	3			N						N									N									N			
22	4				N					N									N									N			
23	N									N									N									N			
24	N									N									N									N			
25	N									N									N									N			
26	N								N	8								N	8								N	8			
27	P		N	N	N	N	N	N	8	P		N	N	N	N	N	N	8	P		N	N	N	N	N	N	8	P		N	N
28																															
29	N									N									N									N			
30	3			N						N									N									N			

The positions where the next player always loses is indicated in the table with a P, these are called the P positions.

Proposition 4.4. The positions where the number of chips on both piles have a value $\equiv 0 \pmod{k+1}$ are always P positions.

Proof. Because we have $k = 8$ in the table this happens when both piles have a size of $0, 9, 18, 27, \ldots$ etcetera. This works for every even value of k. The situation $(0, 0)$ will always be a P position because neither player can make a move, so the next player to make a move loses. When one of the piles has $k+1$ chips, and the other pile 0, the second player will always make the last move. The next player has to take any number of 1 through k chips and because the total number of chips is $k+1$ the second player can always take the remaining number of chips. The first player cannot take the entire pile because the pile is just one chip too big. The first player also cannot take half of the pile, which would restrict taking the other half for the second player, because a pile of size $k+1$ will always have an odd number of chips since k is even.
The same strategy also works if there are two piles of size $k+1$. The first player has to take a certain number of chips from one pile. The second player can now empty that pile, no matter what number of chips was taken, reducing the situation to the same one described before.
This is also why every position that has two piles with a value $\equiv 0 \pmod{k+1}$ is a P position. The second player can always take a number of chips to put the first player in a

new, smaller position where again both piles have a value $\equiv 0 \pmod{k+1}$. This will go on until the position $(0, 0)$ is reached. The second player will have made the last move and wins.

□

Figure 2 shows the example with one pile consisting of 9 chips. Here the first player takes 5 chips, allowing the second player to take the remaining 4 chips. No matter what number of chips the first player takes, the second player can take the rest.

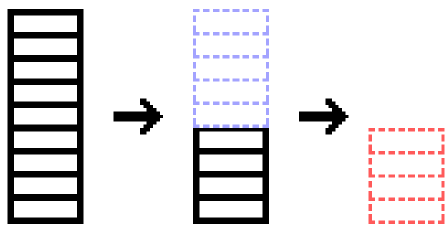

Figure 2: With a single Nim pile of size 9 and $k = 8$, the second player can always win by taking the remaining number of chips.

These positions are also the only P positions that occur. We will show this by proving what the outcomes of the other positions are, and why these are not P positions.

Proposition 4.5. When one of the two piles has only 1 chip left, the first player can always win if taking 1 chip is not restricted, regardless of the number of chips on the other pile. If taking 1 chip is restricted, the first player will lose.

Proof. If the other pile is empty the first player can take the 1 chip from the pile and win. If both piles have 1 chip each, the first player can take 1 chip from a pile, the second player is not allowed to take 1 chip, so he cannot make a move and the first player wins. If one pile has 1 chip and the other pile has more than 1 chip, the first player should always take a single chip from the other pile. As long as he keeps doing this the second player can never empty the pile with 1 chip. If the second player empties the other pile, the first player can win by emptying the pile with 1 chip. If the situation $(1, 1)$ is reached the first player can win by emptying either pile by taking 1 chip.
The first player will lose if taking 1 chip is restricted, because this means that the second player can take 1 chip during his next turn and win using the same strategy. □

Figure 3 illustrates how this strategy could work in the situation $(8, 1)$. The first player always takes a single chip from the left pile. It does not matter how many chips the second player takes, he is not allowed to take 1 chip. When the left pile is empty the first player takes the last chip from the right pile and wins.

We will now look at the positions where both piles have strictly less than $k + 1$ chips.

Proposition 4.6. If one of the piles is empty and the other pile has an odd number of chips, the first player can only win if this odd number is not restricted, by taking the entire pile.

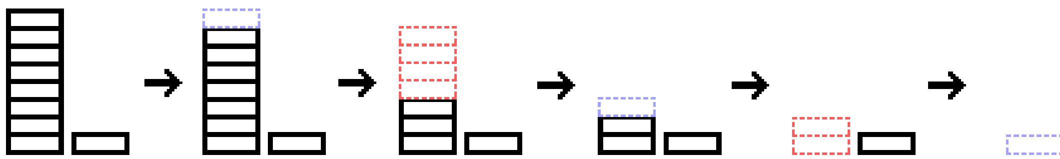

Figure 3: The first player can win in position $(8, 1)$ by always taking 1 chip.

Proof. If this number is restricted he has to take a smaller number, and because the total number of chips was odd, the second player can then empty the pile and win. $\square$

The same goes for when one pile is empty and the other pile has an even number of chips left, but there are exceptions here.

Proposition 4.7. Suppose the first pile is empty, if the number of chips on the second pile is smaller than $k+1$ and the number of chips divided by 2^m is an odd natural number when m is an odd number, the first player can win even when taking the entire pile is restricted. If m is even the first player can only win by taking the entire pile, so he will lose if doing this is restricted.

Proof. The first player can take exactly half of the chips from the second pile. The second player is not allowed to take the remaining chips of the pile now. So if the number of remaining chips is an odd number the second player will lose. He has to take a number of chips so that there will be at least 1 chip left and he cannot divide the pile in half again, the first player can now take the remainder of the chips. If the number of remaining chips after the first player divided the pile in half is an even number, the second player can now divide the pile in half again. If the remaining number of chips is now an odd number, the first player will lose. However if it is even the first player can divide it in half again and so on. This is why if the value of m is odd when the number of chips on the second pile divided by 2^m is odd the first player can win. If the value of m is even, the first player cannot win by dividing the pile in half. Figure 4 illustrates this for a pile of size 8. We get $8/2^3 = 1$, so in this case $m = 3$. Because 3 is an odd number the first player will win by always taking half of the remaining chips.
If m was an even number then the second player would have been the last one able to divide the pile in half. This would cause the first player to be in a situation where he cannot divide the pile in half or take the rest of the chips in the pile, the first player would lose. $\square$

Proposition 4.8. If both piles have more than 0 chips, but less than $k+1$, the first player can always win by removing 1 chip.

Proof. The first player can win by always taking 1 chip, thus restricting the second player from ever taking 1 chip. The first player has to make sure to never be the one who empties a pile. When the second player has to remove all the chips from a pile, the first player can take all the chips from the other pile and win. This does mean that the first player also has to make sure that both piles will not have the same number of chips. Otherwise he will not be able to take all the chips from the second pile, as this would require taking the

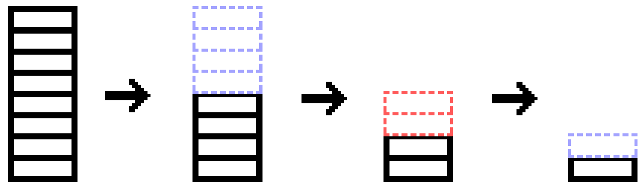

Figure 4: The first player can win in position $(8, 0)$ by always taking half of the remaining chips.

same number of chips the second player just took to empty the first pile. However, the first player can do this by always taking 1 chip. The only way for both piles to have the same number of chips during the second players turn while the first player only takes 1 chip, is if the difference in the number of chips between both piles is only 1 chip. In this case the first player can just take 1 chip from the lower pile, so both piles will not be of equal size. The only exception would be the situation $(2, 1)$, the first player should not empty a pile first, so he has to move to the situation $(1, 1)$. Because the second player is not allowed to take 1 chip, the situation $(1, 1)$ is a winning position for the first player.
The position $(1, 1)$ is also the only position in which the first player should empty a pile, he is forced to do this. The first player will still win because it will lead to the position $(1, 0)$, where the second player cannot make a move as he is not allowed to take 1 chip.

□

There are some cases in which the next player cannot always win when both piles end up with the same number of chips. These are the cases where both piles have an even number of chips that divided by 2^m gives an odd number of chips with m odd. These are the same numbers where there were two winning options if there was only one pile. This is because if one of the two piles gets emptied the other player can divide the remaining pile in half and win using the strategy described before. We have now in general explained all values in the 8×8 upper left hand side square from Table 4.

Now we look at the situations where one of the piles has $k + 1$ chips and the other pile has less than $k + 1$ chips, but more than 0 chips. The first player now always has two options to win, so all these positions are N positions. He can either completely remove the pile smaller than $k + 1$, leaving only the pile of $k + 1$ chips, which is a losing position for the other player, or he can remove 1 chip from the pile with $k + 1$ chips, which puts him in the situation described in Proposition 4.8.
There is one exception to this, if removing 1 chip from the pile with $k + 1$ chips causes both piles to have the same number of chips, so both piles have k chips, and this number is not an even number where the number divided by 2^m is odd with m odd, the first player will not win by removing only 1 chip. This means he can only win if taking the number of chips on the pile smaller than $k + 1$ is not restricted. If both piles have k chips after the move of the first player, the second player can completely remove all chips from one of the piles. This leaves only one pile with k chips, while taking k chips is restricted. The first player is now forced to take half of the remaining chips, as taking any other number of chips will cause the second player to empty the pile during his next turn. However if k is an even number where the number divided by 2^m is odd with m even, the second player will be able to divide the pile in half again and the second player can do this so many

times that the first player will be the one who ends up with an odd number of chips on the pile where he is not allowed to take the entire pile during his turn. This causes the first player to lose.

If both piles have more than $k+1$ chips and at least one of the piles gives a rest higher than 0 when divided by $k+1$, the first player can always win by taking 1 chip. The strategy is similar to when both piles were smaller than $k+1$. The first player has to make sure to never bring a pile down to a value $\equiv 0 \pmod{k+1}$. If he does this, the second player will bring the other pile down to a value $\equiv 0 \pmod{k+1}$, causing the first player to be in a P position. If the first player is forced to bring one of the piles down to a value $\equiv 0 \pmod{k+1}$, the second player cannot do this to the other pile, as he would also need to remove 1 chip. This way the first player can keep removing 1 chip until either one of the situations $(0,0)$, $(0,1)$ or $(1,1)$ is reached, in which the second player cannot make a move because playing 1 is restricted. The first player has to remove 1 chip in many situations because if he does not remove 1 chip, the second player will remove 1 chip and use the strategy just described to win.
There are a few exceptions where removing 1 chip does not win the game. This can happen when the position where both piles have a value $\equiv k \pmod{k+1}$ is an N position. If the first player is in a position where one pile has a value $\equiv k \pmod{k+1}$ and the other pile has a value $\equiv 0 \pmod{k+1}$ and removing 1 chip from the pile with $k+1$ chips leads to an N position, the first player can only win by removing k chips from the pile with a value $\equiv k \pmod{k+1}$. If this is restricted he will lose. An example is the position $(9,17)$ in Table 4, where $k=8$.

If only one of the two piles has more than $k+1$ chips and the other pile has less than $k+1$ chips, the same strategy of always taking 1 chip will work for the first player. However sometimes the first player can also win by taking away the entire pile smaller than $k+1$. If playing with just the pile with more than $k+1$ chips can only be won by taking the number of chips that is on the pile with less than $k+1$ chips, then when playing with both piles, the first player can win by taking the entire smaller pile. This leaves the second player in the situation with only the bigger pile, where taking the number of chips that were in the smaller pile is restricted. These positions are N positions and they only occur when one of the piles is smaller than $k+1$, as otherwise the first player would not be able to take the entire pile, as he can only take a maximum of k chips.

4.3.2 Odd Values of k, Case 1

We divide the odd values of k into two groups. One group consists of the values where $(k+1)/2^m$ is an odd natural number for an even number m. An example is $k=11$, where $k+1=12$ and $12/4=3$. So $(k+1)/2^m$ is an odd natural number when $m=2$. Other examples are $k=15$, where $m=4$ and $16/16=1$, and $k=19$, where $m=2$ and $20/4=5$. Table 5 shows the results when $k=11$ and both piles can have up to 50 chips.

The pattern works the same way as with even values of k. Even though $k+1$ is not an odd number but an even number, the positions where both piles have have a value $\equiv 0 \pmod{k+1}$ are still all P positions. This means that the second player can always win in these positions. For example, if one pile is empty and the other pile has 12 chips, the second player can win by taking the remaining number of chips after the first player has removed some chips from the pile, as no matter how many chips the first player takes there will be at least 1 and at most 11 chips left. The first player can however take half of the pile, so take 6 chips, leaving the same number of chips which is now restricted for

Table 5: Results of the restricted version of Nim with two piles of up to 50 chips and a maximum number of chips that can be taken of 11.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
0	P		N	3	4	5	N	7	N	9	N	11	P		N	3	4	5	N	7	N	9	N	11	P		N	3	4	5	N	7	N	9	N	11	P		N	3	4	5	N	7	N	9	N	11	P		N
1																																																			
2	N												N												N												N												N		
3	3			N									N			N									N			N									N			N									N		
4	4				N								N				N								N				N								N				N								N		
5	5					N							N					N							N					N							N					N							N		
6	N												N												N												N												N		
7	7							N					N							N					N							N					N							N					N		
8	N												N												N												N												N		
9	9									N			N									N			N									N			N									N			N		
10	N												N												N												N												N		
11	11											N	11											N	11											N	11											N	11		
12	P		N	N	N	N	N	N	N	N	N	11	P		N	N	N	N	N	N	N	N	N	11	P		N	N	N	N	N	N	N	N	N	11	P		N	N	N	N	N	N	N	N	N	11	P		N
13																																																			
14	N												N												N												N												N		
15	3			N									N												N												N												N		
16	4				N								N												N												N												N		
17	5					N							N												N												N												N		
18	N												N												N												N												N		
19	7							N					N												N												N												N		
20	N												N												N												N												N		
21	9									N			N												N												N												N		
22	N												N												N												N												N		
23	11											N	11											N	11											N	11											N	11		
24	P		N	N	N	N	N	N	N	N	N	11	P		N	N	N	N	N	N	N	N	N	11	P		N	N	N	N	N	N	N	N	N	11	P		N	N	N	N	N	N	N	N	N	11	P		N
25																																																			
26	N												N												N												N												N		
27	3			N									N												N												N												N		
28	4				N								N												N												N												N		
29	5					N							N												N												N												N		
30	N												N												N												N												N		
31	7							N					N												N												N												N		
32	N												N												N												N												N		
33	9									N			N												N												N												N		
34	N												N												N												N												N		
35	11											N	11											N	11											N	11											N	11		
36	P		N	N	N	N	N	N	N	N	N	11	P		N	N	N	N	N	N	N	N	N	11	P		N	N	N	N	N	N	N	N	N	11	P		N	N	N	N	N	N	N	N	N	11	P		N
37																																																			
38	N												N												N												N												N		
39	3			N									N												N												N												N		
40	4				N								N												N												N												N		
41	5					N							N												N												N												N		
42	N												N												N												N												N		
43	7							N					N												N												N												N		
44	N												N												N												N												N		
45	9									N			N												N												N												N		
46	N												N												N												N												N		
47	11											N	11											N	11											N	11											N	11		
48	P		N	N	N	N	N	N	N	N	N	11	P		N	N	N	N	N	N	N	N	N	11	P		N	N	N	N	N	N	N	N	N	11	P		N	N	N	N	N	N	N	N	N	11	P		N
49																																																			
50	N												N												N												N												N		

the second player. But because m is an even number the second player is able to divide the pile in half again, leaving 3 chips. Now there is an odd number of chips left, so the first player cannot divide the pile in half again, he has to take a number of chips that will cause him to lose, because the second player can take the remaining chips.
When both piles have more than 0 chips but do have a multiple of $k+1$ chips, the next player always loses. If the first player does not take $(k+1)/2$ chips from a pile the second player can reduce the number of chips on that same pile to a multiple of $k+1$ again and repeat the process. So the first player has to take $(k+1)/2$ chips. However now the second player can take 1 chip from the other pile and use the strategy described before for even k and win by always taking 1 chip.

One thing to note is that the position (k, k) will always be an N position, because the first player can either take 1 chip or take k chips and empty a pile. After the first player took k chips, since k is odd, if there is only one pile of k chips left the first player will always be able to empty this pile after the second player made a move, as the second player is not allowed to take k chips. This also causes all positions where both piles have a value $\equiv$

$k \pmod{k+1}$ to be N positions. Other than that, the same strategies apply as for an even k.

4.3.3 Odd Values of k, Case 2

The behaviour when k is odd and $(k+1)/2^m$ is an odd natural number for an odd m is noticeably different from the behaviour of the other two cases. Table 5 shows the result for $k = 7$ for two piles with up to 30 chips.

Table 6: Results of the restricted version of Nim with two piles up to 30 chips and a maximum number of chips that can be taken of 7.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	P		N	3	4	5	N	7	4	P	N	2	N	N	N	6	7	P		N	3	N	5	N	7	P		N	3	4	5
1																															
2	N									N		N						N								N					
3	3			N						N								N			N					N			N		
4	4				N				N	4				N				N								N				N	
5	5					N				N								N					N			N					N
6	N									N						N		N								N					
7	7							N		N							N	7							N	7					
8	4				N					N																					
9	P		N	N	4	N	N	N	N	P		N	N	N	N	N	N														
10	N																														
11	2		N							N																					
12	N									N																					
13	N				N					N																					
14	N									N								N								N					
15	6						N			N																					
16	7							N		N																					
17	P		N	N	N	N	N	7							N																
18																															
19	N																														
20	3			N																											
21	N																														
22	5					N																									
23	N																														
24	7							N																							
25	P		N	N	N	N	N	7							N																
26																															
27	N																														
28	3			N																											
29	4				N																										
30	5					N																									

When both piles have k chips or less the positions are the same as in the other two cases.

Proposition 4.9. If one of the piles is empty and the other pile has $k+1$ chips, the first player can win if taking half the chips is not restricted.

Proof. The first player can keep dividing the pile in half and because m is odd when the number of remaining chips reaches an odd value the first player will become the winner. □

Proposition 4.10. If one pile is empty but the other pile has $k+2$ chips, this position will be a P position.

Proof. The first player will always lose because if he takes more than 1 chip the second player can always take the remaining chips, $k+2$ is always an odd number. If the first player does take 1 chip, he will put the second player in the position described in Proposition 4.9, and the second player will be allowed to take half of the chips. □

Proposition 4.11. If one pile has $k+1$ chips and the other has $(k+1)/2$ chips the next player will always win.

Proof. The first player can either take 1 chip, and win by always taking 1 chip as before, or empty the pile with $(k+1)/2$ chips. Doing this puts the second player in the position with only one pile with $k+1$ chips, where taking half of the chips from this pile is restricted. $\square$

Proposition 4.12. When both piles have exactly $k+2$ chips the next player will always lose.

Proof. If the first player takes more than 1 chip from a pile, the second player will take the remaining chips from that pile, putting the first player in the position $(k+2,0)$, which he will always lose as described before. If the first player does take 1 chip, the second player will be in the position $(k+1,k+2)$. The second player can now take half of the chips from the pile with $k+1$ chips. The first player is now in the position $((k+1)/2,k+2)$. If he plays on the pile with $(k+1)/2$ chips, the second player will be the one who takes the last chip from this pile, leaving the first player in the losing position $(0,k+2)$. The first player has to take chips from the pile with $k+2$ chips. If he does not take 1 chip, the second player will take 1 chip in his next turn and win using the strategy to always take 1 chip. If the first player does take 1 chip he puts the second player in the position $((k+1)/2,k+1)$, which the next player will always win, as we said before. Since the second player will be the next player, the second player will win. This means that the first player always loses, no matter which moves he makes. $\square$

The biggest reason for the differences between these odd values of k and the even values of k has to do with the fact that the first losing position when playing with one pile is at $k+2$ instead of at $k+1$. This causes an irregular pattern of losing positions to form when playing with a single pile. For instance, the third losing position when playing with one pile is at 17 chips for $k=7$. The first losing position is at 0 chips and the second losing position is at 9 chips, so after 9 chips the first player loses again. However the increase of chips from the second to third losing positions is 8 instead of 9. This can make it more difficult to predict a losing position. Also, different values of k can have different patterns for their losing positions, for instance when $k=9$ the increase in chips from the first to second losing positions is $k+2$, as well as the increase from the second to third losing positions, however the increase from the third to fourth losing positions is only $k+1$. This pattern differs from the one found with $k=7$.
We have just shown that the position $(k+2,k+2)$ is a P position. For $k=7$ this is the position $(9,9)$. We can see in Table 6 that the position $(17,0)$ is a P position. One might think that this would mean that the position $(17,9)$ would also be a P position. This is however not the case. A player in this situation can win by taking 1 chip. Because 9 is equal to $k+2$ instead of $k+1$ the first player can take 1 chip from this pile while the second player will not be able to take the remainder, as the remainder is $k+1$. Now the first player has to make sure that the other pile is smaller than 17 after his next turn. By doing this he can use the same strategy of always taking 1 chip as before to win. By lowering the pile with 17 chips he makes sure that the second player can never play to the situation $(17,0)$, and if the first player plays the strategy of taking 1 chip correctly he can always avoid a situation in which the second player can win.
The same strategy can be used for the other positions that might be expected to be P positions, for example the same strategy can be used in position $(25,9)$ for $k=7$. In fact, because there now are a lot less P positions, every position where both piles have more than 17 chips can be won by playing 1 chip, and can only be won by playing 1 chip. The

reason why it can only be won by playing 1 chip is because if a player does not take 1 chip, the other player will do so and will be able to win. The player who always takes 1 chip can win by never putting the other player on an N position or a position that can be won without playing 1 chip. This works essentially in the same way as with the even values of k, but because of the lack of P positions when both piles have more than 17 chips, it is possible to win by playing 1 chip on every position here.
This can be generalized to different values of k by looking at the losing positions when playing with a single pile. The third losing position, where the position 0 is the first, is the minimum value both piles need to have for a position to always be able to be won by playing 1 chip.

4.4 Adding Games with Restrictions

In this section we will look at the outcome of a game of Nim that consists of two components, where both components are also games of Nim. This means that during their turn a player has to choose in which component he makes a move, however they can only make a move in one component at a time [1, p.10–11]. If we take the same restrictions as before where a player is not allowed to take the same number of chips as the other player made during their previous turn and we add two of these games together, we now get a new situation where there are two restricted values. For example, if we have a game played with one pile of chips, then after a player removes a certain number of chips from the pile the other player is not allowed to take that same number of chips from that pile. So by taking a number of chips from the pile a player creates a situation where that number is restricted. If we take two of these games, both played with one pile, they both have a value that is restricted while playing these games. Now if we add the games together we get a new game played with two piles and two values that are restricted. The two piles are the piles from the separate games, which both had their own restricted value. This differs from the version with two piles we played before, where there was only one restricted value for both piles.
When adding two games of regular Nim together we can explain the behaviour of the new game using the earlier mentioned Sprague-Grundy (SG) values. Both games individually have a certain SG value. When adding the games together the SG value of the new game will be the Nim-sum of the SG values of the components.
We now want to see what happens when we add two games of restricted Nim, where players are not allowed to take a certain number of chips that is restricted. This means that if we add two games played with one pile, each pile will have their own restricted value in the new game. So the restricted value is not necessarily based on the number of chips the previous player took. The restricted value will be the last move made on that pile, without taking into consideration which of the player made this move. Table 7 shows the Sprague-Grundy values of the games that are created by adding two games, both played with one pile, together. The columns indicate the number of chips on the pile of one game with the restricted value for that pile in brackets. The rows indicate the number of chips on the pile for the other game, with the restricted value in brackets. The first column shows the SG values of these single games with one pile. So these are the SG values of the games indicated by the rows. The maximum number of chips a player can take during their turn is 3. The table shows the results for one or two piles with up to 6 chips.

Using the values of the first column of the table it is possible to get the Nim-sum for adding two of these games together. When doing so the values of the Nim-sum will all

Table 7: Sprague-Grundy values when combining two games of one pile with their own restricted values. If there are no restrictions, a 0 is used in brackets. The maximum move is 3.

	-	0(0)	0(1)	0(2)	0(3)	1(0)	1(1)	1(2)	1(3)	2(0)	2(1)	2(2)	2(3)	3(0)	3(1)	3(2)	3(3)	4(0)	4(1)	4(2)	4(3)	5(0)	5(1)	5(2)	5(3)	6(0)	6(1)	6(2)	6(3)
0(0)	0	0	0	0	0	1	0	1	1	1	1	1	1	2	2	2	0	0	0	0	0	3	0	2	1	1	1	1	1
0(1)	0	0	0	0	0	1	0	1	1	1	1	1	1	2	2	2	0	0	0	0	0	3	0	2	1	1	1	1	1
0(2)	0	0	0	0	0	1	0	1	1	1	1	1	1	2	2	2	0	0	0	0	0	3	0	2	1	1	1	1	1
0(3)	0	0	0	0	0	1	0	1	1	1	1	1	1	2	2	2	0	0	0	0	0	3	0	2	1	1	1	1	1
1(0)	1	1	1	1	1	0	1	0	0	0	0	0	0	3	3	3	1	1	1	1	1	2	1	3	0	0	0	0	0
1(1)	0	0	0	0	0	1	0	1	1	1	1	1	1	2	2	2	0	0	0	0	0	3	0	2	1	1	1	1	1
1(2)	1	1	1	1	1	0	1	0	0	0	0	0	0	3	3	3	1	1	1	1	1	2	1	3	0	0	0	0	0
1(3)	1	1	1	1	1	0	1	0	0	0	0	0	0	3	3	3	1	1	1	1	1	2	1	3	0	0	0	0	0
2(0)	1	1	1	1	1	0	1	0	0	0	0	0	0	3	3	3	1	1	1	1	1	2	1	3	0	0	0	0	0
2(1)	1	1	1	1	1	0	1	0	0	0	0	0	0	3	3	3	1	1	1	1	1	2	1	3	0	0	0	0	0
2(2)	1	1	1	1	1	0	1	0	0	0	0	0	0	3	3	3	1	1	1	1	1	2	1	3	0	0	0	0	0
2(3)	1	1	1	1	1	0	1	0	0	0	0	0	0	3	3	3	1	1	1	1	1	2	1	3	0	0	0	0	0
3(0)	2	2	2	2	2	3	2	3	3	3	3	3	3	0	0	0	2	2	2	2	2	1	2	0	3	3	3	3	3
3(1)	2	2	2	2	2	3	2	3	3	3	3	3	3	0	0	0	2	2	2	2	2	1	2	0	3	3	3	3	3
3(2)	2	2	2	2	2	3	2	3	3	3	3	3	3	0	0	0	2	2	2	2	2	1	2	0	3	3	3	3	3
3(3)	0	0	0	0	0	1	0	1	1	1	1	1	1	2	2	2	0	0	0	0	0	3	0	2	1	1	1	1	1
4(0)	0	0	0	0	0	1	0	1	1	1	1	1	1	2	2	2	0	0	0	0	0	3	0	2	1	1	1	1	1
4(1)	0	0	0	0	0	1	0	1	1	1	1	1	1	2	2	2	0	0	0	0	0	3	0	2	1	1	1	1	1
4(2)	0	0	0	0	0	1	0	1	1	1	1	1	1	2	2	2	0	0	0	0	0	3	0	2	1	1	1	1	1
4(3)	0	0	0	0	0	1	0	1	1	1	1	1	1	2	2	2	0	0	0	0	0	3	0	2	1	1	1	1	1
5(0)	3	3	3	3	3	2	3	2	2	2	2	2	2	1	1	1	3	3	3	3	3	0	3	1	2	2	2	2	2
5(1)	0	0	0	0	0	1	0	1	1	1	1	1	1	2	2	2	0	0	0	0	0	3	0	2	1	1	1	1	1
5(2)	2	2	2	2	2	3	2	3	3	3	3	3	3	0	0	0	2	2	2	2	2	1	2	0	3	3	3	3	3
5(3)	1	1	1	1	1	0	1	0	0	0	0	0	0	3	3	3	1	1	1	1	1	2	1	3	0	0	0	0	0
6(0)	1	1	1	1	1	0	1	0	0	0	0	0	0	3	3	3	1	1	1	1	1	2	1	3	0	0	0	0	0
6(1)	1	1	1	1	1	0	1	0	0	0	0	0	0	3	3	3	1	1	1	1	1	2	1	3	0	0	0	0	0
6(2)	1	1	1	1	1	0	1	0	0	0	0	0	0	3	3	3	1	1	1	1	1	2	1	3	0	0	0	0	0
6(3)	1	1	1	1	1	0	1	0	0	0	0	0	0	3	3	3	1	1	1	1	1	2	1	3	0	0	0	0	0

correspond with the SG values for the additions of two games in the table. This means that the Nim-sum can also be used to get the value when adding two games together just as in regular Nim. The extra added restriction does not hinder this principle.
When looking at a single game of restricted Nim played with two piles and one restricted value, the SG values will be completely different from the values when adding two games with one pile together. Playing with two piles and one restricted value can be seen as a different game of Nim. We can however still use the Nim-sum to add a different game which has its own restricted value. If we have a game with two piles and one restricted value and we add a game with one pile and one restricted value, we now get a game with three piles, where two piles share one restricted value and the other pile has its own restricted value. The SG value of this new game will be the Nim-sum of the SG values of the game with two piles and the game with one pile. Two games with two piles can also be added this way, to create a game with four piles and two restricted values, each shared by two piles. The resulting SG values will again be the Nim-sum of the SG values of its components, which were the two games both with two piles each.

4.5 Playing with N Players

The same version of Nim we have played before can also be played with more than two players [8], [9]. There are multiple ways to define the restriction that a player is not allowed to take the same number of chips as the previous player. We will only look at the version where a player is not allowed to take the same number of chips that the player before him took. This means that it does not matter how many chips all the other players took, only the player that played right before the current player matters. There will also

be a maximum number of chips the players are allowed to take during their turn, called k. For now we only look at games that are played using one pile of chips.
It is more difficult to define who is the winner of a game of Nim with more than two players than when playing with two players. There are multiple ways to interpret this. One way is to give all the players a ranking once it is impossible for a player to make a move. The players will aim to get the best possible ranking. The first player who is unable to make a move will get the worst ranking and is the biggest loser. The player who actually made the final move will get the best ranking and is the biggest winner. The other rankings are based on how close a player was to making the final move. So the player that made the second to last move will get the second best ranking, and the player who made the third to last move will get the third best ranking, etcetera.
Table 8 shows the results of a game of Nim played with three players, where a player is not allowed to take the same number of chips that was taken in the previous turn. This restriction does not apply to the first move by the starting player, as there has not been a previous turn. The value of k is 3, so players are only allowed to take up to 3 chips. The game is played on one pile which can have up to 12 chips. A 0 in the the table indicates that the next player who has to make a move will lose. Any number higher than 0 indicates the player who will be able to make the final move given optimal play. So a 1 in the table means that the next player will be able to make the final move, a 2 in the table means that the second player to make a move will make the final move, a 3 means the third player will make the final move, etcetera. Due to this numbering it is optimal for a player to move to a position with a value as low as possible, as this will give them the best ranking in the end.

Table 8: Values indicate which player will win when playing restricted Nim with three players, and a maximum move of $k = 3$, for piles of up to 12 chips.

Chips\Restricted	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	1	1	1	1
3	1	1	1	2
4	2	2	2	2
5	2	2	2	2
6	0	0	0	0
7	1	0	1	1
8	1	1	1	1
9	1	1	1	2
10	2	2	2	2
11	2	2	2	2
12	0	0	0	0

Using the same reasoning we used earlier we can say with certainty that these values will keep repeating. We can see in the table that there are certain positions which will always cause the next player to be the biggest loser, no matter which value is restricted. In this case for three players and $k = 3$ these losing positions repeat after adding 6 chips to the total pile starting with the empty pile.
In order to show why this is the case we first look at the pile containing 6 chips. The first player can take either 1, 2, or 3 chips, which leaves 5, 4, or 3 chips respectively. The second player is unable to empty the pile in any of these cases, because he is only allowed to take a maximum of 3 chips and when there are only 3 chips left, taking 3 chips will be

restricted. It is now in the second player's best interest to allow the third player to take all of the remaining chips, in fact in the case of $k = 3$ the second player does not have another option. The third player will gladly take all of the remaining chips making him the biggest winner and making the first player the biggest loser. For any pile with a size that is a multiple of 6 this same strategy can be repeated by the third player by always reducing the pile to a value $\equiv 0 \pmod 6$.
If we consider the case where $k = 4$ and there are three players, the losing positions are the piles with a size $\equiv 0 \pmod 8$. If the pile has a size of 8 chips the first player can move to the positions 7(1), 6(2), 5(3), or 4(4), where the number in brackets is the restricted number for the next player. The second player again is unable to take the remaining number of chips, as $k = 4$. The second player however is able to take a number of chips for which the third player is unable to take the remaining number of chips and end the game. For instance, in the position 7(1) the second player can move to 5(2), where the third player is unable to finish the game. However, by doing this the best move for the third player is to allow the first player to be the biggest winner, causing the second player to be the biggest loser. The second player does not want this, so he would rather let the third player be the biggest winner.
When playing with three players this strategy will apply to any value of k. This results in the losing positions being the piles with a value $\equiv 0 \pmod{2k}$. The reason why it repeats after $2k$ chips is because the maximum number of chips the second and third player can take is $k + (k - 1)$. If the second player takes k chips, the third player is not allowed to also take k chips, so taking $k - 1$ chips is the second highest number of chips he can take. If the second player is not allowed to take k chips, he will take $k - 1$, which will allow the third player to take k chips. So the second and third player together can always take $k + (k - 1)$ chips. Because the first player always has to take at least 1 chip a pile of size $2k$ can always be emptied by the third player. This also means that a pile of size $\equiv 0 \pmod{2k}$ can always be brought down to a pile of size $\equiv 0 \pmod{2k}$ by the third player.

If we look at the losing positions of this game played with more than three players we get the results from Table 9. The values in this table indicate after how many chips a position occurs that will make the next player the biggest loser. The table shows these values for 3 players and up to 9 players. The values of k start at 3 and go up to 6.

Table 9: The number of chips after which the losing positions repeat starting at 0, for 3 to 9 players with k values of 3 to 6.

Players\Maximum move	3	4	5	6
3	6	8	10	12
4	9	12	15	18
5	11	15	19	23
6	14	19	24	29
7	16	22	28	34
8	19	26	33	40
9	21	29	37	45

When playing with $n \geq 3$ players and a maximum move of $k \geq 3$ we need the following value for defining the losing positions:

$$A(n, k) = ((n - 1) \times k) - (\lceil n/2 \rceil - 2))$$

So when the pile of chips has a value $\equiv 0 \pmod{A(n, k)}$ the first player will always be the biggest loser. In such a losing position the first player makes a move after which all the other players together can either empty the pile or bring the pile to another losing position.

So the player playing before the first player will make the last move. This explains the first part of the formula $(n-1) \times k$, the $n-1$ players after the first player will take as many chips as they can to empty the pile before the first player can move again. This means that the $n-1$ players will always take k chips if possible or else take $k-1$ chips. This means that they alternately take either k or $k-1$ chips and because the first part of the formula $(n-1) \times k$ assumes that players always take k chips we need to decrease the value of the losing positions. This is why in the second part of the formula $\lceil n/2 \rceil - 2$ is used. For every two players added, one of these two can only take $k-1$ chips, so the number of chips at which the losing positions repeat is decreased by 1.

4.6 Three Players without Ranking

Previously we based the results of playing Nim with multiple players on a ranking, where the last player to make a move gets the best ranking and the first player unable to make a move gets the worst ranking. We can also disregard this ranking, in this case the player to make the last move is the winner and all the other players lose "equally". This can get complicated because players will be able to choose which player will win, even though they themselves are unable to win in any case. Table 10 shows the results for three players, where the players are not allowed to take the same number of chips the previous player took and the players can only take a maximum of 3 chips. The results are for a single pile of up to 10 chips. Nothing is restricted during the very first turn, in the column indicated with 0 there are no restricted values.

Table 10: The player who will win when playing on one pile with three players, without a ranking system.

Chips\Restricted	0	1	2	3
0	3	3	3	3
1	1	3	1	1
2	1	1	1	1
3	1	1	1	2
4	2	2	2	2
5	2/3	2	2/3	2/3
6	3	3	3	3
7	1	0.5 * 1/3	1	1
8	1	1	(0.5 * 1/3) / (0.5 * 1/2)	1
9	1	1	1	2
10	0.5 * ((0.5 * 1/2) / (2/3))	0.5 * ((0.5 * 1/2) / (2/3))	2	0.5 * ((0.5 * 1/2) / (2/3))

The table shows which player will win in a certain situation. A 1 in the table means that the first player to make a move will win. In the same way a 2 or 3 in the table means the second player or third player will win, respectively. When the pile has 5 chips we see the value 2/3. This value indicates that the first player gets to choose whether the second or third player will win. In this situation the first player is unable to win, but he is able to choose who of the other players will win. At 7 chips we see the situation $0.5 * 1/3$. This indicates that the second player can choose whether the first or third player wins. This means that the first player is dependent on the second player whether he will win or lose. If the second player were to randomly make a move, there would be a 50 percent chance for the first player to win, as well as for the third player.
It gets a little more complicated for a pile of 8 chips, where we see the situation $(0.5 * 1/3)/(0.5 * 1/2)$. Now the first player has a choice, he can choose whether the second or third player chooses who wins. In both situations the first player has a 50 percent chance of winning, so the choice is really about which other player will not be able to win.
With 10 chips it gets even more complicated. In the situation with value $0.5 * ((0.5 *$

1/2)/(2/3)), the second player gets to choose whether the first or third player chooses who wins. This means that there is a 25 percent chance for the first player to win. For the first player to win the second player needs to choose to let the third player decide who wins, and then the third player needs to make a move that allows the first player to win. This gives the first and third player a 25 percent chance to win, and the second player has a 50 percent chance to win, if the decisions would be made randomly.
For piles with more chips this can get even more complicated, where players have to make choices about which players have to make choices, etcetera.

5 Wythoff's Game

Wythoff's game is a variation of Nim that is played with two piles of chips [10]. It differs from regular Nim in that the players are allowed to take chips from both piles at the same time during their turn. The only restriction when doing so is that the number of chips that is taken from each pile needs to be the same.
We first look at the results of Wythoff's Game, as it is normally played. The results for piles with up to 30 chips can be seen in Table 11, see [1, p. 197–205].

Table 11: The losing P positions when playing Wythoff's game.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	P	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
1	-	-	P	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
2	-	P	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
3	-	-	-	-	-	P	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
4	-	-	-	-	-	-	-	P	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
5	-	-	-	P	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
6	-	-	-	-	-	-	-	-	-	-	P	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
7	-	-	-	-	P	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
8	-	-	-	-	-	-	-	-	-	-	-	-	-	P	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
9	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	P	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
10	-	-	-	-	-	-	P	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
11	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	P	-	-	-	-	-	-	-	-	-	-	-	-
12	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	P	-	-	-	-	-	-	-	-	-	-
13	-	-	-	-	-	-	-	-	P	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
14	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	P	-	-	-	-	-	-	-
15	-	-	-	-	-	-	-	-	-	P	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
16	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	P	-	-	-	-
17	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	P	-	-
18	-	-	-	-	-	-	-	-	-	-	-	P	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
19	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
20	-	-	-	-	-	-	-	-	-	-	-	-	P	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
21	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
22	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
23	-	-	-	-	-	-	-	-	-	-	-	-	-	-	P	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
24	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
25	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
26	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	P	-	-	-	-	-	-	-	-	-	-	-	-	-	-
27	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
28	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	P	-	-	-	-	-	-	-	-	-	-	-	-	-
29	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
30	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-

A P in the table indicates that the first player will lose in that situation, so a P position. All other positions are indicated with a $-$ for visibility of the table, and these are all N positions where the first player wins. Since there are no restricted values the first player can win if there is a P position either straight up, straight to the left, or diagonally to the top left of the table, from the position indicated by the height of the two piles of chips. This starts from the position $(0, 0)$, which is always a losing position for the first player, as no moves can be made. The P positions do not come across any other P positions when moving in one of these three directions in the table. So taking chips from either pile, or from both piles at the same time, will not lead to a P position for the other player. Of course, the table is symmetric in the main diagonal.

Now we will look at Wythoff's game with the added restriction that a player is not allowed to take the same number of chips that was taken by the other player in the previous turn. This means that a player cannot take this number of chips from one pile, or from both piles simultaneously. Table 12 shows the results for two piles of up to 30 chips.

Table 12: The results when playing Wythoff's game with an added restriction.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	P	+	N	3	4	5	N	7	N	9	N	11	12	13	N	15	16	17	N	19	20	21	N	23	N	25	N	27	28	29	N
1	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
2	N	+	N	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
3	3	+	+	N	+	+	N	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
4	4	+	+	+	N	+	+	+	N	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
5	5	+	+	+	+	N	+	+	+	+	N	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
6	N	+	+	N	+	+	N	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
7	7	+	+	+	+	+	+	N	+	+	+	+	+	+	N	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
8	N	+	+	+	N	+	+	+	N	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
9	9	+	+	+	+	+	+	+	+	N	+	+	+	+	+	+	+	+	N	+	+	+	+	+	+	+	+	+	+	+	+
10	N	+	+	+	+	N	+	+	+	+	N	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
11	11	+	+	+	+	+	+	+	+	+	+	N	+	+	+	+	+	+	+	+	+	+	N	+	+	+	+	+	+	+	+
12	12	+	+	+	+	+	+	+	+	+	+	+	N	+	+	+	+	+	+	+	+	+	+	+	N	+	+	+	+	+	+
13	13	+	+	+	+	+	+	+	+	+	+	+	+	N	+	+	+	+	+	+	+	+	+	+	+	+	N	+	+	+	+
14	N	+	+	+	+	+	+	N	+	+	+	+	+	+	N	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
15	15	+	+	+	+	+	+	+	+	+	+	+	+	+	+	N	+	+	+	+	+	+	+	+	+	+	+	+	+	+	N
16	16	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	N	+	+	+	+	+	+	+	+	+	+	+	+	+	+
17	17	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	N	+	+	+	+	+	+	+	+	+	+	+	+	+
18	N	+	+	+	+	+	+	+	+	N	+	+	+	+	+	+	+	+	N	+	+	+	+	+	+	+	+	+	+	+	+
19	19	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	N	+	+	+	+	+	+	+	+	+	+	+
20	20	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	N	+	+	+	+	+	+	+	+	+	+
21	21	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	N	+	+	+	+	+	+	+	+	+
22	N	+	+	+	+	+	+	+	+	+	+	N	+	+	+	+	+	+	+	+	+	+	N	+	+	+	+	+	+	+	+
23	23	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	N	+	+	+	+	+	+	+
24	N	+	+	+	+	+	+	+	+	+	+	+	N	+	+	+	+	+	+	+	+	+	+	+	N	+	+	+	+	+	+
25	25	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	N	+	+	+	+	+
26	N	+	+	+	+	+	+	+	+	+	+	+	+	N	+	+	+	+	+	+	+	+	+	+	+	+	N	+	+	+	+
27	27	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	N	+	+	+
28	28	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	N	+	+
29	29	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	N	+
30	N	+	+	+	+	+	+	+	+	+	+	+	+	+	+	N	+	+	+	+	+	+	+	+	+	+	+	+	+	+	N

A P in this table indicates a P position and an N represents an N position. A number in the table indicates that in that position the first player can win if playing that particular number is not restricted, otherwise the first player will lose. If this number is 1, the table shows a + symbol, which is done to make the table easier to read.
Even though the results look similar to the non-restricted version of Wythoff, they are very different. In the restricted version, there is only one P position, which is $(0, 0)$. Every single position where both piles have more than 0 chips can be won by the first player by removing 1 chip. The first player has to make sure to never remove all chips from a pile, and in this way the moment the second player empties one of the piles, the first player can empty the other. In the situation $(1, 1)$, where the first player would be forced to empty a pile, he can just remove 1 chip from either pile, or from both piles, to win. Whenever both piles have the same number of chips and have at least two chips, the first player can also win by removing all chips from both piles, if this is not restricted. Otherwise he can still win by removing 1 chip. There are a few more N positions in the table. Here one of the piles has twice as many chips as the other pile and the first player can win by emptying the smaller pile and also removing this number of chips from the larger pile. The other player is now in a situation with only one pile, and removing that pile completely is restricted.

This does not always work for the first player however, as the second player can sometimes still win by dividing the remaining pile in half. This means that these N positions are the positions where one pile is twice as big as the other pile and when dividing the number of chips of the largest pile by 2^m, the result is an odd natural number for an odd m.
When playing Wythoff with or without the extra restriction, almost all positions are winning positions for the first player, however the strategy is very different. Adding the restriction makes taking 1 chip a very strong move, as it also disallows the other player from taking 1 chip during their turn.

5.1 Three Pile Wythoff

We can also play Wythoff with three piles instead of two. We will differentiate two ways to play Wythoff's Nim with three piles. In Wythoff's Nim the only rule that differentiates it from regular Nim is that a player is allowed to take chips from both piles, as long as they take the same number of chips from each pile. So for three piles we can say that a player can take chips from three piles as long as they take the same number of chips from all three piles. This essentially gives the player four options, take chips from each individual pile or take the same number of chips from all three piles.
It is also possible to add the possibility to take chips from two of the three piles, while still requiring that the number of chips is the same. This is the second way to play with three piles. Now the players have seven options in total. They can take chips from the individual piles, they can take the same number of chips from all three piles or they can take the same number of chips from two of the three piles. So they can take from the first and second pile, the second and third pile, or the first and third pile.
We first consider Wythoff's Nim played with three piles without any extra restrictions. Table 13 shows the result of the first version of this game, where players are allowed to take from all three piles at once or from any individual pile. The table shows the results for piles of up to 9 chips. The number in the upper left corner of each sub-table shows the number of chips of the first pile. The number of chips of the other two piles are given by the rows and columns of the sub-tables. A P in the table indicates that the first player will lose and a $-$ symbol indicates that the first player will win.

Table 13: The results when playing Wythoff's game with three piles and no extra restrictions.

0	0	1	2	3	4	5	6	7	8	9
0	P	-	-	-	-	-	-	-	-	-
1	-	P	-	-	-	-	-	-	-	-
2	-	-	P	-	-	-	-	-	-	-
3	-	-	-	P	-	-	-	-	-	-
4	-	-	-	-	P	-	-	-	-	-
5	-	-	-	-	-	P	-	-	-	-
6	-	-	-	-	-	-	P	-	-	-
7	-	-	-	-	-	-	-	P	-	-
8	-	-	-	-	-	-	-	-	P	-
9	-	-	-	-	-	-	-	-	-	P

1	0	1	2	3	4	5	6	7	8	9
0	-	P	-	-	-	-	-	-	-	-
1	P	-	-	-	-	-	-	-	-	-
2	-	-	-	P	-	-	-	-	-	-
3	-	-	P	-	-	-	-	-	-	-
4	-	-	-	-	-	P	-	-	-	-
5	-	-	-	-	P	-	-	-	-	-
6	-	-	-	-	-	-	-	P	-	-
7	-	-	-	-	-	-	P	-	-	-
8	-	-	-	-	-	-	-	-	-	P
9	-	-	-	-	-	-	-	-	P	-

2	0	1	2	3	4	5	6	7	8	9
0	-	-	P	-	-	-	-	-	-	-
1	-	-	-	P	-	-	-	-	-	-
2	P	-	-	-	-	-	-	-	-	-
3	-	P	-	-	-	-	-	-	-	-
4	-	-	-	-	-	-	P	-	-	-
5	-	-	-	-	-	-	-	P	-	-
6	-	-	-	-	P	-	-	-	-	-
7	-	-	-	-	-	P	-	-	-	-
8	-	-	-	-	-	-	-	-	-	-
9	-	-	-	-	-	-	-	-	-	-

3	0	1	2	3	4	5	6	7	8	9
0	-	-	-	P	-	-	-	-	-	-
1	-	-	P	-	-	-	-	-	-	-
2	-	P	-	-	-	-	-	-	-	-
3	P	-	-	-	-	-	-	-	-	-
4	-	-	-	-	-	-	-	P	-	-
5	-	-	-	-	-	-	P	-	-	-
6	-	-	-	-	-	P	-	-	-	-
7	-	-	-	-	P	-	-	-	-	-
8	-	-	-	-	-	-	-	-	-	-
9	-	-	-	-	-	-	-	-	-	-

4	0	1	2	3	4	5	6	7	8	9
0	-	-	-	-	P	-	-	-	-	-
1	-	-	-	-	-	P	-	-	-	-
2	-	-	-	-	-	-	P	-	-	-
3	-	-	-	-	-	-	-	P	-	-
4	P	-	-	-	-	-	-	-	-	-
5	-	P	-	-	-	-	-	-	-	-
6	-	-	P	-	-	-	-	-	-	-
7	-	-	-	P	-	-	-	-	-	-
8	-	-	-	-	-	-	-	-	-	-
9	-	-	-	-	-	-	-	-	-	-

5	0	1	2	3	4	5	6	7	8	9
0	-	-	-	-	-	P	-	-	-	-
1	-	-	-	-	P	-	-	-	-	-
2	-	-	-	-	-	-	-	P	-	-
3	-	-	-	-	-	-	P	-	-	-
4	-	P	-	-	-	-	-	-	-	-
5	P	-	-	-	-	-	-	-	-	-
6	-	-	-	P	-	-	-	-	-	-
7	-	-	P	-	-	-	-	-	-	-
8	-	-	-	-	-	-	-	-	-	-
9	-	-	-	-	-	-	-	-	-	-

6	0	1	2	3	4	5	6	7	8	9
0	-	-	-	-	-	-	P	-	-	-
1	-	-	-	-	-	-	-	P	-	-
2	-	-	-	-	P	-	-	-	-	-
3	-	-	-	-	-	P	-	-	-	-
4	-	-	P	-	-	-	-	-	-	-
5	-	-	-	P	-	-	-	-	-	-
6	P	-	-	-	-	-	-	-	-	-
7	-	P	-	-	-	-	-	-	-	-
8	-	-	-	-	-	-	-	-	-	-
9	-	-	-	-	-	-	-	-	-	-

7	0	1	2	3	4	5	6	7	8	9
0	-	-	-	-	-	-	-	P	-	-
1	-	-	-	-	-	-	P	-	-	-
2	-	-	-	-	-	P	-	-	-	-
3	-	-	-	-	P	-	-	-	-	-
4	-	-	-	P	-	-	-	-	-	-
5	-	-	P	-	-	-	-	-	-	-
6	-	P	-	-	-	-	-	-	-	-
7	P	-	-	-	-	-	-	-	-	-
8	-	-	-	-	-	-	-	-	-	-
9	-	-	-	-	-	-	-	-	-	-

8	0	1	2	3	4	5	6	7	8	9
0	-	-	-	-	-	-	-	-	P	-
1	-	-	-	-	-	-	-	-	-	P
2	-	-	-	-	-	-	-	-	-	-
3	-	-	-	-	-	-	-	-	-	-
4	-	-	-	-	-	-	-	-	-	-
5	-	-	-	-	-	-	-	-	-	-
6	-	-	-	-	-	-	-	-	-	-
7	-	-	-	-	-	-	-	-	-	-
8	P	-	-	-	-	-	-	-	-	-
9	-	P	-	-	-	-	-	-	-	-

9	0	1	2	3	4	5	6	7	8	9
0	-	-	-	-	-	-	-	-	-	P
1	-	-	-	-	-	-	-	-	P	-
2	-	-	-	-	-	-	-	-	-	-
3	-	-	-	-	-	-	-	-	-	-
4	-	-	-	-	-	-	-	-	-	-
5	-	-	-	-	-	-	-	-	-	-
6	-	-	-	-	-	-	-	-	-	-
7	-	-	-	-	-	-	-	-	-	-
8	-	P	-	-	-	-	-	-	-	-
9	P	-	-	-	-	-	-	-	-	-

The results of playing Wythoff's Nim with three piles in this version are actually exactly the same as the results when playing regular Nim with three piles. This means that adding the option to remove the same number of chips from all three piles does not change the outcome of the game relative to the standard version of the game Nim.

Proposition 5.1. The outcome of a game of Wythoff's Nim played with three piles where players can take any number of chips from any individual pile or take the same number of chips from all three piles is the same as the outcome of regular Nim played with three piles.

Proof. For a position in regular Nim to be a losing position, the Nim-Sum of the three piles needs to be exactly 0. This means that for every individual bit of the numbers of chips on each pile, this bit is either 0 for all three piles or it is 1 for two of the three piles and 0 for the other. It has been proven that for every position with a Nim-Sum that is not 0, a certain move can always be made removing chips from one of the piles such that the Nim-Sum of the resulting position will be 0. By removing chips from one pile a position with a Nim-Sum of 0 will always result in a position with a Nim-Sum that is not 0. Since you can still remove any number of chips from one pile in our version of three pile Wythoff, it will still be possible to always move to a position with Nim-Sum 0 from a position without Nim-Sum 0. We only have to show that even with the extra option of removing chips from all three piles it will still not be possible to move from a position with Nim-Sum 0 to another position with Nim-Sum 0. If we remove the same number of chips from all three piles, the same bit positions will be decremented by 1. We only have to look at the rightmost, or least significant, bit position of the number that gets removed from all three piles. Here the bits will change from 1 to 0 and from 0 to 1, without having to worry about the bit being used as a carry. This means when the bits of all three piles in this position were 0, they will now all be 1, and if two of the three were 1 and the other 0, now two of the three will be 0 and the other 1. The resulting Nim-Sum can now never be 0, as there will definitely be a 1 in this bit position. You can still never reach a position with a Nim-Sum of 0 from a position with a Nim-Sum of 0. □

We will now look at the other version, where players are also allowed to take the same number of chips from two of the three piles during their turn. The results can be seen in Table 14. Again the results are for three piles of up to 9 chips.

Table 14: The results when playing the second version of Wythoff's game with three piles and no extra restrictions.

0	0	1	2	3	4	5	6	7	8	9
0	P	-	-	-	-	-	-	-	-	-
1	-	-	P	-	-	-	-	-	-	-
2	-	P	-	-	-	-	-	-	-	-
3	-	-	-	-	-	P	-	-	-	-
4	-	-	-	-	-	-	-	P	-	-
5	-	-	-	P	-	-	-	-	-	-
6	-	-	-	-	-	-	-	-	-	-
7	-	-	-	-	P	-	-	-	-	-
8	-	-	-	-	-	-	-	-	-	-
9	-	-	-	-	-	-	-	-	-	-

1	0	1	2	3	4	5	6	7	8	9
0	-	-	P	-	-	-	-	-	-	-
1	-	-	-	-	P	-	-	-	-	-
2	P	-	-	-	-	-	-	-	-	-
3	-	-	-	P	-	-	-	-	-	-
4	-	P	-	-	-	-	-	-	-	-
5	-	-	-	-	-	-	P	-	-	-
6	-	-	-	-	-	P	-	-	-	-
7	-	-	-	-	-	-	-	-	-	-
8	-	-	-	-	-	-	-	-	-	-
9	-	-	-	-	-	-	-	-	-	-

2	0	1	2	3	4	5	6	7	8	9
0	-	P	-	-	-	-	-	-	-	-
1	P	-	-	-	-	-	-	-	-	-
2	-	-	-	-	-	-	P	-	-	-
3	-	-	-	-	-	-	-	-	P	-
4	-	-	-	-	-	-	-	-	-	-
5	-	-	-	-	-	-	-	-	-	-
6	-	-	P	-	-	-	-	-	-	-
7	-	-	-	-	-	-	-	P	-	-
8	-	-	-	P	-	-	-	-	-	-
9	-	-	-	-	-	-	-	-	-	-

3	0	1	2	3	4	5	6	7	8	9
0	-	-	-	-	-	P	-	-	-	-
1	-	-	-	P	-	-	-	-	-	-
2	-	-	-	-	-	-	-	-	P	-
3	-	P	-	-	-	-	-	-	-	-
4	-	-	-	-	P	-	-	-	-	-
5	P	-	-	-	-	-	-	-	-	-
6	-	-	-	-	-	-	-	-	-	P
7	-	-	-	-	-	-	-	-	-	-
8	-	-	P	-	-	-	-	-	-	-
9	-	-	-	-	-	-	P	-	-	-

4	0	1	2	3	4	5	6	7	8	9
0	-	-	-	-	-	-	-	P	-	-
1	-	P	-	-	-	-	-	-	-	-
2	-	-	-	-	-	-	-	-	-	-
3	-	-	-	-	P	-	-	-	-	-
4	-	-	-	P	-	-	-	-	-	-
5	-	-	-	-	-	-	-	-	-	-
6	-	-	-	-	-	-	-	-	-	-
7	P	-	-	-	-	-	-	-	-	-
8	-	-	-	-	-	-	-	-	-	-
9	-	-	-	-	-	-	-	-	-	-

5	0	1	2	3	4	5	6	7	8	9
0	-	-	-	P	-	-	-	-	-	-
1	-	-	-	-	-	-	P	-	-	-
2	-	-	-	-	-	-	-	-	-	-
3	P	-	-	-	-	-	-	-	-	-
4	-	-	-	-	-	-	-	-	-	-
5	-	-	-	-	-	-	-	P	-	-
6	-	P	-	-	-	-	-	-	-	-
7	-	-	-	-	-	P	-	-	-	-
8	-	-	-	-	-	-	-	-	P	-
9	-	-	-	-	-	-	-	-	-	-

6	0	1	2	3	4	5	6	7	8	9
0	-	-	-	-	-	-	-	-	-	-
1	-	-	-	-	-	P	-	-	-	-
2	-	-	P	-	-	-	-	-	-	-
3	-	-	-	-	-	-	-	-	-	P
4	-	-	-	-	-	-	-	-	-	-
5	-	P	-	-	-	-	-	-	-	-
6	-	-	-	-	-	-	-	-	-	-
7	-	-	-	-	-	-	-	-	-	-
8	-	-	-	-	-	-	-	-	-	-
9	-	-	-	P	-	-	-	-	-	-

7	0	1	2	3	4	5	6	7	8	9
0	-	-	-	-	P	-	-	-	-	-
1	-	-	-	-	-	-	-	-	-	-
2	-	-	-	-	-	-	-	P	-	-
3	-	-	-	-	-	-	-	-	-	-
4	P	-	-	-	-	-	-	-	-	-
5	-	-	-	-	-	P	-	-	-	-
6	-	-	-	-	-	-	-	-	-	-
7	-	-	P	-	-	-	-	-	-	-
8	-	-	-	-	-	-	-	-	-	-
9	-	-	-	-	-	-	-	-	-	-

8	0	1	2	3	4	5	6	7	8	9
0	-	-	-	-	-	-	-	-	-	-
1	-	-	-	-	-	-	-	-	-	-
2	-	-	-	P	-	-	-	-	-	-
3	-	-	P	-	-	-	-	-	-	-
4	-	-	-	-	-	-	-	-	-	-
5	-	-	-	-	-	-	-	-	P	-
6	-	-	-	-	-	-	-	-	-	-
7	-	-	-	-	-	-	-	-	-	-
8	-	-	-	-	-	P	-	-	-	-
9	-	-	-	-	-	-	-	-	-	-

9	0	1	2	3	4	5	6	7	8	9
0	-	-	-	-	-	-	-	-	-	-
1	-	-	-	-	-	-	-	-	-	-
2	-	-	-	-	-	-	-	-	-	-
3	-	-	-	-	-	-	P	-	-	-
4	-	-	-	-	-	-	-	-	-	-
5	-	-	-	-	-	-	-	-	-	-
6	-	-	-	P	-	-	-	-	-	-
7	-	-	-	-	-	-	-	-	-	-
8	-	-	-	-	-	-	-	-	-	-
9	-	-	-	-	-	-	-	-	-	-

These results are similar to the results when playing Wythoff with two piles. A player has seven different options during their turn. They can take chips from each pile individually, they can take chips from two of the three piles, and they can take chips from all three piles at the same time. The position $(0, 0, 0)$ will be a losing position for the first player since it is not possible to make any move. Now any position that can move directly to $(0, 0, 0)$ will be a winning position for the first player, so the positions $(x, 0, 0)$, $(0, x, 0)$, $(0, 0, x)$, $(x, x, 0)$, $(x, 0, x)$, $(0, x, x)$, and (x, x, x), with $x > 0$, will all be winning positions for the first player. A position which cannot reach any losing position in one move will be a losing position, and any position which can reach that position will then be a winning position. This idea is the same as with the two pile version of Wythoff.

We can also add the extra restriction that a player is not allowed to take the same number of chips that was taken in the previous turn to the three pile version of Wythoff. We will still distinguish two versions, one where players can take take from either one pile or all three, and one where players are also allowed to take chips from two piles.
The results of the version where players can only take chips from one pile or take the same number of chips from all three piles at once, with the extra restriction, can be seen in Table 15. In this table an N indicates that the first player will always win, no matter the restricted value, a P indicates that the first player will always lose and a number indicates that the first player can only win if he is allowed to take that number of chips. For readability, a + symbol is used to indicate that the first player can only win if he is allowed to take 1 chip. Again the number in the top left of the sub-tables indicates the number of chips of the first pile. The results are for three piles of up to 9 chips.

Table 15: The results when playing Wythoff's game with three piles and the extra restriction that players are not allowed to take the same number of chips that was taken during the previous turn.

0	0	1	2	3	4	5	6	7	8	9
0	P	+	N	3	4	5	N	7	N	9
1	+	+	+	+	+	+	+	+	+	+
2	N	+	+	+	+	+	+	+	+	+
3	3	+	+	N	+	+	+	+	+	+
4	4	+	+	+	N	+	+	+	+	+
5	5	+	+	+	+	N	+	+	+	+
6	N	+	+	+	+	+	+	+	+	+
7	7	+	+	+	+	+	+	N	+	+
8	N	+	+	+	+	+	+	+	+	+
9	9	+	+	+	+	+	+	+	+	N

1	0	1	2	3	4	5	6	7	8	9
0	+	+	+	+	+	+	+	+	+	+
1	+	+	+	+	+	+	+	+	+	+
2	+	+	+	+	+	+	+	+	+	+
3	+	+	+	+	+	+	+	+	+	+
4	+	+	+	+	+	+	+	+	+	+
5	+	+	+	+	+	+	+	+	+	+
6	+	+	+	+	+	+	+	+	+	+
7	+	+	+	+	+	+	+	+	+	+
8	+	+	+	+	+	+	+	+	+	+
9	+	+	+	+	+	+	+	+	+	+

2	0	1	2	3	4	5	6	7	8	9
0	N	+	+	+	+	+	+	+	+	+
1	+	+	+	+	+	+	+	+	+	+
2	+	+	N	+	+	+	+	+	+	+
3	+	+	+	+	+	+	+	+	+	+
4	+	+	+	+	+	+	+	+	+	+
5	+	+	+	+	+	+	+	+	+	+
6	+	+	+	+	+	+	+	+	+	+
7	+	+	+	+	+	+	+	+	+	+
8	+	+	+	+	+	+	+	+	+	+
9	+	+	+	+	+	+	+	+	+	+

3	0	1	2	3	4	5	6	7	8	9
0	3	+	+	N	+	+	+	+	+	+
1	+	+	+	+	+	+	+	+	+	+
2	+	+	+	+	+	+	+	+	+	+
3	N	+	+	N	+	+	N	+	+	+
4	+	+	+	+	+	+	+	+	+	+
5	+	+	+	+	+	+	+	+	+	+
6	+	+	+	N	+	+	+	+	+	+
7	+	+	+	+	+	+	+	+	+	+
8	+	+	+	+	+	+	+	+	+	+
9	+	+	+	+	+	+	+	+	+	+

4	0	1	2	3	4	5	6	7	8	9
0	4	+	+	+	N	+	+	+	+	+
1	+	+	+	+	+	+	+	+	+	+
2	+	+	+	+	+	+	+	+	+	+
3	+	+	+	+	+	+	+	+	+	+
4	N	+	+	+	N	+	+	+	N	+
5	+	+	+	+	+	+	+	+	+	+
6	+	+	+	+	+	+	+	+	+	+
7	+	+	+	+	+	+	+	+	+	+
8	+	+	+	+	N	+	+	+	+	+
9	+	+	+	+	+	+	+	+	+	+

5	0	1	2	3	4	5	6	7	8	9
0	5	+	+	+	+	N	+	+	+	+
1	+	+	+	+	+	+	+	+	+	+
2	+	+	+	+	+	+	+	+	+	+
3	+	+	+	+	+	+	+	+	+	+
4	+	+	+	+	+	+	+	+	+	+
5	N	+	+	+	+	N	+	+	+	+
6	+	+	+	+	+	+	+	+	+	+
7	+	+	+	+	+	+	+	+	+	+
8	+	+	+	+	+	+	+	+	+	+
9	+	+	+	+	+	+	+	+	+	+

6	0	1	2	3	4	5	6	7	8	9
0	N	+	+	+	+	+	+	+	+	+
1	+	+	+	+	+	+	+	+	+	+
2	+	+	+	+	+	+	+	+	+	+
3	+	+	+	N	+	+	+	+	+	+
4	+	+	+	+	+	+	+	+	+	+
5	+	+	+	+	+	+	+	+	+	+
6	+	+	+	+	+	+	N	+	+	+
7	+	+	+	+	+	+	+	+	+	+
8	+	+	+	+	+	+	+	+	+	+
9	+	+	+	+	+	+	+	+	+	+

7	0	1	2	3	4	5	6	7	8	9
0	7	+	+	+	+	+	+	N	+	+
1	+	+	+	+	+	+	+	+	+	+
2	+	+	+	+	+	+	+	+	+	+
3	+	+	+	+	+	+	+	+	+	+
4	+	+	+	+	+	+	+	+	+	+
5	+	+	+	+	+	+	+	+	+	+
6	+	+	+	+	+	+	+	+	+	+
7	N	+	+	+	+	+	+	N	+	+
8	+	+	+	+	+	+	+	+	+	+
9	+	+	+	+	+	+	+	+	+	+

8	0	1	2	3	4	5	6	7	8	9
0	N	+	+	+	+	+	+	+	+	+
1	+	+	+	+	+	+	+	+	+	+
2	+	+	+	+	+	+	+	+	+	+
3	+	+	+	+	+	+	+	+	+	+
4	+	+	+	+	N	+	+	+	+	+
5	+	+	+	+	+	+	+	+	+	+
6	+	+	+	+	+	+	+	+	+	+
7	+	+	+	+	+	+	+	+	+	+
8	+	+	+	+	+	+	+	+	N	+
9	+	+	+	+	+	+	+	+	+	+

9	0	1	2	3	4	5	6	7	8	9
0	9	+	+	+	+	+	+	+	+	N
1	+	+	+	+	+	+	+	+	+	+
2	+	+	+	+	+	+	+	+	+	+
3	+	+	+	+	+	+	+	+	+	+
4	+	+	+	+	+	+	+	+	+	+
5	+	+	+	+	+	+	+	+	+	+
6	+	+	+	+	+	+	+	+	+	+
7	+	+	+	+	+	+	+	+	+	+
8	+	+	+	+	+	+	+	+	+	+
9	N	+	+	+	+	+	+	+	+	N

In most cases the first player can win by taking 1 chip. He uses a similar strategy as before. The first player takes 1 chip and makes sure to never empty a pile. If the second player empties one of the three piles, the first player can apply the same strategy as before when playing with two piles. If the first player is forced to empty a pile the situation $(1, 1, 1)$ is reached, and here the first player can just take all the remaining chips and win the game. There are a few positions which are N positions, where there is is another option besides taking 1 chips to win the game. First, all positions of the form (x, x, x) with $x > 1$ are N positions, as the player can take either 1 chip from one pile or take x chips from all three piles to win. The other N positions are the positions which can lead to a position with only one pile, of which the number of chips is not an even number which gives an odd natural number for an odd m when divided by 2^m. For example in the position $(4, 4, 8)$ a player can take 4 chips from all three piles to leave $(0, 0, 4)$, and since 4 gives 1 when divided by 2^2, the position $(4, 4, 8)$ is an N position.

Now we will look at the version where players are also allowed to take the same number of chips from two piles during their turn. The results can be seen in Table 16. The notation works the same as in the previous table.

Table 16: The results when playing the second version of Wythoff's game with three piles and the extra restriction that players are not allowed to take the same number of chips that was taken during the previous turn.

0	0	1	2	3	4	5	6	7	8	9
0	P	+	N	3	4	5	N	7	N	9
1	+	+	+	+	+	+	+	+	+	+
2	N	+	N	+	+	+	+	+	+	+
3	3	+	+	N	+	+	N	+	+	+
4	4	+	+	+	N	+	+	+	N	+
5	5	+	+	+	+	N	+	+	+	+
6	N	+	+	N	+	+	N	+	+	+
7	7	+	+	+	+	+	+	N	+	+
8	N	+	+	+	N	+	+	+	N	+
9	9	+	+	+	+	+	+	+	+	N

1	0	1	2	3	4	5	6	7	8	9
0	+	+	+	+	+	+	+	+	+	+
1	+	+	+	+	+	+	+	+	+	+
2	+	+	+	+	+	+	+	+	+	+
3	+	+	+	+	+	+	+	+	+	+
4	+	+	+	+	+	+	+	+	+	+
5	+	+	+	+	+	+	+	+	+	+
6	+	+	+	+	+	+	+	+	+	+
7	+	+	+	+	+	+	+	+	+	+
8	+	+	+	+	+	+	+	+	+	+
9	+	+	+	+	+	+	+	+	+	+

2	0	1	2	3	4	5	6	7	8	9
0	N	+	N	+	+	+	+	+	+	+
1	+	+	+	+	+	+	+	+	+	+
2	N	+	N	+	+	+	+	+	+	+
3	+	+	+	+	+	+	+	+	+	+
4	+	+	+	+	+	+	+	+	+	+
5	+	+	+	+	+	+	+	+	+	+
6	+	+	+	+	+	+	+	+	+	+
7	+	+	+	+	+	+	+	+	+	+
8	+	+	+	+	+	+	+	+	+	+
9	+	+	+	+	+	+	+	+	+	+

3	0	1	2	3	4	5	6	7	8	9
0	3	+	+	N	+	+	N	+	+	+
1	+	+	+	+	+	+	+	+	+	+
2	+	+	+	+	+	+	+	+	+	+
3	N	+	+	N	+	+	N	+	+	+
4	+	+	+	+	+	+	+	+	+	+
5	+	+	+	+	+	+	+	+	+	+
6	N	+	+	N	+	+	+	+	+	+
7	+	+	+	+	+	+	+	+	+	+
8	+	+	+	+	+	+	+	+	+	+
9	+	+	+	+	+	+	+	+	+	+

4	0	1	2	3	4	5	6	7	8	9
0	4	+	+	+	N	+	+	+	N	+
1	+	+	+	+	+	+	+	+	+	+
2	+	+	+	+	+	+	+	+	+	+
3	+	+	+	+	+	+	+	+	+	+
4	N	+	+	+	N	+	+	+	N	+
5	+	+	+	+	+	+	+	+	+	+
6	+	+	+	+	+	+	+	+	+	+
7	+	+	+	+	+	+	+	+	+	+
8	N	+	+	+	N	+	+	+	+	+
9	+	+	+	+	+	+	+	+	+	+

5	0	1	2	3	4	5	6	7	8	9
0	5	+	+	+	+	N	+	+	+	+
1	+	+	+	+	+	+	+	+	+	+
2	+	+	+	+	+	+	+	+	+	+
3	+	+	+	+	+	+	+	+	+	+
4	+	+	+	+	+	+	+	+	+	+
5	N	+	+	+	+	N	+	+	+	+
6	+	+	+	+	+	+	+	+	+	+
7	+	+	+	+	+	+	+	+	+	+
8	+	+	+	+	+	+	+	+	+	+
9	+	+	+	+	+	+	+	+	+	+

6	0	1	2	3	4	5	6	7	8	9
0	N	+	+	N	+	+	N	+	+	+
1	+	+	+	+	+	+	+	+	+	+
2	+	+	+	+	+	+	+	+	+	+
3	N	+	+	N	+	+	+	+	+	+
4	+	+	+	+	+	+	+	+	+	+
5	+	+	+	+	+	+	+	+	+	+
6	N	+	+	+	+	+	N	+	+	+
7	+	+	+	+	+	+	+	+	+	+
8	+	+	+	+	+	+	+	+	+	+
9	+	+	+	+	+	+	+	+	+	+

7	0	1	2	3	4	5	6	7	8	9
0	7	+	+	+	+	+	+	N	+	+
1	+	+	+	+	+	+	+	+	+	+
2	+	+	+	+	+	+	+	+	+	+
3	+	+	+	+	+	+	+	+	+	+
4	+	+	+	+	+	+	+	+	+	+
5	+	+	+	+	+	+	+	+	+	+
6	+	+	+	+	+	+	+	+	+	+
7	N	+	+	+	+	+	+	N	+	+
8	+	+	+	+	+	+	+	+	+	+
9	+	+	+	+	+	+	+	+	+	+

8	0	1	2	3	4	5	6	7	8	9
0	N	+	+	+	N	+	+	+	N	+
1	+	+	+	+	+	+	+	+	+	+
2	+	+	+	+	+	+	+	+	+	+
3	+	+	+	+	+	+	+	+	+	+
4	N	+	+	+	N	+	+	+	+	+
5	+	+	+	+	+	+	+	+	+	+
6	+	+	+	+	+	+	+	+	+	+
7	+	+	+	+	+	+	+	+	+	+
8	N	+	+	+	+	+	+	+	N	+
9	+	+	+	+	+	+	+	+	+	+

9	0	1	2	3	4	5	6	7	8	9
0	9	+	+	+	+	+	+	+	+	N
1	+	+	+	+	+	+	+	+	+	+
2	+	+	+	+	+	+	+	+	+	+
3	+	+	+	+	+	+	+	+	+	+
4	+	+	+	+	+	+	+	+	+	+
5	+	+	+	+	+	+	+	+	+	+
6	+	+	+	+	+	+	+	+	+	+
7	+	+	+	+	+	+	+	+	+	+
8	+	+	+	+	+	+	+	+	+	+
9	N	+	+	+	+	+	+	+	+	N

When comparing the results from Table 16 to those of Table 15, we can see that the only difference is that Table 16 has a few more N positions. These extra N positions are all positions where one of the piles is empty and the other two are not. They are either positions where the two non-empty piles have the same number of chips, and a player can win by taking all remaining chips, or one pile has twice as many chips as the other. The player can win by emptying the lower pile and leaving a number of chips on the other pile which is not an even number which gives an odd natural number for an odd m when divided by 2^m. So these lead to the same positions as some of the N positions described before, only now by removing chips from two piles at the same time instead of three.

6 Other Variants

In this section we look at some other variants for playing Nim. We consider games where the players have to restrict moves before playing Nim, where players have to play Nim where the number of chips on the initial piles is unknown and where players are allowed to place chips back on a pile after taking chips from a different pile. We first look at a Misère variant, where the condition for winning the game is reversed.

6.1 Misère Play

Nim can also be played in its Misère variant [11]. What this means is that the condition for winning the game is reversed. Before, if a player was unable to make a move they would lose the game. In the Misère variant being unable to make a move means that the player wins the game of Nim. We take a look at how this variant changes the strategy for regular and restricted Nim.

6.1.1 Misère Regular Nim

We look at regular Nim played with two piles, but now examine the Misère variant. In regular Nim a player would lose if both piles have the same number of chips during their turn and would win otherwise. Table 17 shows both the results of regular two-pile Nim and the Misère variant of two-pile Nim. The results are for piles of up to 9 chips. A 0 in the table means the current player would lose and a 1 means the current player would win.

Table 17: The first table shows the results of playing regular Nim with two piles of up to 9 chips, the second pile shows the results of the Misère version of this game.

Pile 1\Pile2	0	1	2	3	4	5	6	7	8	9
0	0	1	1	1	1	1	1	1	1	1
1	1	0	1	1	1	1	1	1	1	1
2	1	1	0	1	1	1	1	1	1	1
3	1	1	1	0	1	1	1	1	1	1
4	1	1	1	1	0	1	1	1	1	1
5	1	1	1	1	1	0	1	1	1	1
6	1	1	1	1	1	1	0	1	1	1
7	1	1	1	1	1	1	1	0	1	1
8	1	1	1	1	1	1	1	1	0	1
9	1	1	1	1	1	1	1	1	1	0

Pile 1\Pile2	0	1	2	3	4	5	6	7	8	9
0	1	0	1	1	1	1	1	1	1	1
1	0	1	1	1	1	1	1	1	1	1
2	1	1	0	1	1	1	1	1	1	1
3	1	1	1	0	1	1	1	1	1	1
4	1	1	1	1	0	1	1	1	1	1
5	1	1	1	1	1	0	1	1	1	1
6	1	1	1	1	1	1	0	1	1	1
7	1	1	1	1	1	1	1	0	1	1
8	1	1	1	1	1	1	1	1	0	1
9	1	1	1	1	1	1	1	1	1	0

If we compare the results we can see there are very few differences between the two versions. The only differences are the cases $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. In the Misère version $(0,0)$ is a losing position because no moves can be made, which means the starting player wins immediately. This means that $(1,0)$ and $(0,1)$ must be losing positions because only the move to $(0,0)$ can be made. Now it also makes sense that $(1,1)$ is a winning position as it can move to either $(0,1)$ or $(1,0)$.
All other cases give the same results, however. The reason for this is that the mirror image strategy can still be used, with a minor modification. If both piles have the same number of chips and both have more than 1 chip, the second player should always copy the move of the first player on the other pile. However, if the first player now empties a pile, the other player should not empty the other pile, but reduce the number of chips on that pile to 1. And if the first player reduces the number of chips of a pile to 1, the second player should remove all chips from the other pile. This way the second player makes sure the situation $(1,1)$ never occurs and the fist player will eventually be in either the situations $(1,0)$ or $(0,1)$, which are losing.

6.1.2 Misère Subtraction Game

Next, we will look at the Misère version of the Subtraction Game. Now the players have a maximum number of chips they are allowed to take in one turn. Table 18 shows the results of the Subtraction Game and the Misère variant. The game is played with two piles of up to 9 chips and the maximum number of chips a player can take in one turn is 3.

Table 18: The first table shows the results of playing the Subtraction Game with two piles of up to 9 chips, the second pile shows the results of the Misère version of this game.

Pile 1\Pile2	0	1	2	3	4	5	6	7	8	9
0	0	1	1	1	0	1	1	1	0	1
1	1	0	1	1	1	0	1	1	1	0
2	1	1	0	1	1	1	0	1	1	1
3	1	1	1	0	1	1	1	0	1	1
4	0	1	1	1	0	1	1	1	0	1
5	1	0	1	1	1	0	1	1	1	0
6	1	1	0	1	1	1	0	1	1	1
7	1	1	1	0	1	1	1	0	1	1
8	0	1	1	1	0	1	1	1	0	1
9	1	0	1	1	1	0	1	1	1	0

Pile 1\Pile2	0	1	2	3	4	5	6	7	8	9
0	1	0	1	1	1	0	1	1	1	0
1	0	1	1	1	0	1	1	1	0	1
2	1	1	0	1	1	1	0	1	1	1
3	1	1	1	0	1	1	1	0	1	1
4	1	0	1	1	1	0	1	1	1	0
5	0	1	1	1	0	1	1	1	0	1
6	1	1	0	1	1	1	0	1	1	1
7	1	1	1	0	1	1	1	0	1	1
8	1	0	1	1	1	0	1	1	1	0
9	0	1	1	1	0	1	1	1	0	1

The differences between the normal and the Misère version are similar to the case of regular Nim. If we play the normal Subtraction Game with the maximum number of chips of 3, the results of a game do not change if we add exactly 4 chips to a pile. This is because the first player will only lose if both piles have a value $\equiv 0 \pmod 4$, the second player can now always return to a similar situation until $(0,0)$ is reached.
In the Misère variant it is also true that adding 4 chips to either or both piles does not change the outcome of the game. Every block of 4×4 in the table is changed in a similar way as with regular Misère Nim. For instance here the situations $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$ are changed in the same way, but now the positions $(4,0)$, $(4,1)$, $(5,0)$ and $(5,1)$ are also changed in the same way, which are the positions with 4 extra chips on the first pile.

6.1.3 Misère Restricted Nim

Finally, we will look at the Misère version of Restricted Nim played with two piles. Table 19 shows the results side by side. Players are allowed to take a maximum of 3 chips in these cases.

Table 19: The first table shows the results of playing Restricted with two piles of up to 10 chips, the second pile shows the results of the Misère version of this game.

Pile 1\Pile2	0	1	2	3	4	5	6	7	8	9	10
0	P		N	3	P		N	3	P		N
1											
2	N				N				N		
3	3			N	3			N	3		
4	P		N	3	P		N	3	P		N
5											
6	N				N				N		
7	3			N	3			N	3		
8	P		N	3	P		N	3	P		N
9											
10	N				N				N		

Pile 1\Pile2	0	1	2	3	4	5	6	7	8	9	10
0	N	X	P	N	N	3	P		N	3	P
1	X	X	N	N	3	P		N	3	P	
2	P	N	2	3	2		N	P		N	N
3	N	N	3	P		N	3			N	3
4	N	3	2		N	3	P	N	N	3	P
5	3	P		N	3	P		N	3	P	
6	P		N	3	P		N	3	P		N
7		N	P		N	N	3	P		N	3
8	N	3			N	3	P		N	3	P
9	3	P	N	N	3	P		N	3	P	
10	P		N	3	P		N	3	P		N

The results are displayed in the same way as in Section 4.3. There is now one extra case,

however. When playing the Misère version and the situation is either $(0,1)$, $(1,0)$ or $(1,1)$, an X is shown in the table. This indicates that in order for the current player to win, taking exactly 1 chip must be restricted. This case can only occur in the Misère version and will only occur in the aforementioned situations. In these situations the current player is unable to make a move because taking 1 chip is restricted, allowing him to win the game. If taking 1 chip was not restricted the current player would have to take 1 chip, leaving the other player in a situation where they are unable to do anything, so the current player loses.
Unlike with regular Nim, playing the game in Misère causes significant differences for Restricted Nim. The strategy of always taking 1 chip becomes less favourable in the Misère version since there are now some cases that can only be won by having the value 1 restricted.

6.2 Banning Moves Before Playing

So far, we have looked at moves being restricted while a game of Nim is being played. However, we can also consider the case where players get to choose the restricted moves beforehand, so before the game of Nim even started. Both players will be allowed to ban a certain number of moves for their opponent, they will not be allowed to make these moves during the entire game of Nim played afterwards. The players do know the number of piles and number of chips on these piles before deciding which moves they want to ban for their opponent. They also know who the starting player will be of the game of Nim. The order in which players ban moves can differ and it is important to note that players have knowledge over which moves have already been banned. Players are allowed to ban the same moves as their opponent.
As an example we will look at the results of the game of Nim played with one pile of chips when both players are allowed to ban two moves of their opponent. We will also give a maximum value for the number of chips a player is allowed to take in one turn. Table 20 shows the results for pile sizes of up to 25 chips where players are never allowed to take more than 5 chips. The starting player first bans a move for the second player, then the second player bans a move for the starting player, then the starting player again and finally the second player, both players get to ban two moves for their opponent.

Table 20: Results of playing Nim where both players are allowed to ban two moves for the other player in advance. The players take turns banning moves and the starting player makes the first ban. The players can only take a maximum of five chips.

Pile Size	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
Winning Player	2	2	2	1	1	1	2	2	2	2	2	1	2	1	2	2	2	2	2	1	1	2	2	1	2	2

The cases where the pile has less than three chips are easily won by the second player. The second player can make sure that the first player is not able to make any move during the very first turn of the game of Nim. It is also easy for the first player to win the games where the pile has three to five chips. The first player can ban the moves of taking one or two chips for the second player. Now the first player always has three options to win; they can either remove the entire pile, remove enough chips so there is one chip left, or remove enough chips so there are two chips left. The second player is only able to ban two of these three options, so the first player can always make one of these three moves which allow him to win.
When playing with more than five chips most of the cases are won by the second player.

It seems like being the last player to ban a move of their opponent gives an advantage for the game of Nim played. In order to test this we reverse the order in which the players are banning moves. So now the second player first bans a move of the first player, then the first player bans a move of the second player, then the second player again and finally the first player makes the last ban. Table 21 shows the results, again the maximum number of chips a player is allowed to take is five and the results are for one pile of up to 25 chips.

Table 21: Results of playing Nim where both players are allowed to ban two moves for the other player in advance. The players take turns banning moves and the second player makes the first ban. The players can only take a maximum of five chips.

Pile Size	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
Winning Player	2	2	2	1	1	1	1	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

The results for piles of up to five chips are the same as described before. However, if we look at the results for piles larger than five chips, the starting player will almost always win. This shows that being the last player to ban a move of their opponent and being the first player to make a move during the game of Nim is a big advantage.
In order to further test the importance of being able to make the final ban we look at two more different banning orders. Now one player will make the first ban and then the other player will make two bans in a row, the final ban will be done by the player who also banned first. The results of both the banning order $1-2-2-1$ and $2-1-1-2$ can be seen in Table 22, again for piles of up to 25 chips and players not being allowed to take more than 5 chips during their turn. The numbers of the banning order indicate the player that is choosing a number to ban for the other player, so in the order $1-2-2-1$ the starting player makes the first and final bans.

Table 22: Results of playing Nim where both players are allowed to ban two moves for the other player in advance. The order of bans is given. The players can only take a maximum of five chips.

Pile Size	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
Winning Player $2-1-1-2$	2	2	2	1	1	1	1	2	2	2	2	1	2	1	2	2	2	2	2	1	1	2	2	1	2	2
Winning Player $1-2-2-1$	2	2	2	1	1	1	1	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

If we look at which player was the last to ban a move the results are almost identical to the results before the change in order. There is only one difference, which is the case where the second player makes the last ban and there are six chips on the pile. In the case where the order of banning is $2-1-1-2$ the first player can win, as opposed tot the case where the order of banning is $1-2-1-2$ where the second player can win. In all other instances the results are the same, showing the importance of being the last player to ban a move of the other player.

We can change the order of banning moves one step further and make it so that one player has to first decide all or their bans and then the other makes all of their bans. We will look at the games the first player is able to win even though he has to make all of his bans before the bans of the second player. In these games the order of bans essentially did not matter, as the first player is able to always ban certain values to win. Table 23 shows which games the first player can win if he has to first ban moves for the second player. The game of Nim is played with one pile and the results shows how many chips need to be on this pile for the first player to win. We look at the games where the players are allowed to ban two, three or four of their opponents moves. The results are given for piles of up to 35 chips. The players are only allowed to take a maximum of five chips during the game of Nim.

Table 23: Games won by the first player for piles of up to 35 chips when the first player has to make all their bans first.

Number of Bans	Number of chips of games won by first player
2	3, 4, 5, 11, 13, 19, 20, 23, 27, 31
3	4, 5, 13, 23, 31
4	5, 34

If we compare these results to the results of Table 20 we can see that when both players get to ban two moves, the games the first player will win are the same. So the order of banning moves did not matter as long as the second player was able to be the last player to ban a move.
It looks like the games won by the first player strictly decreases as more moves are being banned, the games that can be won by the first player with three bans can also be won with only two bans, but there are cases in which the first player can win with only two bans but not with three bans. With four bans the second player wins almost all of the games. However, there is now a game which the first player could not win before, the game with 34 chips on the pile. This means that the games the first player can win will not strictly decrease if more values are getting banned. If we would increase the number of bans to five, the first player will lose every game because all of his options will be banned, so he can never make a move during the first turn.

Now the question is what moves the first player will ban in order to win whenever possible. Table 24 shows the moves the first player has to ban to guarantee victory in the game of Nim played afterwards. In all of these cases the order of banning moves is $1-2-1-2$, so the starting players makes the first ban. The results are for pile sizes of up to 30 chips with maximum moves ranging from three to seven chips. The table shows which two moves the first player will need to ban for the second player. Sometimes multiple combinations are possible for the first player to win. If no combinations will lead to a winning game of Nim for the first player, as $-$ is shown in the table.

Table 24: Winning strategy for the starting player to win the game of Nim, depending on the pile size and maximum allowed move.

Chips\Max Move	3	4	5	6	7
3	1+2	1+2	1+2	1+2	1+2
4	-	1+2 or 1+3	1+2 or 1+3	1+2 or 1+3	1+2 or 1+3
5	-	-	1+2 or 1+3 or 1+4 or 2+3	1+2 or 1+3 or 1+4 or 2+3	1+2 or 1+3 or 1+4 or 2+3
6	-	-	-	1+2 or 1+3 or 1+4 or 1+5 or 2+3	1+2 or 1+3 or 1+4 or 1+5 or 2+3
7	-	-	-	-	1+2 or 1+3 or 1+4 or 1+5 or 1+6 or 2+3
8	-	-	-	-	-
9	1+2	-	-	-	-
10	-	-	-	-	-
11	2+3	1+2	1+2 or 1+3	-	-
12	-	-	-	1+2 or 1+3	-
13	-	-	1+2	1+2 or 1+4	1+2 or 1+3 or 1+4
14	1+3	-	-	-	1+2 or 1+4
15	-	-	-	1+2	1+2 or 1+3 or 1+4
16	-	-	-	-	-
17	1+2	1+2	-	-	1+2
18	-	-	-	-	-
19	1+3	-	1+2	-	-
20	-	-	1+2	-	-
21	-	-	-	-	-
22	1+2	-	-	1+2	-
23	1+2 or 1+3 or 2+3	-	1+2	1+2	1+2
24	-	-	-	-	-
25	-	-	-	-	1+2
26	-	1+2	-	1+2	1+2
27	1+2	1+2	1+2	-	-
28	-	-	-	-	-
29	1+2	-	-	-	1+2
30	-	-	-	1+2	-

It seems that banning the move of taking one chip is very important in order to win the game of Nim. If it is possible for the starting player to win it is almost always required that he bans taking one chip for the other player. The only cases in the table where it is not required is when the number of chips on the pile is smaller than or equal to the number of maximum chips a player is allowed to take during their turn. There is one more case, when there are 23 chips on the pile and the maximum number of chips a player can take is 3. Here any possible combination of bans will result in a win for the first player, however there are only three possible combinations.
In most of the cases using the second ban to restrict taking two chips for the second player seems to be the ban that results in the starting player winning. This is not always the case, though, sometimes the second ban can be, or has to be, taking three or four chips. It does certainly seem that banning the lower values gives an advantage during the game of Nim however. This is likely due to the fact that a player will not only be unable to make a move when the pile has no chips at all but also when the pile has only one or two chips. Which will also make it easier for the other player to move to a losing position.

6.2.1 Extra Restrictions During Play

We can combine the rule of restricting moves before playing the game of Nim with the rule that a player is not allowed to take the same number of chips that was taken during the previous turn. Now players will ban two moves for their opponent in advance just like before, and the number of chips a player takes during the game of Nim will be restricted for the other player during their next turn. Sometimes a player can take a number of

chips that is already restricted for the other player, in this case it does not matter that the value is now double restricted, the player will just have the same restricted values that were decided before the game of Nim.
Table 25 shows the result for this version, where both players restrict two values beforehand in the order $1-2-1-2$. The number of chips a player takes while playing Nim will be restricted for the next player, the maximum number of chips a player is allowed to take ranges from 3 to 7 chips. The results are for piles of up to 30 chips.

Table 25: Results of combining banning two moves in advance with restricting the previous moves during play.

Max Move\Chips	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
3	2	2	2	1	1	2	2	1	1	1	2	1	2	1	1	2	2	1	1	1	2	2	1	1	2	2	2	1	1	1	2
4	2	2	2	1	1	2	1	1	1	1	2	1	1	1	1	2	1	1	1	1	2	2	1	1	1	2	1	1	1	1	2
5	2	2	2	1	1	1	1	2	1	2	2	1	1	1	1	2	2	1	2	1	1	2	2	1	2	2	2	2	2	2	1
6	2	2	2	1	1	1	1	2	1	2	2	1	1	1	2	1	2	2	2	1	1	2	1	1	2	2	1	2	2	2	1
7	2	2	2	1	1	1	1	1	1	2	2	2	2	1	1	1	1	1	2	2	2	2	2	1	1	1	1	2	2	1	2

Even though the second player will be the last player to ban a move, the first player still wins a good number of games. This is the main difference with the version without the extra restriction. The number of games won by the first player is clearly higher than before. A big reason for this is that the first player can sometimes restrict a third move for the second player, while the second player has not yet made a move. The first player is allowed to ban two moves before playing the game, and can then ban a third move by removing this number of chips during the very first turn. This can heavily limit the number of options for the second player and force them to make a move that is advantageous for the first player.
Table 26 shows the winning strategies for the first player using the same rules for the results in Table 25.

Table 26: Winning strategy for the starting player to win the game of Nim with the added restriction, depending on the pile size and maximum allowed move.

Chips\Max Move	3	4	5	6	7
3	1+2	1+2	1+2	1+2	1+2
4	1+3	1+2 or 1+3 or 1+4	1	1	1
5	-	-	1+2 or 1+3 or 1+4 or 3+4	1+2 or 1+3 or 1+4 or 3+4	1+2 or 1+3 or 1+4 or 3+4
6	-	1+2	1+2	1 or 2+5	1 or 2+5
7	1+2	2+4	-	-	1+2 or 1+3 or 1+4 or 1+5 or 1+6 or 4+5
8	1+3	1+4 or 2+3 or 3+4	2+3	2+3	1+2 or 3*
9	1+2	2+3	-	-	-
10	-	-	-	-	-
11	2+3	1+2 or 1+3	1	1+5	-
12	-	1+2	1+2	1+2 or 1+3 or 1+4 or 1+5	-
13	1+2 or 1+3	1+2 or 2+3	1+2 or 2+3	1+2 or 1+3 or 1+4 or 1+6	1+2 or 1+3 or 1+4 or 1+6
14	1+2 or 1+3	2+3 or 3+4	2+3	-	1+2 or 1+4 or 1+6
15	-	-	-	1+2	1+2 or 1+4 or 1+6
16	-	1+3	-	-	1+2
17	1+2	1+2 or 1+3	3*	-	1+2
18	1+2	1+2 or 2+3	-	-	-
19	1+2 or 1+3	1+2 or 2+3 or 2+4	1+2	1+6	-
20	-	-	1+2	1*	-
21	-	-	-	-	-
22	1+2	1+3	-	1+2	-
23	1+2 or 1+3 or 2+3	1+3 or 2+3 or 3+4	1+2 or 2+3	1+2 or 1+6	1+2
24	-	1+2 or 2+3	-	-	1+2
25	-	-	-	-	1+2
26	-	1+2	-	1+2	1+2
27	1+2	1+2	-*	-	-
28	1+3	1+3 or 2+3 or 4*	-	-	-
29	1+2 or 1+3	1+3 or 2+3 or 3+4	-	-	1+2
30	-	-	1+2	1+2	-

Not only does the starting player win a substantial number of games that he previously lost but he also has more options for winning games. It is still usually a good move to ban playing 1 chip for the other player, but there are more cases where this is not necessary. There are even some cases where banning 1 does not result in a win for the first player even though he could have won.

Some cases in the table have a * next to the value that needs to be banned. In these cases the second ban for the first player is dependent of the move that was banned by the second player. So here the first player would not have been able to win if he had to make his second ban before the second player made a single ban. For example the case with a maximum move of 4 and a pile size of 28 the first player can win by first banning 4 for the second player. Now the second ban the first player has to make is dependent on the move that gets banned by the second player. If the second player bans either 1 or 4, the first player needs to ban 1 using his second ban. If the second player bans either 2 or 3 the first player needs to ban 3. The starting player can usually still win by using a different combination of bans even if he has to make all of his bans before the second player, this is not the case when the maximum move is 5 and the number of chips is 17 and when the maximum move is 6 and the number of chips is 20. Table 27 shows the moves the first player needs to ban for all these cases indicated with a *.

Table 27: Winning strategy for the starting player in the cases where it is dependent on the first move banned by the second player.

Max Move-Chips\2nd Player Ban	1	2	3	4	5	6	7
4-28	4+1	4+3	4+3	4+1	-	-	-
5-17	3+2	3+4	3+1 or 3+2	3	3+1 or 3+5	-	-
6-20	1+6	1+6	1+2 or 1+3 or 1+4 or 1+6	1+2 or 1+3 or 1+4	1+2 or 1+3 or 1+4	1+2 or 1+5 or 1+6	-
7-8	3+2	3+1	3+1	3+1	3+1	3+1 or 3+2	3+2
7-15	5+1	5+1	5+1	5+4	5+4	5+1	5+1 or 5+4

There is one more case that is indicated with a *, this is the case where the maximum move is 5 and the number of chips on the pile is 27. This is the only case in the table which used to be a winning position for the starting player but is not any more with the extra restriction of not being allowed to take the same number of chips that was taken in the previous turn. This means that this extra restriction is not always just an advantage for the starting player.

6.2.2 Banning Moves Simultaneously

Before, we have seen that the player who gets to make the final ban will usually win the following game of Nim. In order to make sure that there is no final ban it is possible to let the players decide their bans at the same time. This means that both players have to decide on all their bans before playing the game of Nim with no information on which moves the other player has banned for them. In a lot of cases the moves a player needs to ban are dependent on the moves the other player has banned. Because the players do not know which moves the other player has banned they will need to give a probability distribution for all possible combinations of bans, which maximizes their own chance of winning. The players do know the outcome of the game of Nim for every possible combination of banned moves, they need to use this information to decide on their probability distribution.
Depending on the number of bans each player is allowed to make, the maximum number of chips a player is allowed to take in one turn and the number of chips on the starting pile, every game will have a certain value. This value indicates which player has a greater chance of eventually winning the game, so if the same game is played multiple times using the optimal probability distributions for both players, the value of the game indicates which player will win the most times. We let this value range from -1 to 1, where 1 indicates that player 1, the player who will make the starting move in the game of Nim, will always be able to win and -1 indicates that player 2 will always be able to win. A value of 0 means that both players have and equal 50/50 chance of winning, a positive value indicates an advantage for the first player and a negative value indicates an advantage for the second player.
We will now look at the same games played in Table 24, but now the players have to decide on their bans beforehand. Table 28 shows the values of these games. The game of Nim is played with only a single pile and both players get to ban two different moves for the other player.

Table 28: Value of the games where both players have to decide on their bans simultaneously, depending on the pile size and maximum allowed move.

Chips\Max Move	3	4	5	6	7
3	1	1	1	1	1
4	0	1	1	1	1
5	0	0.333	1	1	1
6	-1	0	0.333	1	1
7	0	-1	0	0.333	1
8	0	0	-1	0	0.333
9	1	0	0	-1	0
10	0	0.2	0	0	-1
11	1	1	1	0	0
12	-1	0	0.333	1	0
13	0	0	1	1	1
14	1	0	0	0.333	1
15	0	0	0	1	1
16	-1	0	0	0	0.333
17	1	1	0	0.2	1
18	0	0.2	0	0	0
19	1	0.2	1	0	0.2
20	-1	0	1	0	0
21	-1	0	0	0.25	0
22	1	0	0.2	1	0
23	1	0	1	1	1
24	-1	0	0	0	0.25
25	-1	0	0	0	1
26	-1	1	0	1	1
27	1	1	1	0	0.2
28	0	0	0.2	0	0
29	1	0	0.2	0	1
30	-1	-1	0	1	0

From the results we can see that it is usually an advantage to be the starting player in the game of Nim. When the maximum move players are allowed to make is 3, the values of the games will always be either 0, indicating that both players have a 50% chance of winning if they play optimally, 1 or -1, which means that player 1 or player 2 will always be able to win respectively.

However, if we look at the results where the maximum move is 4 or higher, the values are mostly either 0 or a positive value. There are still cases where player 2 will always be able to win but if no player is guaranteed a victory the odds are either equal or in favour of player 1. A value of 0.2 indicates that player 1 has a 60% chance of winning and player 2 has a 40% chance of winning if both player use optimal play. A value of 0.25 indicates a 62.5% win chance for player 1 and a value of 0.333 indicates a 66.667% win chance for player 1. There are also still quite some games with a value of 1 and only very few with a value of -1.

The probability distributions of both players are in a Nash equilibrium [12], [13]. This means that if a player decides to deviate from the distribution, their results will not improve. They can get worse, however. If both players deviate from their distribution the results can change, but because the results for a player will not improve in the Nash equilibrium they will not change their probability distribution. If they did change it, this would allow the other player to change their distribution as well to improve their own results.

6.3 Playing with Unknown Pile Sizes

In this section we will take a look at Nim played with unknown initial pile sizes. This means that at the start of the game both players do not know how many chips there are on every pile. They do however know a certain range of possible values each pile can have. They also know the number of piles of chips, if a pile exists it cannot contain exactly 0 chips. There will also be a maximum number of chips a player is allowed to take in one turn. In order to prevent a player to take more chips than the number of chips on a pile, the range of possible values for the size of a pile will always be at least as high as the maximum move players are allowed to make.

Once a move has been made on an unknown pile both players will know what the new size of the pile is. The players also know this, so they have to plan their moves with the knowledge that if they take chips from an unknown pile the other player will know the size of that pile in the next turn.

The results for this game can be calculated by first looking at the instances where both piles have a known number of chips. In these instances both players know exactly how many chips there are on each pile and there are no unknown values. The results of these instances are just normal Subtraction Game, players would play the same way since there are no unknown values. When these results are known we look at the instances where one pile is unknown and the rest is known. Because we know all possible values for the size of this unknown pile, we know the number of outcomes that exist if the current player decides to take chips from the unknown pile. Say there are n possibilities for the unknown pile, taking chips from this unknown pile could lead to n different known outcomes. The result of taking chips from the unknown pile becomes the average of the results of the possible outcomes. If the player would win in a known outcome the result of that outcome is 1 and otherwise the result is 0. So say $n = 4$ and there are 4 different outcomes when taking a certain number of chips, say 1 chip for example, from the unknown pile and two of those outcomes will lead to a win for the current player, the result of taking 1 chip from the unknown pile will be 1/2. Table 29 shows the results for two unknown piles with a maximum move of 3. The values indicate the chance of winning for the first player. The table shows the possible values for the first pile X and the second pile Y and which moves the first player could make to get the mentioned result.

Table 29: Results for the starting player when playing Nim with two unknown piles with a maximum move of 3.

Pile X Range	Pile Y Range	Result	Move to play
3,4,5	3,4,5	4/9	X: 1,3 Y: 1,3
3,4,5	4,5,6	4/9	X: 1,3 Y: 1,2
3,4,5	5,6,7	4/9	X: 1,3 Y: 1,2,3
3,4,5	6,7,8	4/9	X: 1,3 Y: 2,3
3,4,5	7,8,9	4/9	X: 1,3 Y: 1,3
4,5,6	4,5,6	4/9	X: 1,2 Y: 1,2
4,5,6	5,6,7	4/9	X: 1,3 Y: 1,2,3
4,5,6	6,7,8	4/9	X: 1,2 Y: 2,3
4,5,6	7,8,9	4/9	X: 1,2 Y: 1,3
5,6,7	5,6,7	4/9	X: 1,2,3 Y: 1,2,3
5,6,7	6,7,8	4/9	X: 1,2,3 Y: 2,3
5,6,7	7,8,9	4/9	X: 1,2,3 Y: 1,3
6,7,8	6,7,8	4/9	X: 2,3 Y: 2,3
6,7,8	7,8,9	4/9	X: 2,3 Y: 1,3
7,8,9	7,8,9	4/9	X: 1,3 Y: 1,3
3,4,5,6	3,4,5,6	3/8	X: 1,2,3 Y: 1,2,3
3,4,5,6	4,5,6,7	3/8	X: 1,2,3 Y: 1,2,3
3,4,5,6	5,6,7,8	3/8	X: 1,2,3 Y: 1,2,3
3,4,5,6	6,7,8,9	3/8	X: 1,2,3 Y: 1,2,3
4,5,6,7	4,5,6,7	3/8	X: 1,2,3 Y: 1,2,3
4,5,6,7	5,6,7,8	3/8	X: 1,2,3 Y: 1,2,3
4,5,6,7	6,7,8,9	3/8	X: 1,2,3 Y: 1,2,3
5,6,7,8	5,6,7,8	3/8	X: 1,2,3 Y: 1,2,3
5,6,7,8	6,7,8,9	3/8	X: 1,2,3 Y: 1,2,3
6,7,8,9	6,7,8,9	3/8	X: 1,2,3 Y: 1,2,3
3,4,5,6,7	3,4,5,6,7	12/25	X: 3 Y: 3
3,4,5,6,7	4,5,6,7,8	11/25	X: 3 Y: 1,2,3
3,4,5,6,7	5,6,7,8,9	12/25	X: 3 Y: 1
4,5,6,7,8	4,5,6,7,8	8/25	X: 1,2,3 Y: 1,2,3
4,5,6,7,8	5,6,7,8,9	11/25	X: 1,2,3 Y: 1
5,6,7,8,9	5,6,7,8,9	12/25	X: 3 Y: 3
3,4,5,6,7,8	3,4,5,6,7,8	4/9	X: 3 Y: 3
3,4,5,6,7,8	4,5,6,7,8,9	4/9	X: 3 Y: 1
4,5,6,7,8,9	4,5,6,7,8,9	4/9	X: 1 Y: 1
3,4,5,6,7,8,9	3,4,5,6,7,8,9	20/49	X: 1,3 Y: 1,3

The starting player always has a disadvantage in these instances, it is usually not favourable to play on a pile with an unknown number of chips. Since the starting player is forced to do this in the first turn he is at a disadvantage. The denominator of the fraction for the win chance of the starting player is always equal to the number of possibilities of pile X times the number of possibilities of pile Y, although simplified in the case of 16 and 36, to 8 and 9 respectively.

The results do not seem to change much when the range of values shifts. The results when both piles can have four different continuous values are always the same. Only when both piles can have five values is there some variation in the results.

6.4 Putting Chips Back

In this section we look at a version of Nim where it is allowed to put a certain number of chips that were taken from a pile back onto another pile, there will be no other restrictions. If a player takes n chips from a pile he can choose if he wants to place up to $n-1$ chips back on another pile, if any at all. Because a player is allowed to place back a maximum of $n-1$ chips the game will be closer to ending after every turn, because there will always be more chips taken than put back.
If we play this game with only one pile of chips the starting player can always win by taking the entire pile, he has no reason to put chips back. We will first look at the version played with two piles. In this case the results of the game will be the same as when playing standard Nim with two piles. That is that a player will always lose if both piles have the same number of chips and otherwise he will win. In standard Nim the second player can use a mirror-image strategy, where he will always take the same number of chips the first player took from the other pile, which means that the two piles will have the same number of chips again when the first player has to make a move. This way the second player will make the last move. This same strategy can also be used if the players are allowed to put chips back. If both piles have the same number of chips the first player has to remove chips from one pile and is allowed to move chips to the other pile. Now the second player can still just remove chips from the larger pile to make it as big as the smaller pile, and the second player can choose to not put any chips back.

Next consider the version played with three piles. It turns out that when playing with three piles the starting player will always win. When there are only two piles left the starting player will lose if both piles have the same number of chips. If there are three non empty piles and at least two of these piles have the same number of chips the first player can just remove the other pile so there are two piles with the same number of chips left. If all three piles have a different number of chips the first player can completely remove the largest pile. If this largest pile had n chips, he is now allowed to put back as many chips as he wants onto one of the other piles up to $n-1$ chips. If all three piles had a different number of chips the difference in size between the smallest and second smallest piles can be at most $n-2$, if we assume none of the piles were empty. This means that the first player can always raise the number of chips on the smallest pile to be the same as that of the second smallest pile by first removing the largest pile, this will leave two piles of the same size, a losing position.

So what would happen if we increase the number of piles to four? Again a solid strategy can be found. The starting player will lose if two pairs of piles have the same number of chips and will win otherwise. We just showed that a game with three piles can always be won by the starting player. This means that when playing with four piles, the first player to empty one of the piles will lose, as they leave their opponent in a situation with three piles. The situation in which a player is forced to remove a pile is when all piles have only one chip left. In this situation there are two pairs of piles with the same number of chips. Now say that two of the piles have exactly one chip and the other two piles have exactly two chips left. The starting player will lose because he can only bring one of the piles with two chips down to one. In a situation where there are two pairs of piles with the same number of chips the starting player will always lose because it is never possible to move to a similar position. In all other cases the first player will win because it is always possible to move to a position where there are two pairs of piles with the same number of chips. If there are at least three piles with a different number of chips the starting player can remove chips from the largest pile so that is will be as big as the smallest pile. Now he can raise the number of chips in the second smallest pile so it will be as big as the second largest pile. This way there are two pairs of piles with the same number of chips. This will always work if there at least three piles with a different number of chips because the difference in size between the largest and smallest pile will always be bigger than the difference between the second largest and second smallest piles. If there are two pairs of piles with the same number of chips this will be impossible because the difference in chips between the largest and smallest piles and between the second largest and second smallest piles will be the same, and the player will not be allowed to put back enough chips.

If we look at the game with five piles of chips the results will be a bit more vague. There is not really one solid strategy that will always work in a pattern. One thing we do know is that when there are two piles with the same size, the starting player can always win. In this case he can always create a situation with four piles where two pairs of piles have the same number of chips. This can be done by completely removing the largest pile that is not part of the pair with the same number of chips and adding chips to the smallest pile to be the same size as the second smallest pile, where again these piles are not part of the pair with the same number of chips. This means that any position with five non empty piles that is a sub position of the position $(1, 2, 3, 4, 5)$ is a winning position for the first player, there need always be two piles with the same number of chips. The position $(1, 2, 3, 4, 5)$ itself is a losing position for the first player, as it has to lead to either one of these sub positions or a position with four piles, but without two pairs of piles with the same number of chips. However, not every position with five piles where the number of chips on each pile is different is a losing position. For instance, the position $(1, 2, 3, 4, 6)$ is a winning position for the first player as it can lead to the position $(1, 2, 3, 4, 5)$ by removing one chip from the fifth pile. For five piles of up to 10 chips the following position are losing positions for the first player:

- $(1, 2, 3, 4, 5)$
- $(1, 2, 3, 6, 7)$
- $(1, 2, 3, 8, 9)$
- $(1, 4, 5, 6, 7)$
- $(1, 4, 5, 8, 9)$
- $(1, 6, 7, 8, 9)$
- $(2, 4, 6, 8, 10)$
- $(2, 5, 7, 9, 10)$
- $(3, 4, 7, 9, 10)$
- $(3, 5, 6, 9, 10)$
- $(3, 5, 7, 8, 10)$

Once a losing position is found at least two pile sizes need to change to find another losing position.

7 Conclusion and Future Work

In this thesis we have looked at different versions of the game of Nim by restricting the players options. We mostly looked at the version where the players are not allowed to

remove the same number of chips that were taken in the previous turn and only allowing a maximum number of chips to be taken. By doing this it becomes less clear what moves need to be made than when playing regular Nim. It turns out that only removing one chip is a very good move, as it will lead a player to win the game in a lot of situations. The reason is that a player can restrict the other player from taking one chip, so not only will that player lose when there are no more chips left but also when there is only one chip left.
We can also see this when the players have to ban moves for the other player before playing. Restricting the other player to take only one chip is usually the winning strategy, if it is possible to win. When adding the restriction to playing the previous move to Wythoff's Game, it also becomes favourable to take only one chip in a lot of instances.
We have also looked at the results of playing with more than two players, playing with unknown pile sizes and playing a version where players are allowed to put some of the chips they took back onto a pile.

For future work it is obviously possible to look at the variants we have played with a higher number of piles or higher maximum moves, more players, etcetera. Something more interesting to look at is the process of banning moves for the other player before playing. We have only looked at players individually deciding on their bans before player Nim, and then they just play Nim with those restricted moves. This banning process can also be done more dynamically, more bans are being made during Nim play. The banning process can also be expanded upon by disallowing players to ban the same numbers for example, this can turn the banning process into a game itself.
Something else that can be expanded upon is the version where the pile sizes are not known. In our version a pile is only unknown until a move is made on that pile. To make matters a lot more complicated we can make it so that the size of a pile stays unknown until it is completely empty. The number of possible outcomes for a player is now a lot more and the optimal moves become more difficult to determine.

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