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Opleidingen Wiskunde en Informatica

Hex Circles

Winning Strategies for variants of Hex using Pure Monte-Carlo

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BACHELOR THESIS

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Abstract

This thesis discusses the games HEX, and two variants CYLINDRICAL HEX and TORUS HEX. We start by giving the rules of the games, and showing that no tie can take place, meaning that there will always be a winner. After that we discuss some existing strategies for CYLINDRICAL HEX and program a Pure Monte-Carlo player to play this game. From smart strategies observed from the Pure Monte-Carlo player, a new strategy is determined for CYLINDRICAL HEX. To test this new strategy, experiments are carried out and discussed.

Contents

1 Introduction

Most people like to play a game once in a while. Lucky for you, this thesis discusses one such game. The game of Hex is a two-player game played on a parallelogram-shaped board that is made of hexagonal fields. Both player's objective is to make a path between two opposite sides of the board. The winner is the player that makes this connection between the two opposing edges the fastest. The rules of the game are in essence quite simple, so what if we find a way to make it more complicated and interesting. We can do this by taking our board and connecting the left and right sides of the board to form a cylinder, giving form to the variant of the game named Cylindrical Hex. But what if we were to take it one step further, and also connect the top and bottom sides of our cylindrical board to form a torus, thus making another variant called Torus Hex.

Just like with any other game, every player wants to win. To do this, we need a strategy. In this thesis we research winning strategies for Cylindrical Hex, and see if we can find a winning strategy for a board of width 5.

We will start in Section 2 by explaining the rules for Hex, Cylindrical Hex and Torus Hex. In Section 3 we show that there is always a winner in all variants of the game, giving a new proof for Torus Hex. Section 4 discusses existing winning strategies for Hex and Cylindrical Hex. We also give a new proof for the winning strategy for width 3 for Cylindrical Hex. In Section 5 we define a Pure Monte-Carlo strategy and use this for some experiments. Using observations made from surveying the games, we define a new strategy in Section 6. In Section 7 we do experiments on this new strategy, comparing it to the performance of the Pure Monte-Carlo strategy. Section 8 concludes.

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2 Rules of the Game

In this section we give the rules of the base game Hex and the two versions Cylindrical Hex and Torus Hex.

2.1 Hex

Hex is a two-player game played on a parallelogram-shaped board with m columns and n rows that is made of hexagonal fields called *cells*. The classical board uses $m = n$, but any size can be used. The two players, called Red and Blue, alternate turns placing a stone of their colour on an empty cell. The starting player can be either Red or Blue. Both player's objective is to make a path between two opposite sides of the board, wherein Red has to make a path between the top and bottom edges and Blue a path between the left and right edges (see Figure 1). The winner is the player that makes this connection between the two opposing edges the fastest.

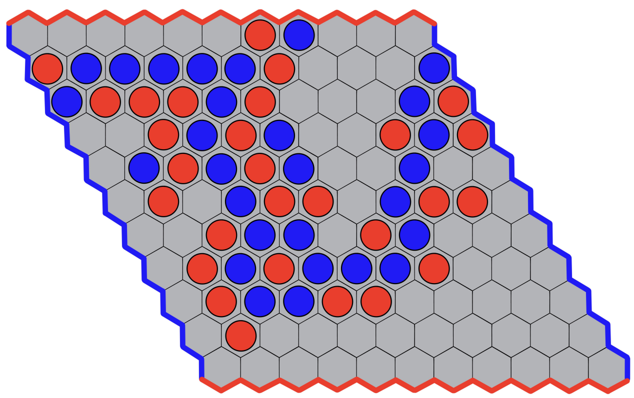


Figure 1: Example of 11×11 Hex board where Blue wins.

We can represent the game Hex as a graph. The hexagons are the vertices where hexagons sharing a common face are called *adjacent*.

Definition 2.1. S is an infinite graph with vertices

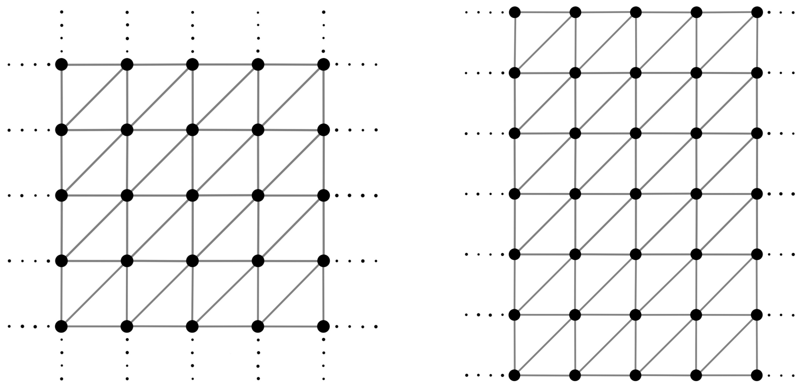
$$V = \{(i, j) : i, j \in \mathbb{Z}\}.$$

Two vertices $z = (z_1, z_2), w = (w_1, w_2) \in V$ are adjacent if and only if $z \neq w$, $|z_1 - w_1| \leq 1,\ |z_2 - w_2| \leq 1$, and $|(z_1 + z_2) - (w_1 + w_2)| \leq 1$.

Definition 2.2. S_n, with $n \in \mathbb{N}_{>0}$, is a horizontal strip of the infinite graph S restricted to the vertices

$$V_n = \{(i, j) : i, j \in \mathbb{Z}, 1 \leq j \leq n\} \subseteq V.$$

A small part of S and S_n can be seen in Figure 2(a) resp. Figure 2(b).



(a) S (b) S_n

Figure 2: Infinite graph S and horizontal strip S_n.

Definition 2.3. The $m \times n$ Hex graph $H_r(m, n)$ with $m, n \in \mathbb{N}_{>0}$ is the subgraph of S_n with vertices

$$V(m, n) = \{(i, j) : i, j \in \mathbb{Z}, 1 \le i \le m, 1 \le j \le n\}.$$

In Figure 3 an example of a Hex graph is given.

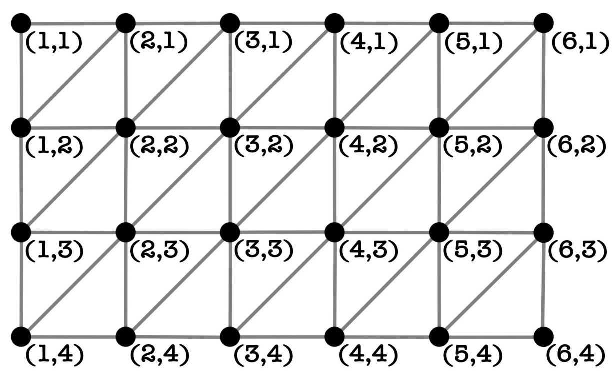


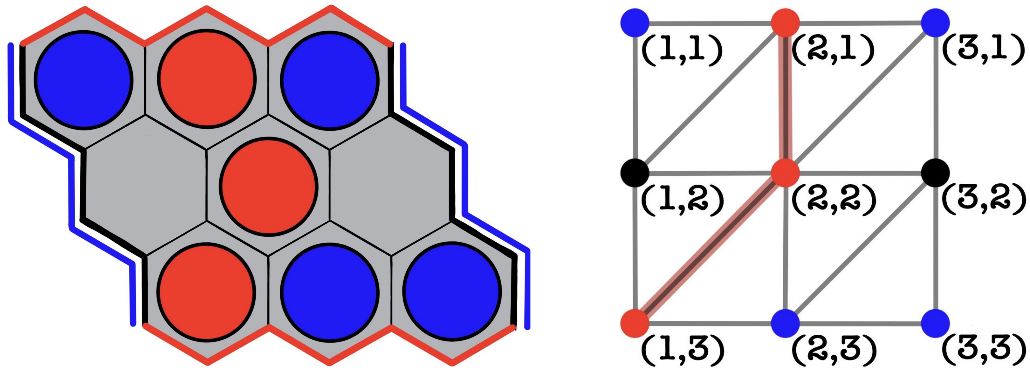
Figure 3: Hex graph $H_r(6, 4)$.

Note that a Hex graph is thus essentially the infinite graph S restricted to a finite width and height.

We are now able to concisely give a way to determine the winner of an $m \times n$ Hex game. Let the (partial) colouring $C \colon V(m, n) \to \{\text{red, blue}\}$ of a graph be any colouring of the vertices of $V(m, n)$.

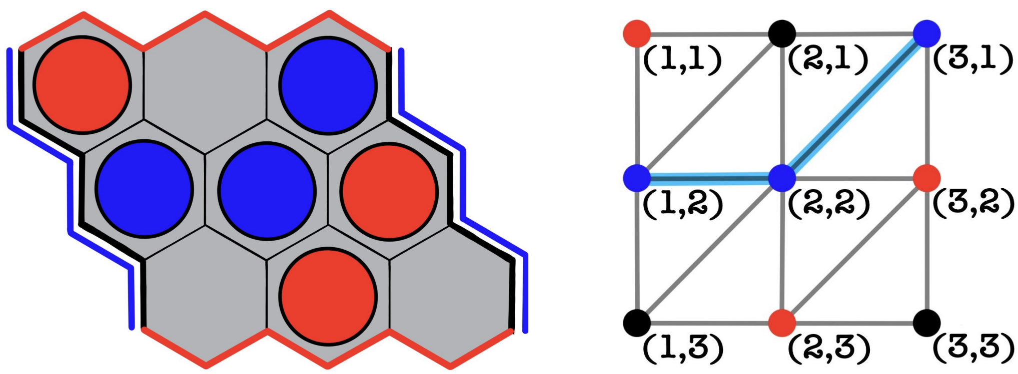
- A C-winning path for Red is any red path in C from some $z = (z_1, 1) \in V(m, n)$ to some $w = (w_1, n) \in V(m, n)$.
- A C-winning path for Blue is any blue path in C from some $z = (1, z_2) \in V(m, n)$ to some $w = (m, w_2) \in V(m, n)$.

An example of a Red (resp. Blue) winning path with their corresponding Hex graph can be found in Figure 4 (resp. Figure 5).



(a) Colouring of 3×3 Hex board (b) Colouring of $H_r(3,3)$

Figure 4: Example of Red winning path.



(a) Colouring of 3×3 Hex board (b) Colouring of $H_r(3,3)$

Figure 5: Example of Blue winning path.

2.2 Cylindrical Hex

The game we specifically look at is an expanded form of Hex, named Cylindrical Hex. In Cylindrical Hex, the board is wrapped around a cylinder making the left and right edges of the board connected. The rules for Cylindrical Hex are for the most part the same as for Hex, the players alternate turns placing a stone of their own colour on a cell until a player makes a winning path. Red still wants to make a path connecting the top and bottom edges. Blue now wants to make a circle around the cylinder, instead of just connecting two opposite edges. Note that the cylinder aspect also gives Red more room to make a winning path.

In order to define the graph for Cylindrical Hex we need a way to preserve the connection between the left and right sides of the board. To achieve this we can reduce the vertices of the infinite graph S_n using the map $\phi \colon V_n \to V(m,n)$ defined by $\phi(z_1, z_2) = (z_1 \bmod m, z_2)$.

Definition 2.4. $H_c(m, n)$ is the $m \times n$ Cylindrical Hex graph with vertices

$$V(m, n) = \{(i, j) : i, j \in \mathbb{Z}, 1 \le i \le m, 1 \le j \le n\}.$$

Two vertices $z = (z_1, z_2), w = (w_1, w_2) \in V(m, n)$ are adjacent if and only if there are two adjacent vertices $z', w' \in S_n$ with $z = \phi(z')$ and $w = \phi(w')$.

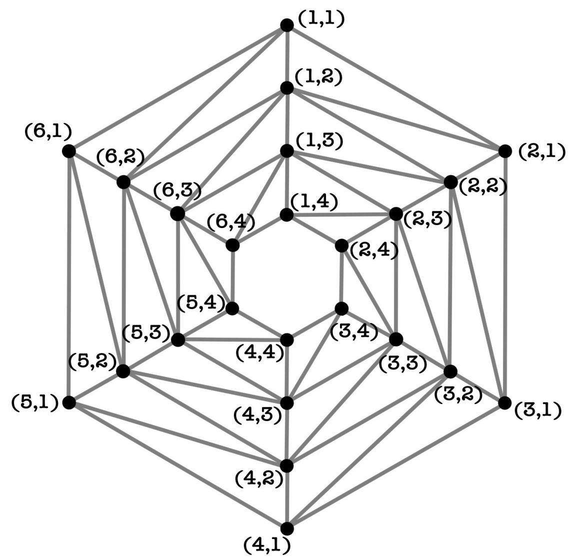


Figure 6: Cylindrical Hex graph $H_c(6, 4)$.

It can be seen that $H_c(m, n)$ has all edges of $H_r(m, n)$ with the addition of the edges between $(1, j)$ and (m, j) for $j = 1, \ldots, n$ and the edges between $(1, j)$ and $(m, j + 1)$ for $j = 1, \ldots, n - 1$. For an example we can compare Figure 3 and Figure 6.

In Cylindrical Hex we can again define a C-winning path for both Red and Blue. A C-winning path for Red is still any path in $H_c(m, n)$ of red vertices from some $z = (1, z_2) \in V(m, n)$ to some $w = (n, w_2) \in V(m, n)$. For Blue a winning path is a bit harder to define.

Definition 2.5. Let $C^*: V_n \to \{\text{red, blue}\}$ be a lifting of a colouring C of $H_r(m, n)$ to S_n by $C^*(z) = C(\phi(z))$ for $z \in V_n$.

Note that C^* is m-periodic in the horizontal coordinates, since for all $(i, j) \in V_n$ we have

$$C^*(i + m, j) = C(\phi(i + m, j)) = C(\phi(i, j)) = C^*(i, j).$$

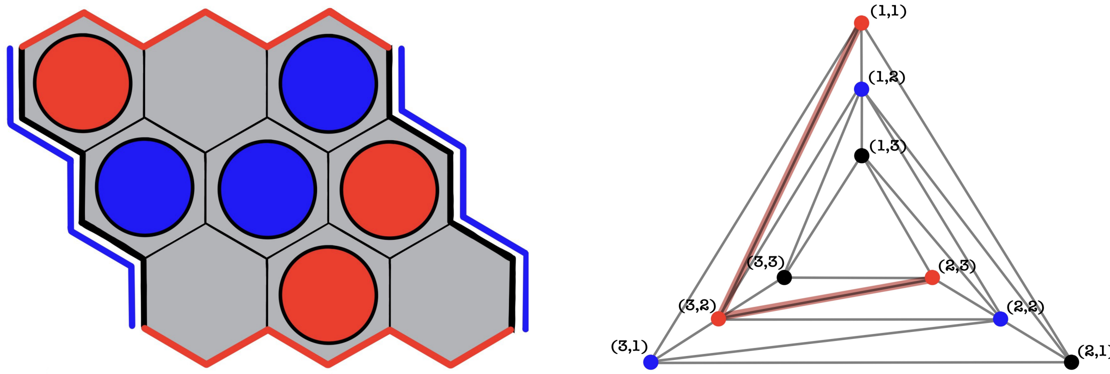
A lifting C^* is thus essentially copying the board C infinitely to the left and right of the graph $H_c(m, n)$ to create the horizontal strip S_n.

A C-winning path for Red and C-winning ring for Blue can now be defined:

- A C-winning path for Red is any red path in the lifting C^* of C from some $z = (z_1, 1) \in V_n$ to some $w = (w_1, n) \in V_n$ with $z_1 \bmod m = w_1 \bmod m$.

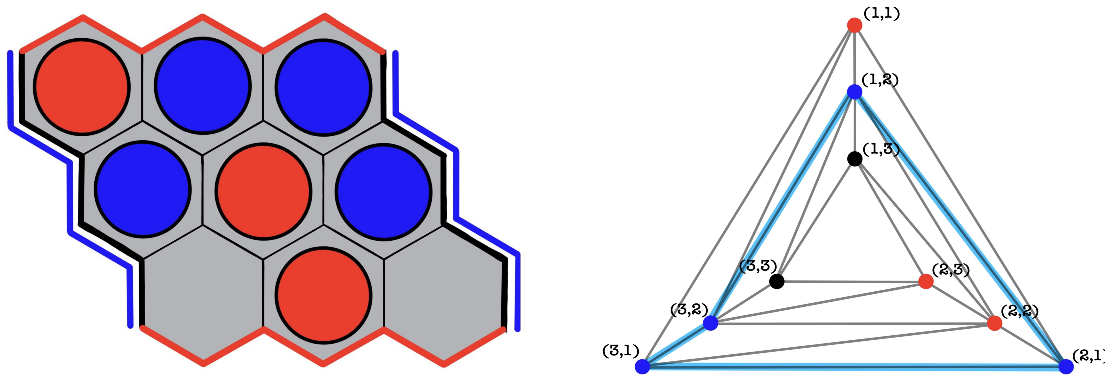
- A C-winning ring for Blue is any blue path in the lifting C^* of C from some $z = (1, z_2) \in V_n$ to some $w = (m + 1, w_2) \in V_n$ with $z_2 = w_2$.

An example of a Red (resp. Blue) winning path (resp. ring) with their corresponding Cylindrical Hex graph can be found in Figure 7 (resp. Figure 8).



(a) Colouring of 3×3 Cylindrical Hex board (b) Colouring of $H_c(3, 3)$

Figure 7: Example of Red winning path.



(a) Colouring of 3×3 Cylindrical Hex board (b) Colouring of $H_c(3, 3)$

Figure 8: Example of Blue winning ring.

It can be seen that the colouring of the boards in Figure 5(a) and Figure 7(a) are the same, while a different player wins depending on if the game is Hex or Cylindrical Hex.

2.3 Torus Hex

A further expansion of Cylindrical Hex which is interesting is Torus Hex. In Torus Hex the board of Cylindrical Hex is wrapped around a cylinder again making the top and bottom edges of the board also connected. In this process a torus is formed. The rules for Torus Hex are for the most part the same as for Cylindrical Hex, the players alternate turns placing a stone of their own colour on a cell until a player makes a winning path. Both Red and Blue now want to make a circle around the torus. Red wants to make a circle in the toroidal direction, and Blue in the poloidal direction (see Figure 9). Note that the torus aspect evens out the odds compared to the cylinder.

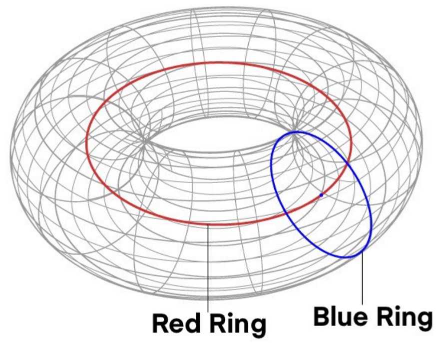


Figure 9: Torus representation of the board.

To define the graph for TORUS HEX we need to preserve both the connection between the left and right edges of the board, as the top and bottom edges of the board. To achieve this we can reduce the vertices of the infinite graph S using the map $\psi\colon V \to V(m,n)$ defined by $\psi(z_1, z_2) = (z_1 \bmod m, z_2 \bmod n)$.

Definition 2.6. $H_t(m,n)$ is the $m \times n$ TORUS HEX graph with vertices

$$V(m,n) = \{(i,j) : i,j \in \mathbb{Z}, 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Two vertices $z = (z_1, z_2), w = (w_1, w_2) \in V(m,n)$ are adjacent if and only if there are two adjacent vertices $z', w' \in S$ with $z = \psi(z')$ and $w = \psi(w')$.

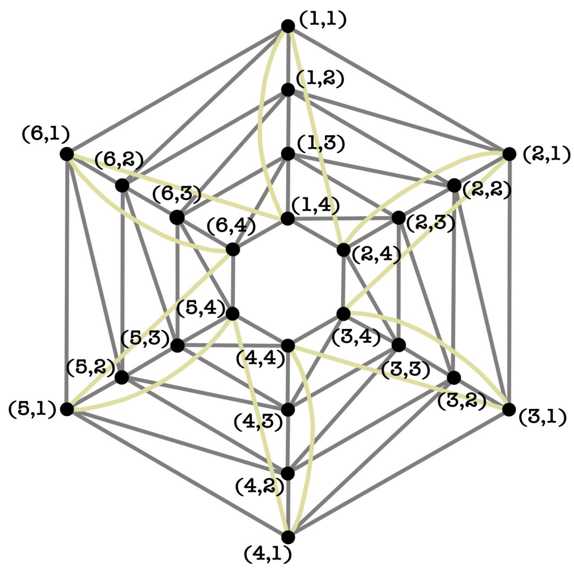


Figure 10: TORUS HEX graph $H_t(6,4)$.

We can see that $H_t(m,n)$ has all edges of $H_c(m,n)$ with the addition of the edges between $(i,1)$ and (i,n) for $i = 1,\ldots,m$ and the edges between $(i,1)$ and $(i+1 \bmod m, n)$ for $i = 1,\ldots,m$. For an example we can compare Figure 6 and Figure 10.

In Torus Hex we can again define a C-winning path for both Red and Blue.

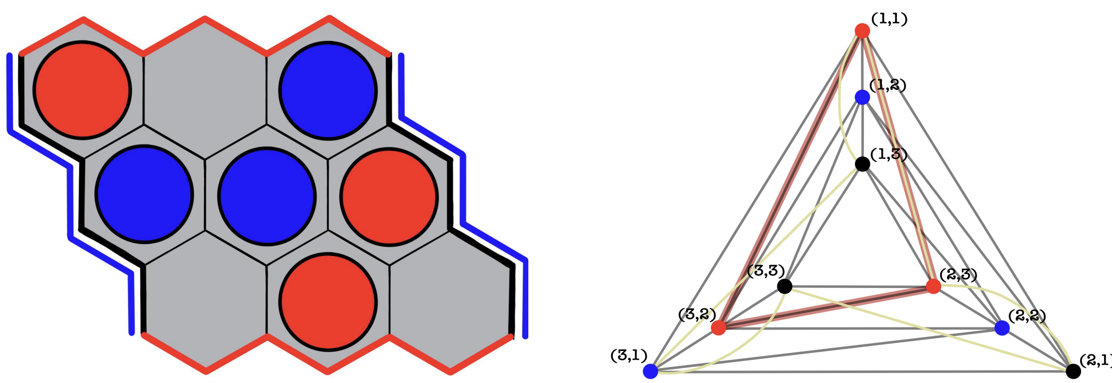
Definition 2.7. Let $C^\dagger \colon V \to \{\text{red}, \text{blue}\}$ be a lifting of a colouring C of $H_t(m,n)$ to S by $C^\dagger(z) = C(\psi(z))$ for $z \in V$.

Note that $C^\dagger$ is both n-periodic in the vertical coordinates, and m-periodic in the horizontal coordinates. A lifting $C^\dagger$ is thus essentially copying the colouring C infinitely to the top, bottom, left and right of the graph $H_t(m,n)$ to create the infinite graph S.

A C-winning ring for both Red and Blue can now be defined:

- A C-winning ring for Red is any red path in the lifting $C^\dagger$ of C from some $x = (z_1, z_2) \in V$ to some $w = (w_1, z_2) \in V$ with $z_1 \bmod m = w_1 \bmod m$ and $z_2 \bmod n = w_2 \bmod n$, but $z_2 \neq w_2$.
- A C-winning ring for Blue is any blue path in the lifting $C^\dagger$ of C from some $x = (z_1, z_2) \in V$ to some $w = (w_1, w_2) \in V$ with $z_1 \bmod m = w_1 \bmod m$ and $z_2 \bmod n = w_2 \bmod n$, but $z_1 \neq w_1$.

An example of a Red (resp. Blue) winning ring with their corresponding Torus Hex graph can be found in Figure 11 (resp. Figure 12).

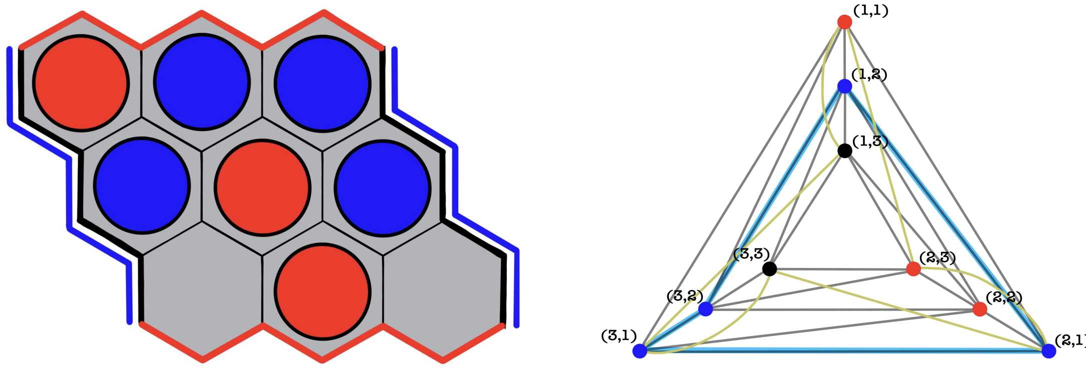


(a) Colouring of 3×3 Torus Hex board

(b) Colouring of $H_t(3,3)$

Figure 11: Example of Red winning ring.

It can be seen that the colouring of the boards in Figure 5(a), Figure 7(a) and Figure 11(a) are the same, just as the colouring of the boards in Figure 4(a) and 12(a). Depending on the variant of Hex, a different player wins each time, showcasing the different winning rules for each variant.



(a) Colouring of 3×3 Torus Hex board (b) Colouring of $H_t(3,3)$

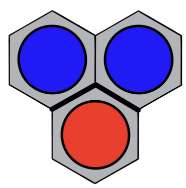
Figure 12: Example of Blue winning ring.

3 Always a Winner

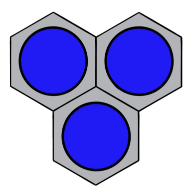
It is interesting to note that Hex, Cylindrical Hex, and Torus Hex always have a winner. So either Red or Blue can make a C-winning path or ring. For Hex a proof was given by J. Li [Ji11], but we will now see a new proof.

Theorem 3.1. For any colouring C of the $m \times n$ Hex graph $H_r(m, n)$ at least one player has a C-winning path.

Proof. Let the colouring C be of a full Hex board (all interior cells are coloured). Add a column of blue stones on the left and right border, and a row of red stones at the top and bottom border of the board. Adding these stones doesn't change the colouring C and thus doesn't change the winner. Note that under rotation and switching of colours, there are only two possible ways to colour three adjacent cells, as can be seen in Figure 13.



(a) Possibility 1



(b) Possibility 2

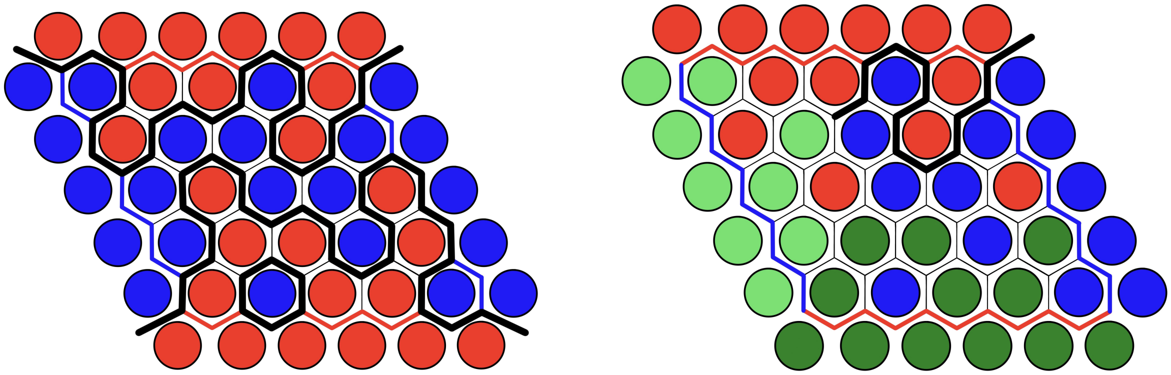
Figure 13: All possible ways to colour three adjacent cells

To show that either Red or Blue can always make a winning path, we recolour our stones. Recolour as dark green all red stones that are either along the bottom border or that reach the bottom border by a red path. Recolour as light green all blue stones that are either along the left border or that reach the left border by a blue path. There are now three mutually exclusive possible cases:

I) The top border is recoloured dark green. There thus exists a path of red stones from the bottom border to the top border and Red thus has a C-winning path.

II) The right border is recoloured light green. There thus exists a path of blue stones from the left border to the right border and Blue thus has a C-winning path.

III) Neither the top border or the right border is recoloured dark/light green, and neither Red or Blue has thus a C-winning path. This case is however not possible.
Starting from the upper right corner, between the red and blue stones, draw a line that separates the red and blue stones.
This line never stops along a border since the left and bottom border are recoloured green. The line can also not circle back on itself to the upper right corner, since from Figure 13 we can see that no possible colouring of three adjacent cells will allow this. So the line stops on an edge where both red, blue and dark/light green meet (an example can be seen in Figure 14(b)). However, if the colour of the third cell is dark green (rep. light green) then the adjacent red (rep. blue) stone would have been recoloured dark green (rep. light green).
The third case thus does not occur (see Figure 14).



(a) Example colouring C of 5×5 Hex board with lines dividing the colours.

(b) Example of case III for $H_r(5,5)$.

Figure 14: Example colouring C of 5×5 Hex board with the added columns of blue stones and rows of red stones along the border.

□

Using the theorem for Hex, we can give a proof for the case of Cylindrical Hex [AB91].

Theorem 3.2. Let C be any colouring of the $m \times n$ Cylindrical Hex graph $H_c(m,n)$. Then either Red has a C-winning path or Blue has a C-winning ring.

Proof. Let the colouring C^* be the lifting of the colouring C of $H_c(m,n)$. Define C' to be the restriction of C^* to the Hex graph $H_r(nm+1,n)$. According to Theorem 3.1, Red or Blue has a winning C'-path in $H_r(nm+1,n)$. We prove that the same player has a C-winning path or ring in $H_c(m,n)$. We have two cases:

1. Let $P = s^1, \ldots, s^k$ be the C'-winning path for Red in $H_r(nm+1,n)$. Since ϕ preserves the adjacency of the vertices, we have that $\phi(P)$ is the C-winning path for Red in $H_c(m,n)$.
2. Let $P = s^1, \ldots, s^k$ be the C'-winning path for Blue in $H_r(nm+1,n)$. Since P is a winning path for Blue, there exists a blue vertex $w^i \in P$ of the form $(im+1, y_i)$ for $i = 0, \ldots, n$ and $1 \leq y_i \leq n$. Since there are $n+1$ y_i variables, there are more of these variables than the number of values (namely n). This means that for some $1 \leq a, b \leq n$ with $a \neq b$ we have that $y_a = y_b$.
   Let $Q = w^a, \ldots, w^b$ be the section of P between w^a and w^b. We have that $\phi(w^a) = (am+1, y_a)$ and $\phi(w^b) = (bm+1, y_b)$. Since ϕ is m-periodic we can rewrite this as $\phi(w^a) = (1, y_a)$ and $\phi(w^b) = (m+1, y_b)$ with $y_a = y_b$. This gives us that $\phi(Q)$ is a C-winning ring for Blue in $H_c(m,n)$.

□

Using the theorem for Hex, we can also give a proof for the case of Torus Hex.

Theorem 3.3. Let C be any colouring of the $m \times n$ Torus Hex graph $H_t(m,n)$. Then either Red or Blue has a C-winning ring.

Proof. Let the colouring $C^\dagger$ be the lifting of the colouring C of $H_t(m,n)$. Define C' to be the restriction of $C^\dagger$ to the HEX graph $H_r(nm+1, nm+1)$. According to Theorem 3.1 either Red or Blue has a winning C'-path in $H_r(nm+1, nm+1)$. We will prove that the same player has a C-winning ring in $H_t(m,n)$. We have two cases:

1. Let $P = s^1, \ldots, s^k$ be the C'-winning path for Red in $H_r(nm+1, nm+1)$. Since P is a winning path for Red, in each row there is at least one red cell. Thus there exists a red vertex $w^i \in P$ of the form $(x_i, in+1)$ for $i = 0, \ldots, m$ and $1 \leq y_i \leq nm+1$. Given that C' is a restriction of the lifting $C^\dagger$, we have that C' is also m-periodic. Thus for the x_i's it follows that $0 \leq x_i \bmod m \leq m-1$.

   Since there are $m+1$ x_i's, there are more x_i's than there are possible places (namely m). This means that for some $1 \leq a, b \leq m$ with $a \neq b$ we have that $x_a \bmod m = x_b \bmod m$. Let $Q = w^a, \ldots, w^b$ be the section of P between w^a and w^b.

   We have that $\psi(w^a) = (x_a \bmod m, an+1)$ and $\psi(w^b) = (x_b \bmod m, bn+1)$. Therefore

$$an+1 = [\psi(w^a)]_2 \neq [\psi(w^b)]_2 = bn+1$$
$$[\psi(w^a)]_2 = an+1 \bmod n = bn+1 \bmod n = [\psi(w^b)]_2$$
$$[\psi(w^a)]_1 = y_a \bmod m = y_b \bmod m = [\psi(w^b)]_1$$

   This gives us that $\psi(Q)$ is a C-winning ring for Red in $H_t(m,n)$.

2. Let $P = s^1, \ldots, s^k$ be the C'-winning path for Blue in $H_r(nm+1, nm+1)$. Using the same reasoning as above but now using columns, we get that Blue has a C-winning ring for Blue in $H_t(m,n)$.

□

4 Strategies

Research has been done on winning strategies by multiple groups, among others Hayward and van Rijswijck [HvR06]; Hayward, Björnsson, Johanson, Kan, Po, van Rijswijck [HBJ+05]; Gardner [Gar59]; van Hees [vH22].

Most research and progress has been made for Hex. Some important findings for $m \times n$ Hex are:

- If $m \neq n$, the player with the smaller side can always win. Red (resp. Blue) should thus always be able to win if $n < m$ (resp. $m < n$).
- If $m = n$, the first player should always be able to win. We can proof this by using a strategy stealing argument. If the second player has a winning strategy, the first player can namely steal this strategy and use this strategy to win himself. This however only shows that the first player can win, but doesn't give a concrete winning strategy. Up to now, a 7×7 board is the largest board for which a concrete winning strategy has been determined [YLP18].

For Torus Hex, no research on winning strategies has been done.

4.1 Strategies for Cylindrical Hex

For Cylindrical Hex, two winning strategies for Red have been determined and proven.

4.1.1 Strategy $2k \times n$

For a board with an even width, a winning strategy for Red has been determined by Alpern and Beck [AB91]. We start with giving the algorithm that defines Red's strategy.

Algorithm 1 (*Red's Winning Strategy*)

Let $H_c(2k, n)$ be an $m \times n$ Cylindrical Hex *graph with $m = 2k$. Red follows the first applicable rule:*

I. If Blue has not played yet, then play anywhere.

II. If Blue's previous move was at cell $z = (i, j)$, play as follows:

1. at cell $T(z) = T(i, j) = (i + k \text{ mod } m, j)$,

2. anywhere.

The idea behind this strategy is that Red copies Blue's move on the other half of the board, thus making it that Blue can never form a winning ring (see Figure 15).

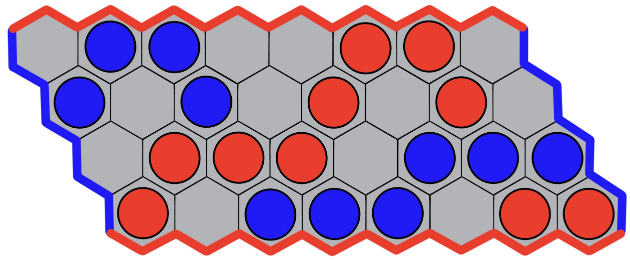


Figure 15: Example of Red using Algorithm 1 on a 8×4 Cylindrical Hex board.

To prove that this strategy ensures that Red wins any game of $2k \times n$ Cylindrical Hex, we introduce the following theorem [Zha19].

Theorem 4.1 (Jordan Curve Theorem)**.** Let C be a simple and closed curve (Jordan curve) in the plane $\mathbb{R}^2$. Then its complement, $\mathbb{R}^2 \setminus C$, consists of exactly two connected components, with C the boundary.

This means that a Jordan curve divides the plane into exactly two regions, one inside the curve and one outside, such that a path from a point in one region to a point in the other region must pass through the curve.

Using this, we can now prove that Algorithm 1 is a winning strategy for Red on a board with even width.

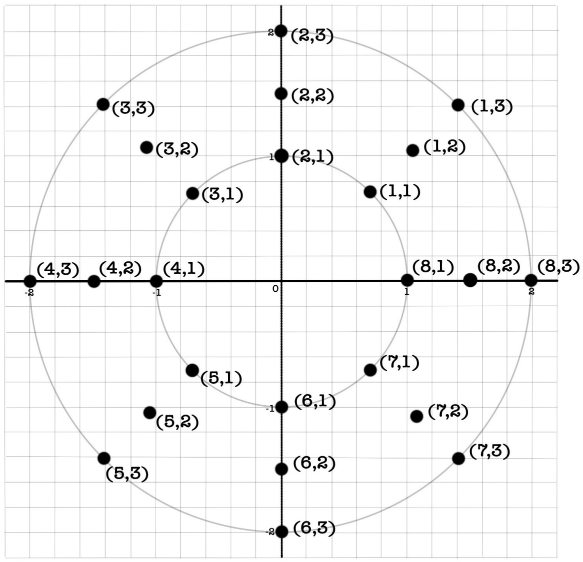
Proof that Algorithm 1 is a winning strategy [AB91]. We first note that this strategy is always feasible. For $z \in V(m, n)$:

- if $T(z)$ is already red, then z was already blue before Blue's move;
- if $T(z)$ is already blue, then z was already red before Blue's move.

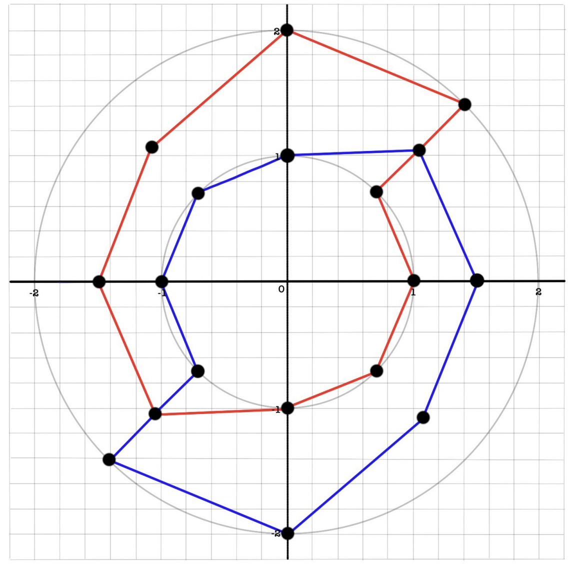
Let w be a winning ring for Blue on $H_c(m, n)$ with z^* the winning move. We know that $T(z^*)$ is uncoloured at the end of the game. Colour $T(z^*)$ red. Then $w^* = T(w)$ is a Red ring since the map T preserves adjacency. We can draw $H_c(m, n)$ on the annulus A with inner radius 1 and outer radius 2 by using the map $\gamma\colon V(m, n) \to \mathbb{R}^2$, $\gamma(i, j) = (r\cos\theta, r\sin\theta)$ with $r = \frac{n-2+j}{n-1}$ and $\theta = \frac{2i\pi}{m}$.

So $\gamma(w)$ and $\gamma(w^*)$ are Jordan curves which are disjoint, otherwise they would have to meet in a vertex which would mean that the vertex is both red and blue (see Figure 16). Theorem 4.1 requires then that one of the curves is inside the other. Since T however preserves the second coordinate j which determines the radius in γ, we have that both $\gamma(w)$ and $\gamma(w^*)$ have the same maximum radius. This means that it is impossible for one of the curves to be inside the other.

It is thus impossible for Blue to make a winning ring if Red follows the strategy, which means that Red is the winner. $\square$



(a) $H_c(8,3)$ drawn on the annulus.



(b) Example of the two circuits w and w^* on the annulus.

Figure 16: Example of an annulus A for Cylindrical Hex graph $H_c(8,3)$.

4.1.2 Strategy $3 \times n$

For a board of size $3 \times n$ a winning strategy for Red has also been determined by Huneke, Hayward, and Toft [HHT14]. We start with giving the algorithm that defines Red's strategy.

Algorithm 2 (*Red's Winning Strategy*)

Add the rings 0 and $n+1$ to the board, and colour them completely red (this allows the strategy to function on the rings 1 and n). Red follows the first applicable rule:

I. *If Blue has not played yet, then play anywhere.*

II. *If Blue's previous move was at cell (j,i), play as follows:*

   1. *in one of the rings $i-1$, i, or $i+1$ such that there is a red cell in ring i touching a red cell in ring $i-1$ and in ring $i+1$,*
   2. *in ring i or $i+1$ such that there is a red cell in ring i touching a red cell in ring $i+1$,*
   3. *in ring i or $i-1$ such that there is a red cell in ring i touching a red cell in ring $i-1$,*
   4. *in ring i,*
   5. *anywhere.*

The idea is that Red makes a locally winning path in every ring that Blue plays in, at the end making a winning path for the game.

It is important to notice that in $3 \times n$ Cylindrical Hex there exist for Blue, under mirroring and rotation, exactly two minimal ways to make a winning path: a ring, or for some ring index i and column index j (reduced mod 3) the set $\{(j+1, i+1), (j+2, i), (j+3, i), (j+3, i+1)\}$. Both are shown in Figure 17.

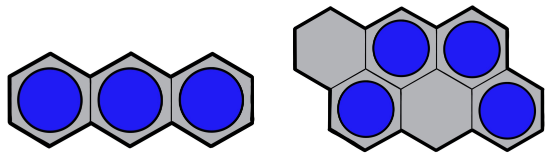


Figure 17: The two minimal blue winning patterns.

We can prove that Algorithm 2 is indeed a winning strategy for Red by going over the whole game tree as is done by Huneke, Hayward and Toft [HHT14]. Since this is very inefficient, we will now give a different new proof that this algorithm indeed guarantees that Red wins any game of $3 \times n$ Cylindrical Hex.

Proof that Algorithm 2 is a winning strategy. Let colouring C^* be the lifting of the colouring C of $H_c(3, n)$, wherein Red has followed Algorithm 2. Suppose that Blue has won.

Suppose that Blue has won by forming a ring via Blue's first minimal winning path. Red would have played a stone in this ring according to rules 1/2/3/4. So Blue can't have won via this winning path.

We thus know that Red has at least one red stone in each ring in the colouring C^*. Determine the longest possible path for Red starting at ring n, and say that the path ends in ring r. If $r = 1$ then Red has a winning path, thus suppose that $r \neq 1$.

Since ring r and ring $r-1$ are not connected by a red stone, we know that there are two blue stones directly above the red stone in ring r. If there are two red stones in ring r, then ring $r-1$ and ring r are connected since ring $r-1$ contains at least one red stone. Ring r would then not be the end of the path. So the other two stones in ring r are blue. We thus have colouring seen in Figure 18. We can see that Blue has a winning ring.

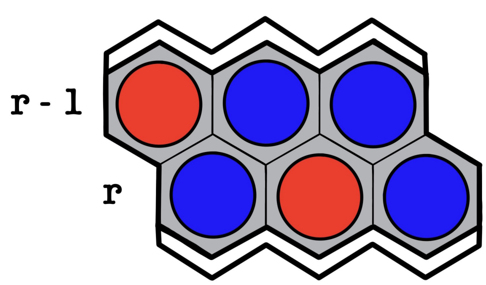


Figure 18: Blue has won.

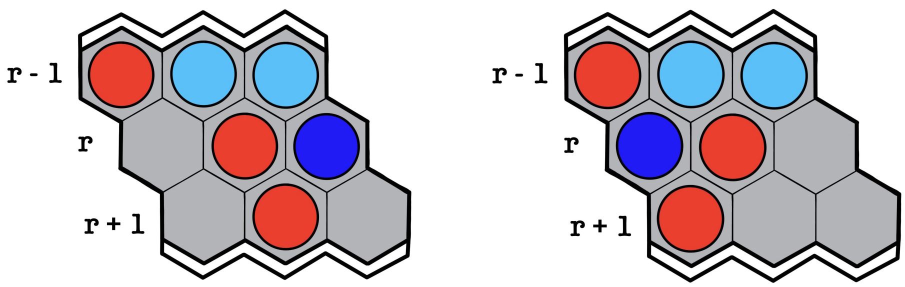
We prove that this situation could not have occurred if Red would have followed Algorithm 2.

There are two cases, namely that the red stone in ring r was placed before the red stone in ring $r-1$, or the other way around.

- Suppose the red stone in ring r was placed first. Then when the blue stone was placed in ring $r-1$, Red would have placed the red stone in ring $r-1$ so that it would connect to ring r according to rule 1/2. The path would have thus not have stopped at ring r. ↯
- Suppose the red stone in ring $r-1$ was placed first. If there is no red stone in ring $r+1$, when Blue places a stone in ring r, there is always a way for Red to place a stone in ring r according to rule 3. Thus suppose that there is a red stone in ring $r+1$.

  If Blue places his stone in ring r such that rule 1 can be followed, then Red would have followed rule 1 and rings $r-1$, r and $r+1$ would have been connected.

  Suppose that Blue would have placed his stone in ring r such that rule 1 can't be followed. There are two possible ways for this to occur (see Figure 19). The light blue stones represent the two places that Blue has placed his stone in ring $r-1$, one of which contains a blue stone and the other is still empty and can still be filled.



(a) Case I (b) Case II

Figure 19: Already a red stone in rings $r-1$, $r+1$ and rule 1 can't be followed.

  Suppose we have case I. When Blue places a stone in ring $r-1$, Red will place a red stone in ring r according to rule 1/2, connecting rings $r-1$ and r. If Blue places a stone in ring r, Red will place a red stone in ring $r-1$ according to rule 1, connecting rings $r-1$ and r. The same reasoning follows for case II. ↯

Blue could have thus not formed his winning ring from Figure 18, which means that Blue does not win. Using Theorem 3.2 we have that Red will win. ☐

4.2 Case $(2k+1) \times n$

We have just seen a winning strategy for Red for $(2k+1) \times n$ Cylindrical Hex with $k = 1$. For $k > 1$ this strategy does not work anymore. As can be seen in Figure 20, Blue is able to win if Red uses Algorithm 2 on a board with width bigger than 3. The whole game progression can be found in Appendix A.

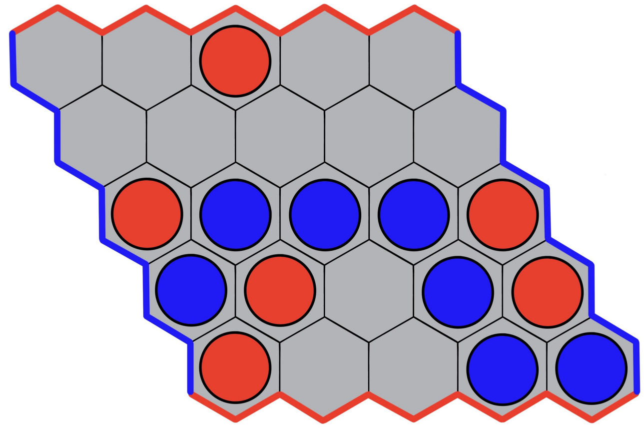


Figure 20: Example of Red using Algorithm 2 on 5×5 Cylindrical Hex board.

On a board of width 3, we have seen that Blue only has 2 minimal ways to form a winning ring. On a board of uneven width bigger than 3, the number of ways that Blue can win has grown much larger. This makes it harder to stop Blue from winning. For an uneven width > 5 no winning strategy has been found yet.

5 Pure Monte-Carlo Player

For our first experiment, we implement a Pure Monte-Carlo (MC) agent [KBTB14]. This agent evaluates each possible move of the player by doing a large number of simulated games called *playouts*. Each simulated playout starts at the current position of the board. A game in which both players do random moves is played until the winner can be determined. The move is then evaluated by the win rate of all the playouts, and at the end a move with the highest win rate is chosen. In the implementation the number of playouts was chosen to be 500. Another version of a Monte-Carlo simulation is the Monte-Carlo Tree Search (MCTS) [Wan21], which in addition to what was described above also pays attention to

We have done multiple experiments with this Monte-Carlo player to test how well the player does, and if we can determine a winning strategy for Red. To do this, we pitch two Monte-Carlo players against each other on a Cylindrical Hex game of differing width and height. We've done this for both Cylindrical Hex and Torus Hex.

5.1 Results Cylindrical Hex

We've run this experiment 100 times for each board size, letting both Red and Blue start. The results can be seen in Figure 21.

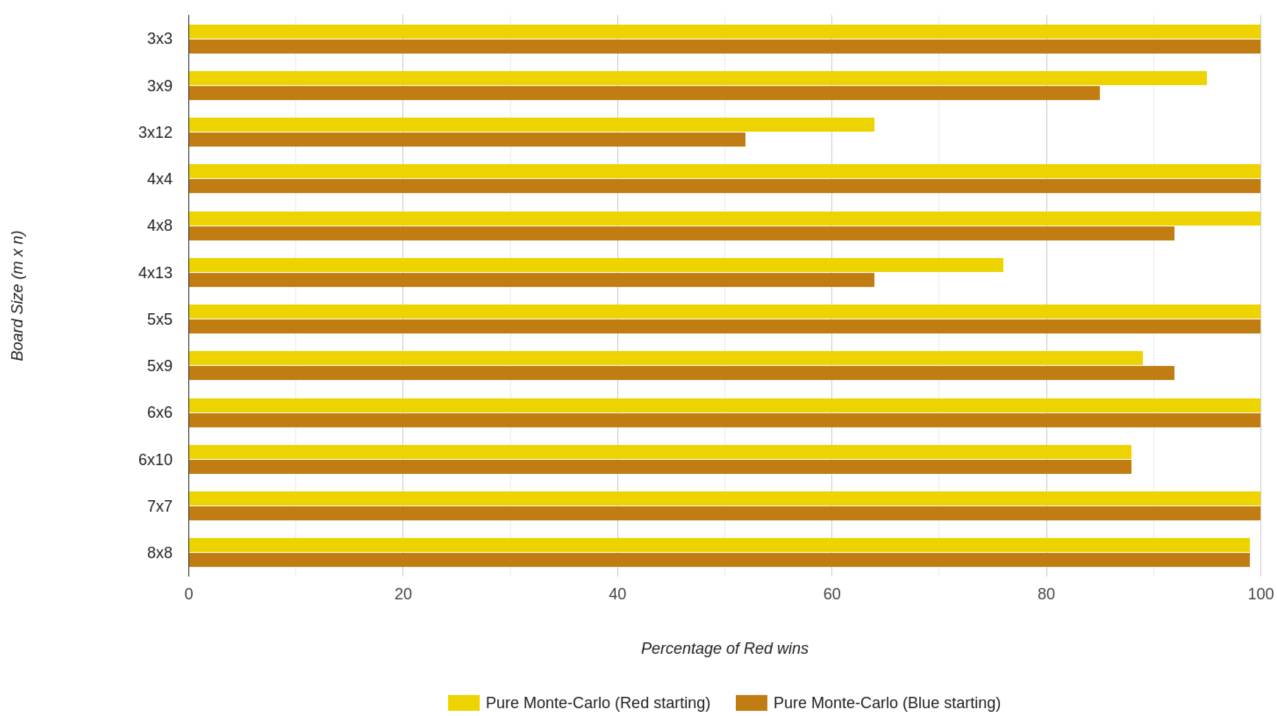


Figure 21: Percentage of Red wins using Monte-Carlo strategy on a Cylindrical Hex board.

It can be seen that Red has a large winning rate, which is consistent with the easier conditions for Red to win compared to Blue. When the height of the board becomes too large compared to the width of the board, Red starts losing more and more. The wider the board is, the smaller this difference between width and height needs to become to make Red start losing. It is also interesting to note that Red has a higher winning rate when it starts, in contrast to when Blue starts.

5.2 Results Torus Hex

We've run the same experiment also for Cylindrical Hex, again running each experiment 100 times and letting both Red and Blue start. The results can be seen in Figure 22.

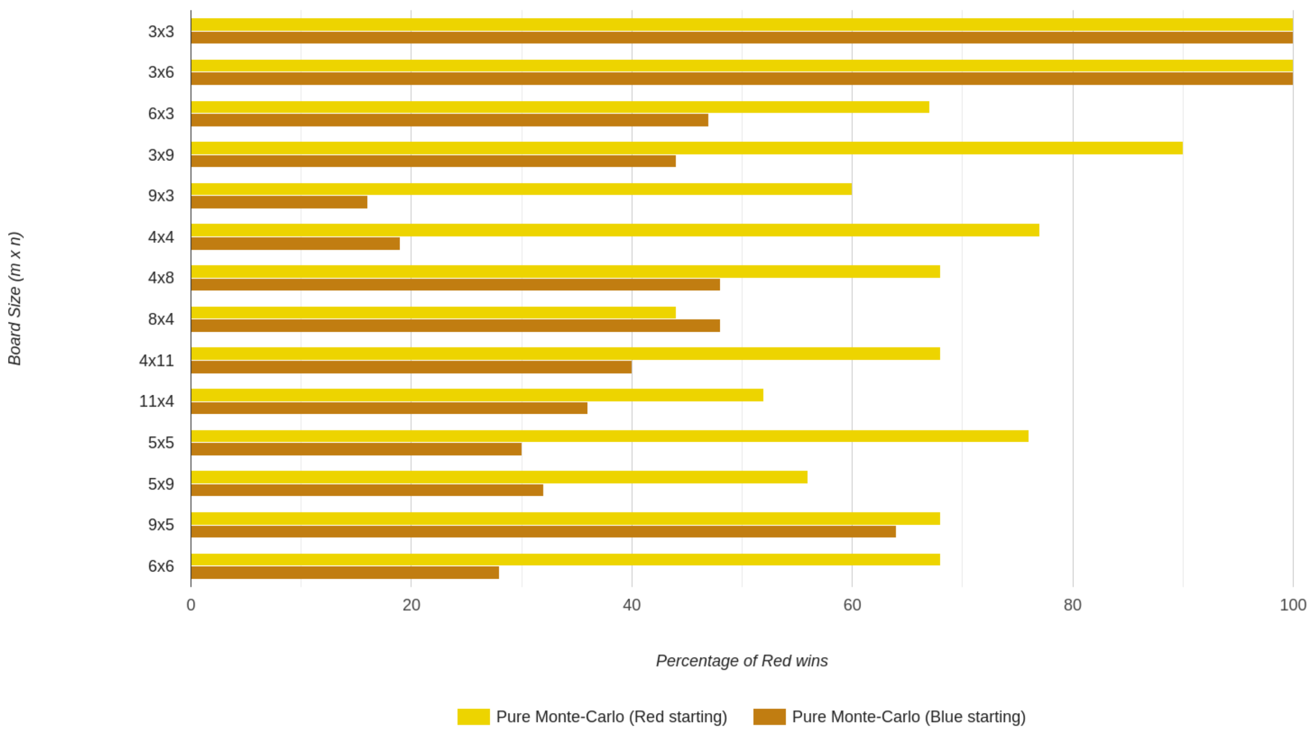


Figure 22: Percentage of Red wins using Monte-Carlo strategy on a Torus Hex board.

In Section 4, we've seen that in Hex the player that has the smaller side should be able to win. From the results in Figure 22 the opposite can be seen for Torus Hex. The winning rate for Red is namely higher when the height is larger, and when Red thus has a larger distance to travel. For boards of the same width as height, we can see that the first player has a higher winning rate, which is the same as in Hex.

5.3 Useful Techniques

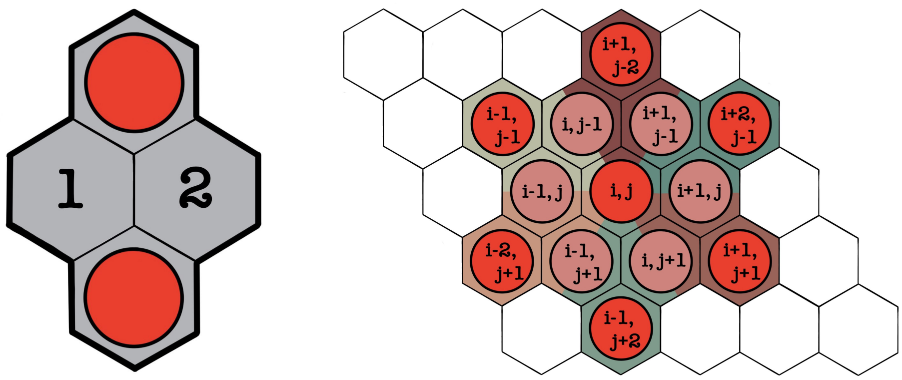
Each game that our Monte-Carlo player played was saved and researched to make out smart techniques. From observing the way that our Monte-Carlo player moves, especially for Cylindrical Hex, some key parts can be observed. It can be noticed that the strategy of the Monte-Carlo player resembles a further enhanced version of Algorithm 2. Unlike Algorithm 2, the Monte-Carlo player looks nothing like Algorithm 1. Since Algorithm 1 is not the most effective or obvious way for Red to win on a board with even width, our Monte-Carlo player chooses more different moves.

Using our observations, we can determine two possible techniques by which we can enhance Algorithm 2 to possibly make a winning strategy for Red on boards of uneven width > 3 for Cylindrical Hex.

- The first useful technique that can be observed from the Monte-Carlo player is the technique of building bridges.

  Definition 5.1. Two non-adjacent vertices $z, w \in V$ on S are connected through a bridge if there are two possible ways to connect the two vertices by colouring only one vertex $p \in V$.

  A bridge is thus essentially an indirect connection between two stones of the same colour that can always be formed. An example of a bridge can be seen in Figure 23(a), with Figure 23(b) showing all possible bridges.



(a) Example of a bridge. (b) All possible bridges of a cell (i, j) divided by background colour.

Figure 23: Bridges on S.

  Bridges are thus an effective method to lengthen an existing path and connecting rings. By using bridges instead of directly connecting two vertices, Red can connect three rings with one move instead of only two rings.

- Another useful technique that the current strategy does not make use of, is lengthening an existing path. If the previous Blue move was in a ring that is already encompassed in an existing Red path, Algorithm 2 gives that Red should place a stone in a random cell. A better way to use this move, however, is by lengthening an existing Red path even further. This makes it so that Red can win using fewer stones, and that Red is blocking Blue in more rings.

6 Pure Monte-Carlo Based Strategy for Red

Using the techniques we observed from the Monte-Carlo player in Section 5, we can deduce a new algorithm that enhances the existing Algorithm 2.

Algorithm 3 (*Red's Strategy*)

Add the rings 0 and $n+1$ to the board, and colour them completely red (this allows the strategy to function on the rings 1 and n). Red follows the first applicable rule:

I. *If Blue has not played yet, then play in the middle of the board.*

II. *If Blue's previous move was at cell (i, j), play as follows:*

1. *in cell (x, y) if it is a winning move,*
2. *if cell (i, j) was part of one of the existing red bridges, connect that bridge,*
3. *in ring $j-2$, $j-1$, j, $j+1$ or $j+2$ such that these rings are connected,*
4. *in ring $j-1$, j or $j+1$ such that these rings are connected,*
5. *in ring j, $j+1$ or $j+2$ such that these rings are connected,*
6. *in ring $j-2$, $j-1$ or j such that these rings are connected,*
7. *in ring j or $j+1$ such that these rings are connected,*
8. *in ring j or $j-1$ such that these rings are connected,*
9. *in cell (x, y) of ring $j-1$ such that rings $j-2$ and $j-1$ are indirectly connected and (x, y) has a possibility of forming a bridge with ring $j+1$,*
10. *in cell (x, y) of ring $j+1$ such that rings $j+1$ and $j+2$ are indirectly connected and (x, y) has a possibility of forming a bridge with ring $j-1$,*
11. *in cell (x, y) in ring $j-1$ such that rings $j-3$ and $j-1$ are indirectly connected and (x, y) has a possibility of forming a bridge with ring $j+1$,*
12. *in cell (x, y) in ring $j+1$ such that rings $j+1$ and $j+3$ are indirectly connected and (x, y) has a possibility of forming a bridge with ring $j-1$,*
13. *in ring j,*
14. *if ring k is the closest end of the existing path encompassing ring j, in ring $k+2$ such that these rings are connected,*
15. *if ring k is the closest end of the existing path encompassing ring j, in ring $k-2$ such that there is a red bridge connecting these rings,*
16. *if ring k is the closest end of the existing path encompassing ring j, in ring $k+1$ such that these rings are connected,*

17. if ring k is the closest end of the existing path encompassing ring j, in ring $k-1$ such that these rings are connected,

18. fill in any of the bridges,

19. anywhere.

This new Algorithm 3 uses as base the same structure as Algorithm 2, but is improved upon by the techniques from Section 5.3. We first of all make our first move (case I) more useful, by playing it in the middle of the board which gives a Red a central position on the board. The most important upgrade to the algorithm is the use of bridges, which allows the player to not only make a direct connection by connecting the stones, but also an indirect connection by forming bridges (both of which are considered connecting the rings). Cases II.14–17 are used to connect our longest Red path, and if a player uses this step it means that the ring in which Blue had played should be locally protected.

Another thing that this new strategy should improve upon is the number of stones Red has to play to win. A minimal number of stones for Red (resp. Blue) to be able to make a winning path/ring, is the length (resp. width) of the board. The closed we get to this number, the more optimal the strategy makes use of its stones to win.

7 Experiments

To test the new Algorithm 3, we again run some experiments for both Cylindrical Hex and Torus Hex.

7.1 Results Cylindrical Hex

To determine the win ratio we put a Red player using Algorithm 3 against a Blue player using the Pure Monte-Carlo strategy. Again we let both Red and Blue be the starting player. The results are shown in Figure 24.

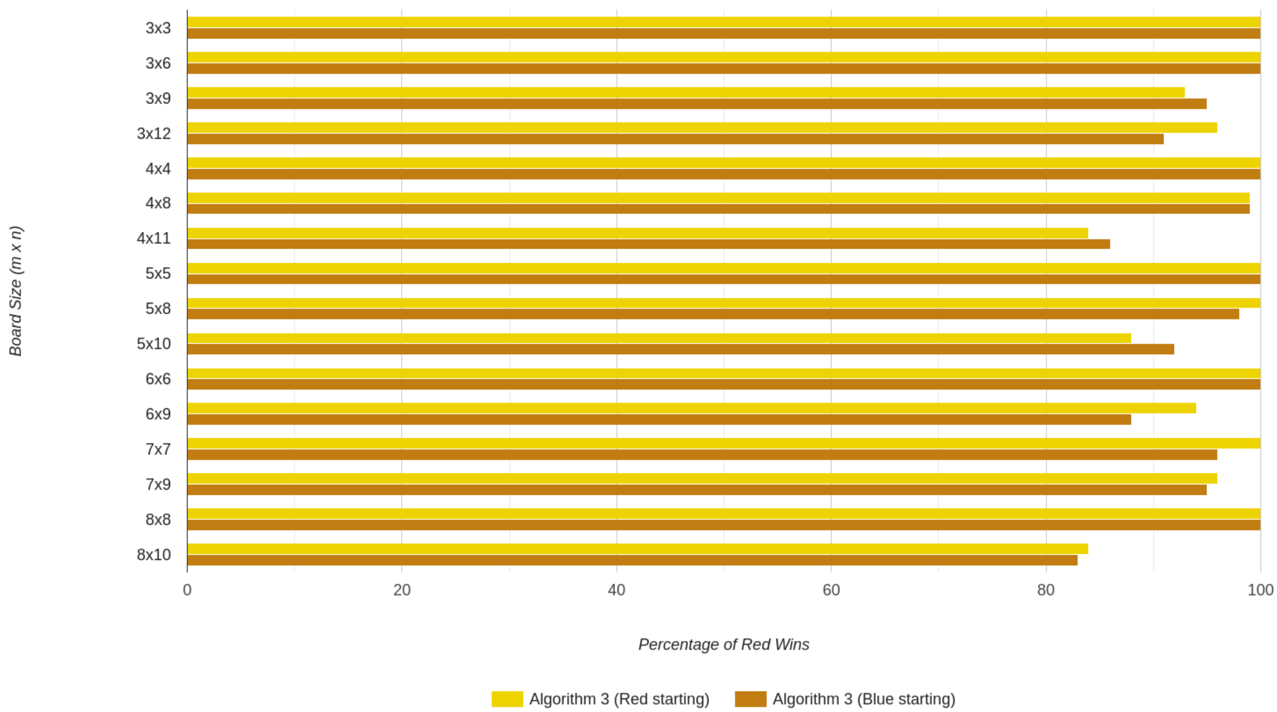


Figure 24: Percentage of Red wins using Algorithm 3 on a Cylindrical Hex board.

Comparing the results from the Monte-Carlo player from Section 5.1 with the results from our new Algorithm 3, we can see that Algorithm 3 gives a much higher winning rate for Red. Red still loses some games when the height becomes bigger than the width, but still has a higher winning rate compared to when it is using the Monte-Carlo strategy. The precise situation in which our Pure Monte-Carlo player loses is further discussed in Section 7.3.

Since Algorithm 3 is an enhanced version of Algorithm 2, it would also be interesting to see how these two algorithms compare to each other, and how much effect the enhancements have had. For this, we run the same experiment again, but now letting the Red player use Algorithm 2 (see Figure 25).

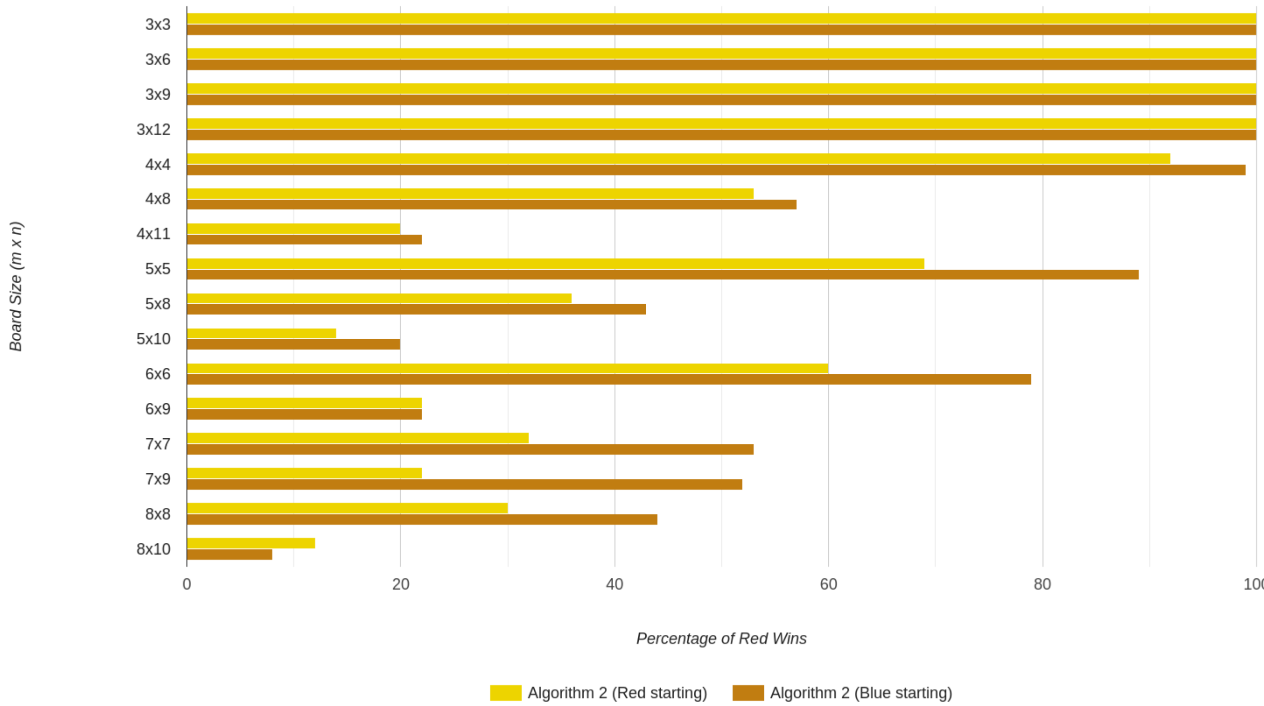


Figure 25: Percentage of Red wins using Algorithm 2 on Cylindrical Hex board.

From comparing Figure 24 and Figure 25, we can see that Algorithm 2 has a higher winning rate when the width is 3. While Algorithm 2 is specifically a winning strategy for width 3, it makes sense to have 100 percent winning rate for Red. When the board becomes wider, Algorithm 2 starts making Red lose more and more.

Another big difference between Algorithm 2 and Algorithm 3, is the difference in winning rate between Red starting and Blue starting. While for Algorithm 3 the winning rate does not differ much, for Algorithm 2 a clearly higher winning rate for Blue starting can be seen. This is because when Red is starting, Algorithm 2 lets Red place its first stone randomly while Algorithm 3 makes Red place its first stone in the middle of the board, which is a strategically better location.

Lastly, we also look at difference in the average number of Red stones played per game between Algorithm 2 and Algorithm 3.

From Figure 26 we can see that the difference in Red stones played isn't that big. This means that when Red loses more (like in Algorithm 2), it does not influence the length of the game that much.

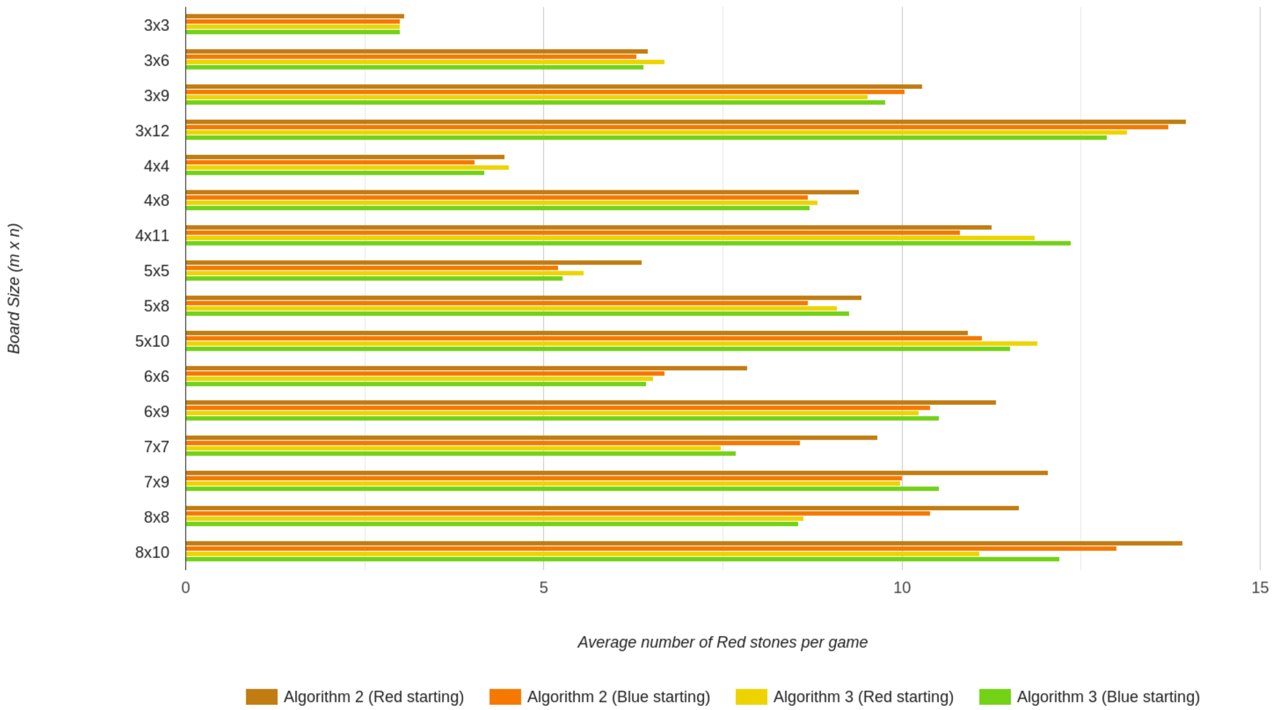


Figure 26: Number of Red stones played using Algorithm 2 and 3 on a Cylindrical Hex board.

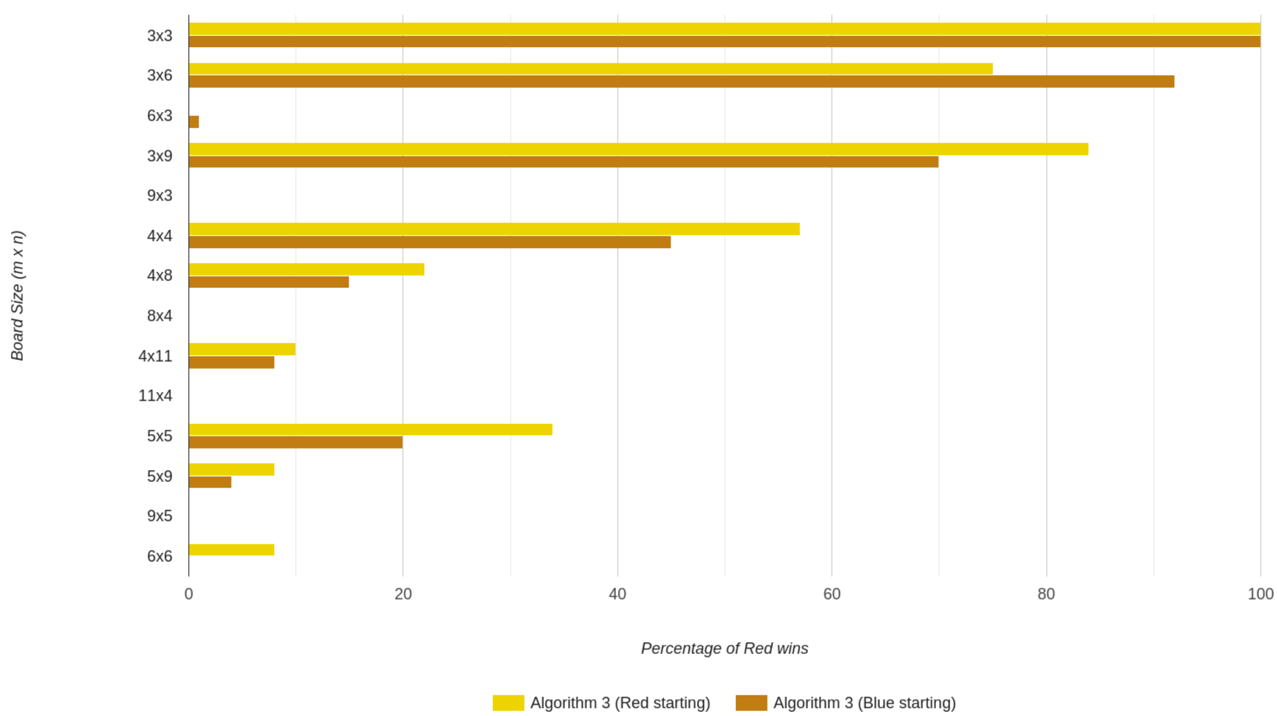


Figure 27: Percentage of Red wins using Algorithm 3 on a Torus Hex board.

7.2 Results Torus Hex

Same as in the experiments for Cylindrical Hex, we want to determine the winning rate for Red when using Algorithm 3 on Torus Hex.

As can be seen in Figure 27, Red loses most of if not all of the games. This is because Algorithm 3 does not take into account that the red path also needs to connect to form a circle around the torus. Without taking this into account, it forms a Red path from the top to the bottom border of the board, but only by chance do these ends connect to form a circle.

7.3 Losing Situation for Cylindrical Hex

When we look at the boards from the experiment on Cylindrical Hex, it can be observed that both Algorithm 2 and 3 have the same issue that Blue makes use of to win. The problem is that both algorithms neglect connecting existing local paths, which gives Blue the opportunity to win. Blue does this by letting Red make two paths (one from the top edge down and one from the bottom edge up) that do not connect in the middle of the board. This gap that then forms gives Blue the opening it needs to make a winning path. Another thing that Algorithm 3 forgets to take into account is that Blue can also make use of bridges.

In Figure 28 an example of such a board is given. The whole game can be found in Appendix B.

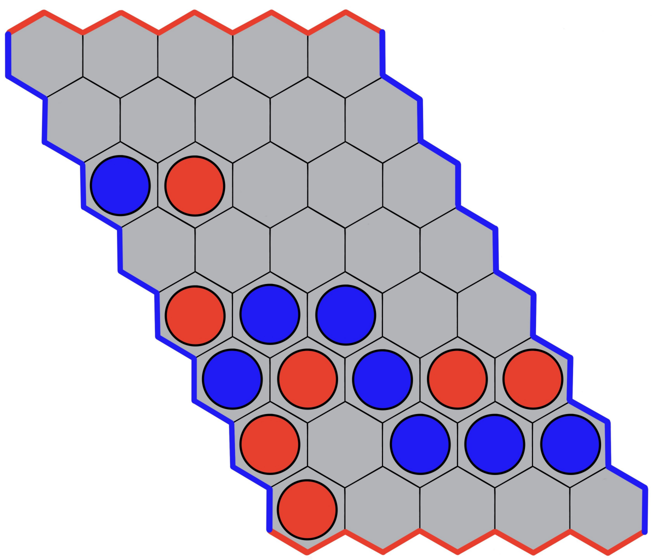


Figure 28: Example of a board in which Blue wins.

Algorithm 3 is less affected by this, since it takes more rings into account when making its decision. Still, when the height/width become too large, the chance for this situation too occur becomes larger and larger.

8 Conclusions and Further Research

In this thesis we considered the base game Hex and two versions of the game, named Cylindrical Hex and Torus Hex, from which Torus Hex hasn't been researched before. We defined graphs to represent these games, with which we proved that all versions always have a winner, giving a new proof for Torus Hex. We saw some existing strategies for Cylindrical Hex for boards of even width and width 3, which we gave a new proof showing that this strategy indeed always wins. After having seen why these strategies for Cylindrical Hex don't work for boards of uneven width > 3 , we started experimenting using a Pure Monte-Carlo player.

Using observations made from the Pure Monte Carlo player, we have determined a new algorithm which builds on an existing algorithm for Cylindrical Hex with width 3. We tested the performance of both the existing algorithm and enhanced algorithm on different board sizes for both Cylindrical Hex and Torus Hex, letting both Red and Blue start. While Algorithm 2 has a possibility of winning for Cylindrical Hex, greater board sizes make it so that Blue is also able to win. In comparison, Algorithm 3 has a much higher winning percentage. But still when the height of the board becomes much bigger than the width, Blue also starts winning. For Torus Hex, Algorithm 3 does not take into account that Red also needs to make a ring around the torus. This leads to Red losing a lot on bigger boards, since the chance that Red accidentally connects the path becomes much smaller.

For future research, it is definitely worthwhile to further improve Algorithm 3. This could lead to a better algorithm that takes connecting multiple paths better into account, which means a higher winning percentage for Red for Cylindrical Hex. For Torus Hex, this also means taking into account the new winning condition for Red. Because of time and hardware limitations, the experiments could not be done on larger boards within reasonable time. For the same reason a better enhanced version of Algorithm 3 has not been deduced.

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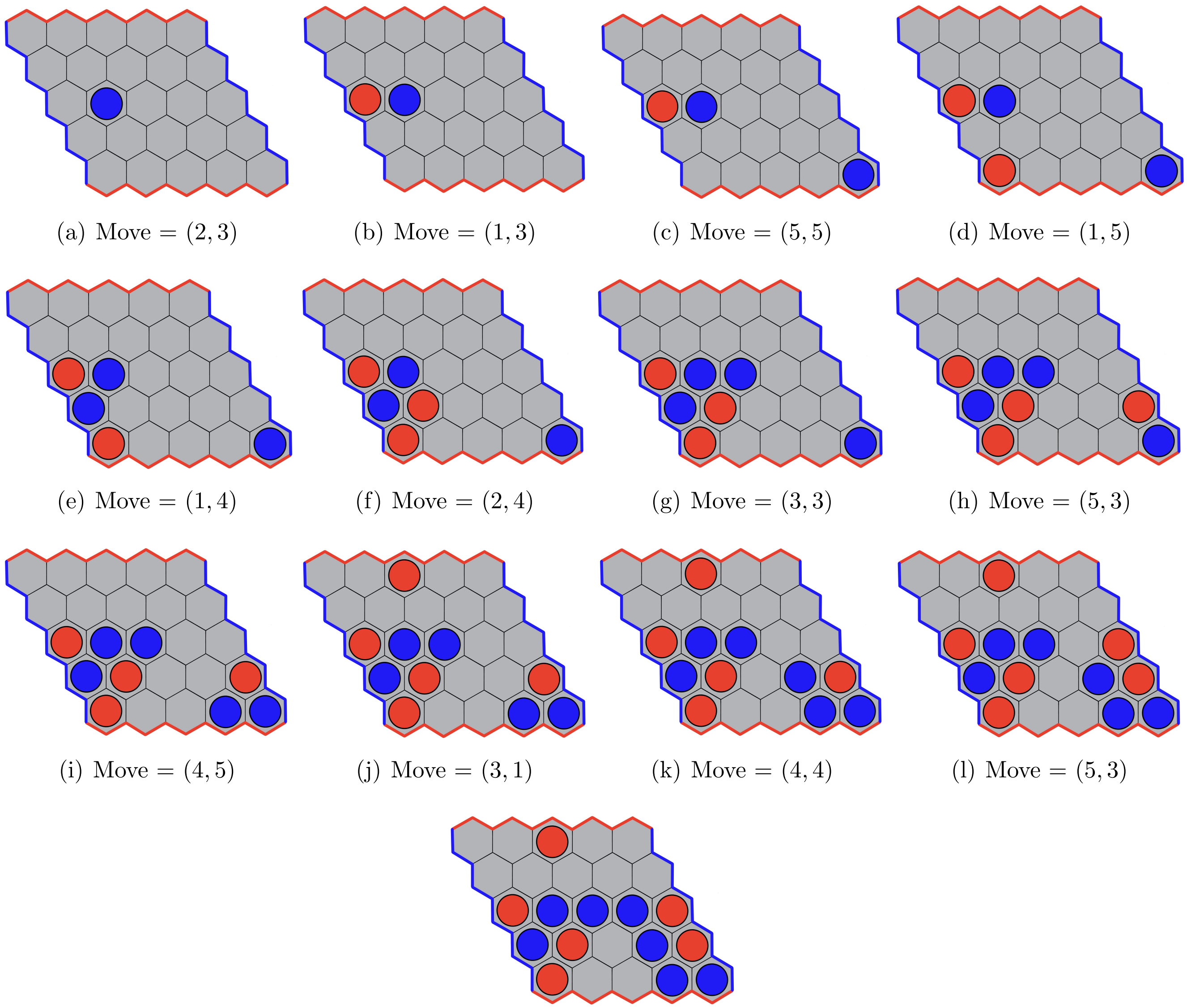
A Game Progression of 5×5 Cylindrical Hex Algorithm 2



(a) Move = $(2, 3)$ (b) Move = $(1, 3)$ (c) Move = $(5, 5)$ (d) Move = $(1, 5)$

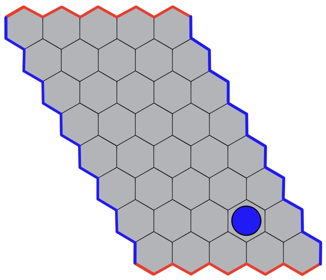
(e) Move = $(1, 4)$ (f) Move = $(2, 4)$ (g) Move = $(3, 3)$ (h) Move = $(5, 3)$

(i) Move = $(4, 5)$ (j) Move = $(3, 1)$ (k) Move = $(4, 4)$ (l) Move = $(5, 3)$

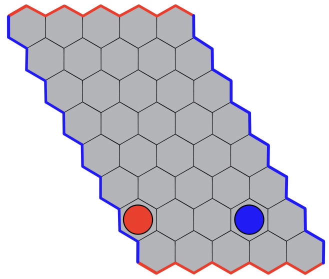
(m) Move = $(4, 3)$

Figure 29: Game progression of 5×5 Cylindrical Hex with Red using Algorithm 2.

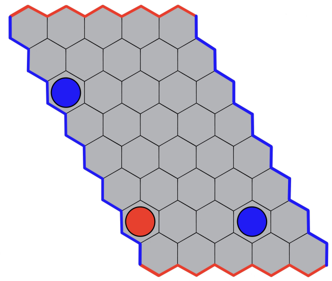
B Game Progression of 5×8 Cylindrical Hex Algorithm 3



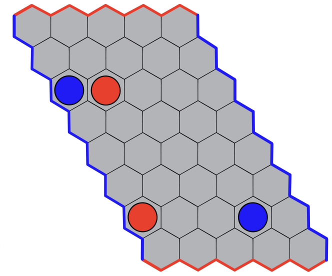
(a) Move = $(4, 7)$



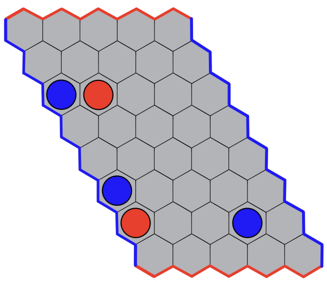
(b) Move = $(1, 7)$



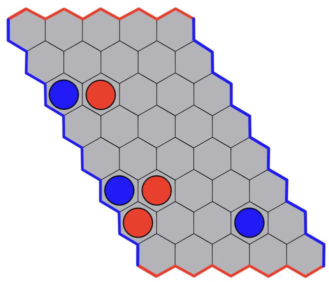
(c) Move = $(1, 3)$



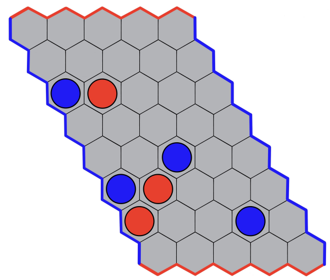
(d) Move = $(2, 3)$



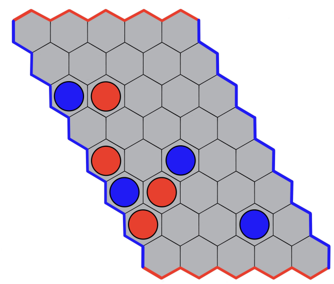
(e) Move = $(1, 6)$



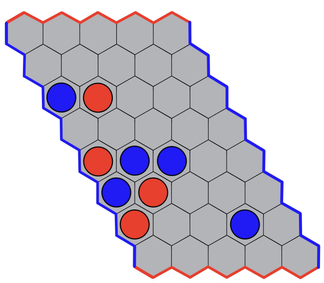
(f) Move = $(2, 6)$



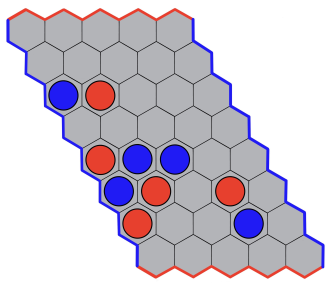
(g) Move = $(3, 5)$



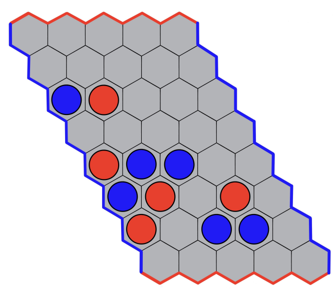
(h) Move = $(1, 5)$



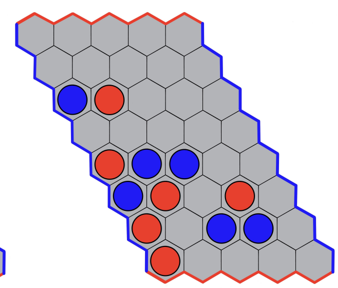
(i) Move = $(2, 5)$



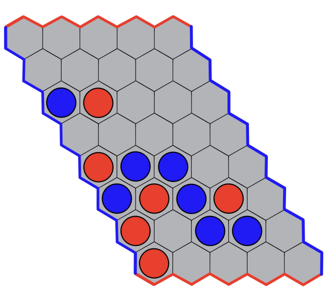
(j) Move = $(4, 5)$



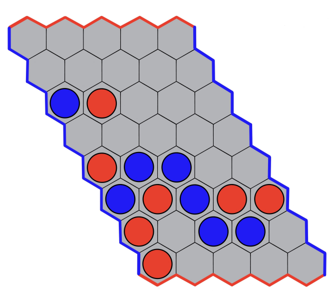
(k) Move = $(3, 6)$



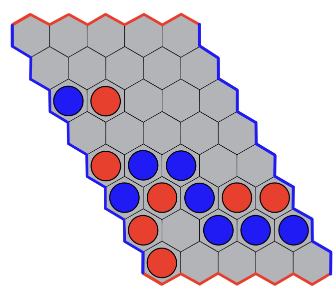
(l) Move = $(1, 7)$



(m) Move = $(3, 5)$



(n) Move = $(5, 5)$



(o) Move = $(5, 6)$

Figure 30: Game progression of 5×8 Cylindrical Hex with Red using Algorithm 3.