Universiteit Leiden
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Opleiding Wiskunde & Informatica

Finite accessibility of Higher-Dimensional Automata and unbounded parallelism of their languages

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BACHELOR THESIS

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08/12/2022

Abstract

Higher-dimensional automata are a formalism to model the behaviour of concurrent systems. They are similar to ordinary automata but allow transitions in higher dimensions, effectively enabling multiple transitions to be executing at the same time. As ordinary automata generate string languages HDA generate languages of pomsets with interfaces. We develop some properties of the category of precubical sets and event consistent precubical sets, which provide the underlying structure of HDA. We show that the category of higher-dimensional automata is not cocomplete but does have all small coproducts and filtered colimits. We show that a HDA is compact if and only if it is finite, that every HDA can canonically be expressed as a filtered colimit of compact/finite HDA and that the category of HDA is finitely accessible. We extend the definition of tracks from finite HDA to all HDA and use this to introduce their languages. We show that the language of a colimit of HDA contains the colimit or union of the languages of the individual HDA, and that in the case of a coproduct or filtered colimit it is equal to it. We define parallel composition in the form of a tensor product and show that the tensor product of colimits of HDA is a colimit of the tensor product of their respective diagrams. Lastly we show that the repeated parallel composition can be expressed as the coproduct or filtered colimit of a chain of finite parallel compositions.

Contents

1 Introduction

Higher-dimensional automata (HDA) form a model of "true" concurrency compared to the interleaving concurrency in ordinary automata. HDA were originally introduced by Pratt [Pra91] and van Glabbeek [vG91]. Where ordinary automata generate languages of strings, HDA generate languages of partially ordered multisets with interfaces (ipomsets).
The string languages that can be generated by finite ordinary automata are called the regular languages. The theorem known as the Kleene theorem states that these regular languages are closed under union, serial composition and serial Kleene star, and that every non-empty regular language can be generated by basic languages (those languages $L = \{"a"\}$ for any symbol "a") under these three operations. This gives rise to the question whether such a Kleene theorem exists for HDA and ipomset languages as well.
This question was explored by Fahrenberg, Johansen, Struth and Ziemiański in their papers [FJSZ21] and [FJSZ22]. For this, two new operations were introduced: the parallel composition and the parallel Kleene star. The parallel composition is what separates the interleaving concurrency of ordinary automata from the "true" concurrency of HDA: If we have strings "a" and "b" then their interleavings would be the strings "ab" and "ba" (as represented by the linearly ordered multisets $(a \to b)$ and $(b \to a)$). However in both cases the strings are still ordered. We say that "a" comes after "b" or "b" comes after "a", they do not take place at the same time. The parallel composition of the languages $\{(a \to b)\}$ and $\{(b \to a)\}$ gives us the ipomset language $\left\{(a \to b), \begin{pmatrix} a \\ b \end{pmatrix}, (b \to a)\right\}$. Instead of linearly ordered sets we have partially ordered sets (which become partially ordered multisets with interfaces later), the elements a and b in $\begin{pmatrix} a \\ b \end{pmatrix}$ are not ordered and are therefore "happening at the same time".

Figure 1: To the left an automata where its language represents the interleavings of the strings "a" and "b" and to the right a HDA where its language represents the parallel composition.

The automaton on the left in the figure above is an ordinary automata (note that all ordinary automata can be represented as HDA, as shown in [vG06]). The automaton on the right is the same automata but with an added 2-dimensional cell, represented by the grey square. For ordinary automata there are nodes and edges which represent the transitions between the nodes. In a HDA an execution can follow any paths admitted by the geometry. The ipomset $\begin{pmatrix} a \\ b \end{pmatrix}$ represents the execution path that starts in the bottom left node, transitions to the surface and ends in the top right node. The parallel composition is the operation on the languages, while the tensor product is the operation on the HDA that generate those languages such that the tensor product of two HDA generates the language that is the parallel composition of their two languages.

The languages of finite HDA are closed under the parallel composition, as shown in [FJSZ21]. However they are not closed under the parallel Kleene star. The tensor product of arbitrary many HDA generally results in cells of arbitrary large dimension, which means that the resulting HDA is not finite. The original goal of this thesis was to find ways to give HDA extra structure such that something similar to the Kleene theorem might be constructed using a condition that is weaker than finiteness. While this goal was not really achieved, we were able to prove some structural properties for the category of HDA and the languages of HDA. Among other things, we show a way to construct a colimit from a diagram of HDA, that the category of HDA is finitely accessible, that a HDA is finite if and only if it is compact and that the category of HDA is not cocomplete. We show that the language of a colimit of HDA contains the colimit or union of the languages of the individual HDA, and that in the case of a coproduct or filtered colimit it is equal to it. We define parallel composition in the form of a tensor product and show that the tensor product of colimits of HDA is a colimit of the tensor product of their respective diagrams. Lastly we show that the repeated parallel composition can be expressed as the coproduct of a chain of finite parallel compositions.

In section 2, we introduce precubical sets and their morphisms, which form the basis for HDA. We develop some of their properties, one of which is that a precubical set is compact if and only if it is finite. With this we also introduce the category of precubical sets $\mathtt{Set}^{\square^{\mathrm{op}}}$, and prove it is locally finitely presentable. In section 3 we introduce the condition of event consistency and see if the properties we proved for precubical sets in the previous section apply to event consistent precubical sets as well. In section 4 we are finally able to introduce higher-dimensional automata. In section 5 we formally introduce ipomsets and define the gluing composition or serial composition. In section 6 we introduce tracks, which are the execution paths mentioned before, of which their labelling generates the ipomsets and therefore the languages of higher-dimensional automata. The languages of HDA and their interactions with colimits are covered in section 7. In section 8 we cover the tensor product of HDA and their colimits.

2 Precubical sets

In this section we introduce precubical sets, which provide the underlying structure of higher-dimensional automata. We prove some theorems about colimits of precubical sets and compact precubical sets. We also prove that the category of precubical sets is locally finitely presentable and explain what this means.

2.1 Definition of precubical sets

We start with a simplified definition:

Definition 2.1. *X is a precubical set if*

- *X is a family of sets $\{X^n\}_{n\in\mathbb{N}}$.*
- *X has elementary face maps $\delta^n_{\nu,a} : X^n \to X^{n-1}$ for all $\nu \in \{0,1\}$, $a, n \in \mathbb{N}_{\geq 1}$ with $a \leq n$. These maps must satisfy the following condition:*

$$\delta^{n-1}_{\nu,a} \circ \delta^n_{\mu,b} = \delta^{n-1}_{\mu,b-1} \circ \delta^n_{\nu,a}$$

for all $n \in \mathbb{N}_{\geq 2}$*,* $\nu, \mu \in \{0, 1\}$ *and* $a, b \in \mathbb{N}$ *with* $1 \leq a \leq b \leq n$*.*

Here X^n for each $n \in \mathbb{N}$ is a set containing the n-dimensional elements of the precubical set X. These X^n are given structure by the elementary face maps δ. It is easiest to look at the elements of X^n as n-dimensional hypercubes, which means that the elements of X^0 are nodes, the elements of X^1 are edges, the elements of X^2 are squares, the elements of X^3 are cubes and the elements of X^n for all $n \in \mathbb{N}_{\geq 4}$ are n-dimensional hypercubes. The elementary face maps then identify the faces of these hypercubes. Applying the elementary face maps on an element of X^3, which is a cube, gives us in total 6 possibly different squares. Applying the elementary face maps on a square gives us 4 edges and applying them again on these edges gives us the nodes.

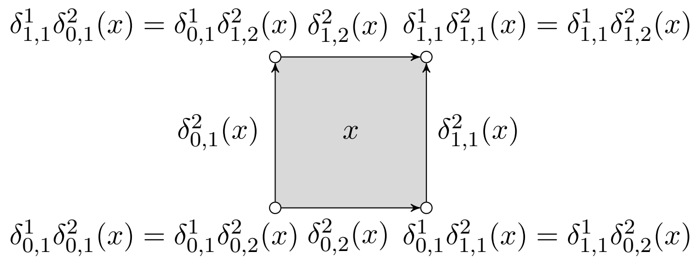


Figure 2: The square $x \in X^2$, its four elementary faces and their four corners.

We will refer to the compositions of zero or more elementary face maps as just face maps and for certain $n \in \mathbb{N}$, $x \in X^n$ we will refer to the elements that can be reached by the elementary face maps as faces. Since there are two possible choices for ν and n possible choices for a we see that every face can be represented by a combination of ν and a for a total of $2n$ possible faces for an element in X^n.

We say possible because these faces might not always be unique. In the example in figure 2 we could identify $\delta^2_{0,1}(x) = \delta^2_{1,1}(x)$ which would make the precubical set into a cylinder or tube. If we were to identify $\delta^2_{0,1}(x) = \delta^2_{0,2}(x)$ instead we would get a confusing shape that looks something like a cone. Note that by identifying edges we also identify the nodes that are attached to the edges. However we can also identify the nodes without identifying the edges, which would give us the case of parallel edges that have the same starting and ending nodes. For cases of precubical sets where not all of the faces are unique it's easier to first imagine the case in which they are unique, and then identify elements with each other until one gets the desired precubical set.

Something we mentioned were the starting and ending nodes of an edge. In the case of figure 2 the bottom edge starts at the bottom-left node and ends at the bottom-right node. Here the starting node is reached by $\delta^1_{0,1}$ and the ending node is reached by $\delta^1_{1,1}$. For the elementary face maps $\nu = 0$ gives us the start node and $\nu = 1$ gives us the end node. The same works for the square $x \in X^2$, only in two directions. Here $a \in \{1, 2\}$ decides which direction, 1 being vertical and 2 being horizontal, and $\nu \in \{0, 1\}$ decides if we go to the start or the end.

For higher-dimensional hypercubes it works similarly. If we have an n-dimensional element there are n possible directions or dimensions with each a start and an end. By taking $\nu = 0$ and $a = 1$ we "remove" the first dimension by moving to its start or end. After that the second dimension becomes the first, the third becomes the second etc. This is why in definition 2.1 we have $b - 1$ on the right side of the equation. For these simple n-dimensional precubical sets it's possible to see every element as an n-dimensional vector of elements in $\{0, \lrcorner\ulcorner, 1\}$, where 0 denotes the beginning of a dimension, 1 the end and $\lrcorner\ulcorner$ being everything in between. Applying this to figure 2 gives us the following:

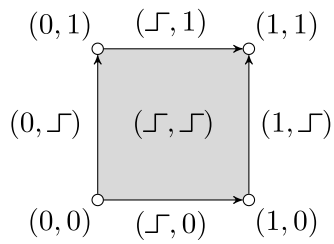


Figure 3: The precubical set of figure 2 imagined as a vector space.

This way of looking at things is most useful when looking at a certain higher-dimensional element and all elements that can be reached by its face maps. However it still works for some more complicated precubical sets.

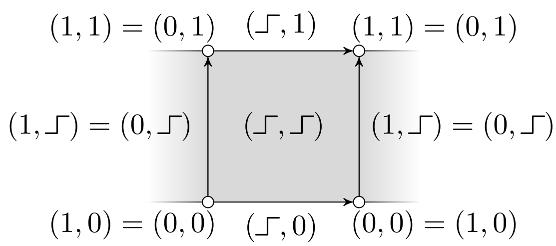


Figure 4: The precubical set of figure 3 with $(0, \lrcorner\ulcorner) = (1, \lrcorner\ulcorner)$ identified.

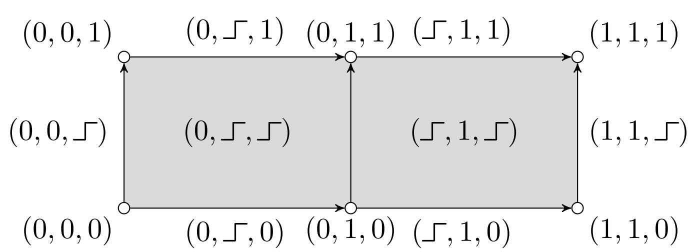


Figure 5: A precubical set that is the combination of two square precubical sets with the rightmost and leftmost edges identified. The unlabelled middle edge is $(0, 1, \lrcorner\ulcorner)$. The dimensions in the vectors are in the order of the second horizontal dimension first and the vertical dimension last.

It is clear how figure 4 works. Due to identifying the vertical edges with each other there is no difference between $(0, s)$ and $(1, s)$, where $s \in \{0, \lrcorner\ulcorner, 1\}$. This precubical set can therefore be seen as a cylinder.
Figure 5 is two 2-dimensional precubical sets glued together on their vertical edges. There are three sets of parallel edges (the two sets of horizontal parallel lines are separate) which gives us three dimensional vectors. We will later in section 3 expand on this idea of parallel edges with the notion of

events, where all parallel edges correspond to the same event. As we will see then these event consistent precubical sets will also rule out cases like if we identified the edges $\delta^2_{0,1}(x)$ and $\delta^2_{0,2}(x)$ in figure 2.

We have now defined precubical sets as families of sets on which elementary face maps are applied. This works for the most part but we still want to use a category theoretical definition instead, as it would automatically give us many properties such as cocompleteness (theorem 2.11) and the Yoneda embedding (remark 2.41.1). We use the following definition:

Definition 2.2. *A precubical set is a functor $X : \square^{op} \to \mathtt{Set}$ and the category of precubical sets is the presheaf category $\mathtt{Set}^{\square^{op}}$.*

The definition for the category $\square$ can be found in [FJSZ21], where it is the skeletal subcategory of the precube category $\boxdot$. We won't go into much detail here but the result is a definition which gives something basically identical to our previous definition but this time defined in category theory. The objects of $\square$ are linear sequences $(1 \to 2 \to ... \to n)$ with $n \in \mathbb{N}_{\geq 1}$ and the morphisms are such that they generate the face maps we mentioned before. While the reduced precube category $\square$ won't really be relevant again the fact that the category of precubical sets is a presheaf category $\mathtt{Set}^{\square^{op}}$ is important and will be used for multiple proofs. An understanding that goes beyond the abstract idea is however not necessary, though not properly defining the (reduced) precube category does have the drawback that we won't be able to use the Yoneda embedding for theorem 2.44.
We are now ready to introduce precubical maps. Category theory wise these are just the natural transformations between precubical sets. In practice we can define them like this:

Definition 2.3. *Suppose that X and Y are precubical sets. A precubical map $f : X \to Y$ is a family of morphisms $(f^n : X^n \to Y^n)_{n\in\mathbb{N}}$ which satisfies the requirement that for all $n \in \mathbb{N}_{\geq 1}$, $\nu \in \{0, 1\}$ and all $a \in \mathbb{N}$ with $1 \leq a \leq n$ we have*

$$f^{n-1} \circ \delta^n_{\nu,a} = \delta^n_{\nu,a} \circ f^n$$

Here for each $n \in \mathbb{N}$ the map f^n is called the component of f at n. By definition the precubical map f preserves the dimension of the elements in its domain and commutes with the elementary face maps (and therefore with all face maps).

Definition 2.4. *Suppose that X and Y are precubical sets and $f : X \to Y$ is a precubical map. We say that f is an injective/surjective/bijective precubical map if for all $n \in \mathbb{N}$ the component $f^n : X^n \to Y^n$ is injective/surjective/bijective.*

Theorem 2.5. *A precubical map is an isomorphism if and only if it is bijective.*

Proof. It is clear that if f is an isomorphism then for all $n \in \mathbb{N}$ the components $f^n : X^n \to Y^n$ must be bijective as well.
Suppose that for all $n \in \mathbb{N}$ the components $f^n : X^n \to Y^n$ are bijective and have the inverse maps $g^n : Y^n \to X^n$ which gives us the family of maps $(g^n : Y^n \to X^n)_{n\in\mathbb{N}}$. For all $n \in \mathbb{N}$, $y \in Y^n$ there exists a unique $x \in X^n$ with $f^n(x) = y$ and therefore $x = g^n(y)$. This gives us that for all $n \in \mathbb{N}_{\geq 1}$, $\nu \in \{0, 1\}$ and $a \in \mathbb{N}$ with $1 \leq a \leq n$ we have

$$g^{n-1} \circ \delta^n_{\nu,a}(y) = g^{n-1} \circ \delta^n_{\nu,a} \circ f^n(x) = g^{n-1} \circ f^{n-1} \circ \delta^n_{\nu,a}(x) = \delta^n_{\nu,a}(x) = \delta^n_{\nu,a} \circ g^n(y)$$

which shows that $(g^n : Y^n \to X^n)_{n\in\mathbb{N}}$ defines a precubical map $g : Y \to X$ that is also the inverse of $f : X \to Y$. This then makes f an isomorphic precubical map. ∎

2.2 Face map theorems

Up to now we have mostly used the elementary face maps, but we also want to introduce easy notations for all face maps. There are many more theorems for the face maps than we cover in this section, which can be found in appendix A.

Definition 2.6. *Let X be a precubical set. Then for all $n \in \mathbb{N}$ we denote the identity face map as $\delta^n_{id} : X^n \to X^n$ which sends every $x \in X^n$ to itself.*

Note that these identity face maps can be used to form the identity map $\mathrm{id}_X : X \to X$, where for each $n \in \mathbb{N}$ we have $\mathrm{id}^n_X = \delta^n_{\mathrm{id}}$.
The face maps are defined as the composition of any amount of elementary face maps, including the identity map which is defined above. For the compositions of elementary face maps we introduce the following notation:

Definition 2.7. *Let X be a precubical set, let $n \in \mathbb{N}_{\geq 1}$ and let $k \in \mathbb{N}_{\geq 1}$ with $1 \leq k \leq n$. Let V be a k-dimensional vector with elements $\nu_i \in \{0, 1\}$ and let A be a k-dimensional vector with elements $a_i \in \mathbb{N}_{\geq 1}$ such that for all $1 \leq i < j \leq k$ we have $1 \leq a_i < a_j \leq n$. Then for all $x \in X^n$ and all $\nu \in \{0, 1\}$ we define*

$$\delta^n_{V,A}(x) = \delta^{n-k+1}_{\nu_1,a_1} \circ \delta^{n-k+2}_{\nu_2,a_2} \circ \ldots \delta^{n-1}_{\nu_{k-1},a_{k-1}} \circ \delta^n_{\nu_k,a_k}(x)$$

We will use $\nu \in \{0, 1\}$ in place of V to denote a vector where all elements are identically ν.

It's important to note that we require the vector A to be strictly increasing (if looking at A as a sequence). This is because of the following theorem:

Theorem 2.8. *Suppose that we have $n, s \in \mathbb{N}_{\geq 1}$, $s \leq n$, $\nu_1, ..., \nu_s \in \{0, 1\}$ and $a_1, ..., a_s \in \mathbb{N}_{\geq 1}$ with $1 \leq a_{s-t} \leq n - t$ for all $0 \leq t \leq s - 1$. Then there exists an s-dimensional vector A with elements $a_i \in \mathbb{N}_{\geq 1}$ such that for all $1 \leq i < j \leq n$ we have $1 \leq a_i < a_j \leq n$ and a sequence $\mu_1, ..., \mu_s \in \{0, 1\}$ for which the following is true:*

$$\delta^n_{(\nu_1,...,\nu_s),(a_1,...,a_s)} = \delta^n_{(\mu_1,...,\mu_s),A}$$

Proof. This follows from the condition that $\delta^{n-1}_{\nu,a} \circ \delta^n_{\mu,b} = \delta^{n-1}_{\mu,b-1} \circ \delta^n_{\nu,a}$ for all $n \in \mathbb{N}_{\geq 2}$, $\nu, \mu \in \{0, 1\}$ and $a, b \in \mathbb{N}_{\geq 1}$ with $1 \leq a < b \leq n$ (note that the statement is trivial for $n = 1$). We have

$$\delta^n_{(\nu_1,...,\nu_s),(a_1,...,a_s)} = \delta^{n-s+1}_{\nu_1,a_1} \circ \delta^{n-s+2}_{\nu_2,a_2} \circ \ldots \circ \delta^{n-1}_{\nu_{s-1},a_{s-1}} \circ \delta^n_{\nu_s,a_s}$$

Suppose that for a certain $t \in \mathbb{N}_{\geq 1}$, $t < n$ we have $a_t \geq a_{t+1}$. Then we get

$$\delta^{n-1}_{\nu_t,a_t} \circ \delta^n_{\nu_{t+1},a_{t+1}} = \delta^{n-1}_{\nu_{t+1},a_{t+1}} \circ \delta^n_{\nu_t,a_t+1}$$

where we have $a_t + 1 > a_{t+1}$.
Suppose that we have a s-dimensional vectors $(\nu_1, ..., \nu_s)$ and $(a_1, ..., a_s)$ as described. We apply the following algorithm:

1. Let $t = 1$.

2. If $a_t \geq a_{t+1}$ then:

(a) Swap ν_t and ν_{t+1}

(b) Increment a_t by 1 and swap $(a_t + 1)$ and a_{t+1}

3. If for all $1 \leq t \leq s-1$ we had $a_t < a_{t+1}$ then stop.

4. Else let $t = t+1$. If $t = s$ jump to step 1 and if $t < s$ jump to step 2.

Let x and y mark two elements in $(a_1, ..., a_s)$ such that if elements are swapped then the marks move with them. If at any point in the algorithm x and y are compared and we have $x < y$ (or $y < x+1$ after swapping) then these marks will never be swapped (again). This is because x and y can only be compared again if they appear consecutively in the sequence which is true if and only if they have both been moved the same amount of steps to the right. Assuming that x marks an element before y we know that x can never overtake y in the sequence before having to be compared to it.
This means that after two elements are compared the algorithm sorts them correctly relative to each other. Because the input sequence is finite it takes a finite amount of steps to either compare every possible pair of elements (which will mean that the output sequence is sorted) or to sort the sequence. Therefore this algorithm will always return a sorted sequence.
The way the algorithm swaps elements is the same as how they are swapped on the face maps. Therefore this algorithm results in vectors $(\nu_1, ..., \nu_s)$ and A as required, which proves the statement. $\square$

Theorem 2.9. *Every face map can be represented as a face map defined in definition 2.6 or definition 2.7.*

Proof. The identity face map is trivial. For the other face maps the statement is proven in theorem 2.8. $\square$

For the specific case of the face maps $X^n \to X^1$ for all $n \in \mathbb{N}_{\geq 2}$ we use the following notation:

Definition 2.10. *Let X be a precubical set, let $n \in \mathbb{N}_{\geq 1}$ and let $a \in \mathbb{N}_{\geq 1}$ with $a \leq n$. We define the $n-1$-dimensional vector $A_a^n = (1, 2, ..., a-1, a+1, ..., n-1, n)$.*

In other words A_a^n is the $n-1$-dimensional vector that contains every element ≥ 1 and $\leq n$ in order except for the element a. Because of theorem 2.9 these form all the vectors needed to construct face maps $X^n \to X^1$ for every $n \in \mathbb{N}_{\geq 2}$.
These cover some of the basic notational shortcuts and face map theorems. There are more advanced face map theorems for which we will refer to appendix A at the end.

2.3 Colimits of precubical sets

Theorem 2.11. *The category of precubical sets $\mathtt{Set}^{\square^{op}}$ is cocomplete.*

Proof. Proposition 8.8 from [Awo06] gives us that given any two categories $\mathcal{C}$ and $\mathcal{D}$, if the category $\mathcal{D}$ is cocomplete then the functor category $\mathcal{D}^{\mathcal{C}}$ is also cocomplete. Since the category $\mathtt{Set}$ is cocomplete the category of precubical sets $\mathtt{Set}^{\square^{op}}$ is cocomplete as well. $\square$

So now that we know that every small diagram of precubical sets has a colimit we want to gain some more insight into the structure of these colimits. In this subsection we will prove some structural theorems. These proofs will be useful later as well as giving us a better understanding of how colimits of precubical sets work. At the end of this subsection we will give an intuitive explanation as to how these colimits work.
We start with a very simple condition for a co-cone to be a colimit.

Theorem 2.12. *Let $X : J \to \mathtt{Set}^{\square^{op}}$ be a small diagram and let (L, ϕ) be a colimit of this diagram. Let (N, ψ) be a co-cone of X. Then (N, ψ) is a colimit of X as well if the unique precubical map $q : L \to N$ such that $q \circ \phi_i = \psi_i$ for all $i \in J$ is an isomorphism.*

Proof. Let (M, θ) be a co-cone of X and let $p : L \to M$ be the unique precubical map such that $p \circ \phi_i = \theta_i$ for all $i \in J$. Then $p \circ q^{-1} : N \to M$ is a precubical map with $p \circ q^{-1} \circ \psi_i = \theta_i$ for all $i \in J$. Suppose that $f : N \to M$ is a different precubical map that satisfies the property with $f \neq p \circ q^{-1}$. Then $f \circ q : L \to M$ is another precubical map that satisfies the property with $f \circ q \neq p$ which is in contradiction with (L, ϕ) being a colimit. Therefore no such f exists which means that $p \circ q^{-1} : N \to M$ gives us an unique precubical map which therefore makes (N, ψ) a colimit of X. $\square$

It is important to note that this condition is stronger than L and N just being isomorphic. Take for example the diagram $X : J \to \mathtt{Set}^{\square^{op}}$ with $\mathrm{obj}(J) = \{1, 2\}$, $\mathrm{mor}(J) = \emptyset$, $X_1^0 = \{1\}$, $X_2^0 = \{2\}$ and $X_1^n = X_2^n = \emptyset$ for all $n \in \mathbb{N}_{\geq 1}$. A colimit of this diagram is the precubical set Y with $Y^0 = \{1, 2\}$ and $Y^n = \emptyset$ for all $n \in \mathbb{N}_{\geq 1}$ with the precubical maps $\phi_1^0(1) = 1$ and $\phi_2^0(2) = 2$. A co-cone of this diagram is the precubical set Z with $Z^0 = \{1, 2\}$ and $Z^n = \emptyset$ for all $n \in \mathbb{N}_{\geq 1}$ but with the precubical maps $\psi_1^0(1) = 1 = \psi_2^0(2)$. Here we clearly have $Y \cong Z$ but it is also clear that (Z, ψ) is not a colimit of X.
In this theorem we proved that (N, ψ) is a colimit if the map $q : L \to N$ is an isomorphism. Something that is true as well is that $q : L \to N$ being an isomorphism is a requirement for (N, ψ) being a colimit. We will leave this proof up to the reader.
Now a quick theorem about the coproduct, which will be useful at some points later.

Theorem 2.13. *Let $X : J \to \mathtt{Set}^{\square^{op}}$ be a small diagram. Then for all $n \in \mathbb{N}$, $x \in \bigsqcup_{i \in J} X_i^n$ there exists unique $j \in J$, $y \in X_j^n$ such that $\varphi_j^n(y) = x$, where φ is the injection map of the coproduct.*

Proof. This follows from the definition of the coproduct in $\mathtt{Set}$, which gives us that all elements in a coproduct or disjoint union $\bigsqcup_{i \in J} X_i^n$ are uniquely injected. $\square$

This is a rather trivial result, but since it is rather useful it's best to have it written down. Now we define a certain equivalence relation:

Definition 2.14. *Let $X : J \to \mathtt{Set}^{\square^{op}}$ be a diagram with J a small category. We define the equivalence relation $\sim$ on X to be generated by the following: For all $n \in \mathbb{N}$, $i, j \in J$, $x \in X_i^n$ and $y \in X_j^n$ we have*

$$x \sim y \Longleftarrow \exists\, (f : i \to j) \text{ with } X_f^n(x) = y$$

We claim that this equivalence relation through quotient sets will give us the colimit of a diagram. To prove this however will take a while. An alternative definition for $\sim$ is the following:

Definition 2.15. *Let $X : J \to \mathbf{Set}^{\square^{op}}$ be a diagram with J a small category. We define the equivalence relation $\sim$ on X as*

$$\left\{ (x,y) \,\middle|\, \begin{array}{c} n \in \mathbb{N},\ i,j \in J,\ x \in X_i^n,\ y \in X_j^n \text{ such that} \\ \exists k \in J,\ (f : i \to k),\ (g : j \to k) \text{ with } X_f^n(x) = X_g^n(y) \end{array} \right\}^T$$

where T denotes taking the transitive closure of the set.

It is clear that the set above (before taking the transitive closure) describes a reflexive and symmetric relation. Note that taking the transitive closure is generally necessary. For example in the case where we have $X_f : X_i \to X_j$ and $X_g : X_i \to X_k$ but where there exists no maps between X_j and X_k for every element $x \in X_i^n$ we wouldn't have $X_f^n(x) \sim X_g^n(x)$ without the transitive closure.

Remark 2.15.1. *As a consequence of theorem 2.13 the equivalence relation works the same whether we are talking about elements of X_i^n and X_j^n or elements of $\coprod_{i \in J} X_i^n$. We can therefore define a relation $\sim$ on $\coprod_{i \in J} X_i^n$ where we have that $\varphi_i^n(x) \sim \varphi_j^n(y)$ is equivalent to $x \sim y$.*

Theorem 2.16. *The canonical quotient map $[-]^n : \bigsqcup_{i \in J} X_i^n \to \bigsqcup_{i \in J} X_i^n / \sim$ exists for all $n \in \mathbb{N}$.*

Proof. This follows from the properties of equivalence relations on sets. $\square$

We eventually want to use the quotient maps $[-]^n$ to construct a precubical map to a precubical set which we will eventually prove is a colimit. We first need to prove that the equivalence relation $\sim$ and the quotient maps $[-]^n$ properly respect the elementary face maps.

Theorem 2.17. *Suppose that $X : J \to \mathbf{Set}^{\square^{op}}$ is a small diagram. For all $n \in \mathbb{N}_{\geq 1}$, $i,j \in J$ $x \in X_i^n$, $y \in X_j^n$, $\nu \in \{0,1\}$ and $a \in \mathbb{N}$ with $1 \leq a \leq n$ we have*

$$x \sim y \implies \delta_{\nu,a}^n(x) \sim \delta_{\nu,a}^n(y)$$

Proof. Suppose that for certain $n \in \mathbb{N}_{\geq 1}$, $i,j \in J$, $x \in X_i^n$ and $y \in X_j^n$ there exists a $f : i \to j$ such that $X_f^n(x) = y$. By definition for all $\nu \in \{0,1\}$ and $a \in \mathbb{N}$ with $1 \leq a \leq n$ we have that

$$\delta_{\nu,a}^n(y) = \delta_{\nu,a}^n \circ X_f^n(x) = X_f^{n-1} \circ \delta_{\nu,a}^n(x)$$

which gives us $\delta_{\nu,a}^n(x) \sim \delta_{\nu,a}^n(y)$. Due to the way the equivalence relation $\sim$ is generated this gives us the above result. $\square$

Theorem 2.18. *For all $[x]^n \in \bigsqcup_{i \in J} X_i^n / \sim$ there exists a unique $[y]^{n-1} \in \bigsqcup_{i \in J} X_i^{n-1} / \sim$, $\nu \in \{0,1\}$, $a \in \mathbb{N}$ with $1 \leq a \leq n$ such that for all $z \in [x]^n$ we have $\delta_{\nu,a}^n(z) \in [y]^{n-1}$.*

Proof. Because we are talking about equivalence classes every element of $\bigsqcup_{i \in J} X_i^n$ is in exactly one equivalence class. Theorem 2.17 gives us that for $u, v \in X_i^n$, $i \in J$, $n \in \mathbb{N}$ if $u \sim v$, then $\delta_{\nu,a}^n(u) \sim \delta_{\nu,a}^n(v)$ for all $\nu \in \{0,1\}$ and $a \in \mathbb{N}$ with $1 \leq a \leq n$. Therefore for all $z \in [x]^n$ the elements $\delta_{\nu,a}^n(z)$ are all sent to the same equivalence class by the quotient map $[-]^{n-1} : \bigsqcup_{i \in J} X_i^{n-1} \to \bigsqcup_{i \in J} X_i^{n-1} / \sim$, which proves the theorem. $\square$

Theorem 2.19. *We can define a precubical set Y such that $Y^n = \bigsqcup_{i \in J} X_i^n / \sim$ and $[-] : \coprod_{i \in J} X_i \to Y$ is a precubical map with $[-]^n$ defined to be the quotient map $\bigsqcup_{i \in J} X_i^n \to \bigsqcup_{i \in J} X_i^n / \sim$ for all $n \in \mathbb{N}$.*

Proof. Using theorem 2.18 we define the face map such that for all $n \in \mathbb{N}_{\geq 1}$, $[x]^n \in \bigsqcup_{i \in J} X_i^n / \sim$ we have that $\delta_{\nu,a}^n [x]^n$, $\nu \in \{0,1\}$, $1 \leq a \leq n$ is the equivalence class with $\delta_{\nu,a}^n(y) \in \delta_{\nu,a}^n[x]^n$ for all $y \in [x]^n$. This gives us $[-]^{n-1} \circ \delta_{\nu,a}^n(x) = \delta_{\nu,a}^n \circ [x]^n$.
Suppose that $\nu, \mu \in \{0,1\}$, $n \in \mathbb{N}_{\geq 2}$ and $a, b \in \mathbb{N}$ with $1 \leq a < b \leq n$. We need to prove that

$$\delta_{\nu,a}^{n-1} \circ \delta_{\mu,b}^n = \delta_{\mu,b-1}^{n-1} \circ \delta_{\nu,a}^n$$

Suppose that $[x]^n \in \bigsqcup_{k \in J} X_k^n / \sim$. Then for all $y \in [x]^n$ we have

$$\delta_{\nu,a}^{n-1} \circ \delta_{\mu,b}^n \circ [y]^n = \left[\delta_{\nu,a}^{n-1} \circ \delta_{\mu,b}^n(y)\right]^{n-2} = \left[\delta_{\mu,b-1}^{n-1} \circ \delta_{\nu,a}^n(y)\right]^{n-2} = \delta_{\mu,b-1}^{n-1} \circ \delta_{\nu,a}^n \circ [y]^n$$

With this Y satisfies all of the necessary conditions for being a precubical set. By definition $[-] : \coprod_{i \in J} X_i \to Y$ is now a precubical map as well. $\square$

Theorem 2.20. *Suppose that $X : J \to \mathtt{Set}^{\square^{op}}$ is a diagram and J a small category. $(Y, [-] \circ \varphi)$ with the precubical set Y as defined in theorem 2.19 is a co-cone of this diagram.*

Proof. Theorem 2.19 gives us a precubical set Y and a precubical map $[-] : \coprod_{i \in J} X_i \to Y$. Combining this precubical map with the injection maps $\varphi_j : X_j \to \coprod_{i \in J} X_i$ for all $j \in J$ gives us the precubical maps $[-] \circ \varphi_j : X_j \to Y$ for all $j \in J$. Suppose that $f : j \to k$ is a morphism in J. Then for all $x \in X_j^n$ we have $x \sim X_f^n(x)$, therefore $\varphi_j^n(x) \sim \varphi_k^n \circ X_f^n(x)$ and therefore $[-]^n \circ \varphi_j^n(x) = [-]^n \circ \varphi_k^n \circ X_f^n(x)$. This makes $(Y, [-] \circ \varphi)$ a co-cone. $\square$

Theorem 2.21. *Suppose that $X : J \to \mathtt{Set}^{\square^{op}}$ is a small diagram and that (N, ψ) is a co-cone of this diagram. For all $n \in \mathbb{N}$, $i, j \in J$, $x \in X_i^n$ and $y \in X_j^n$ we have*

$$x \sim y \implies \psi_i^n(x) = \psi_j^n(y)$$

Proof. For all $n \in \mathbb{N}$, $i, j \in J$ and $x \in X_i^n$ and $y \in X_j^n$ we have that if there exists a $f : i \to j$ such that $X_f^n(x) = y$ then $x \sim y$ and $\psi_i^n(x) = \psi_j^n \circ X_f^n(x) = \psi_j^n(y)$ due to the properties of the co-cone. Because of the way the equivalence relation $\sim$ is generated this gives us the above result. $\square$

The opposite isn't true for every co-cone. However it is true for colimits which we will prove now. Later we will prove that it is true only for colimits.

Theorem 2.22. *Suppose that $X : J \to \mathtt{Set}^{\square^{op}}$ is a small diagram with the colimit (L, ϕ). For all $n \in \mathbb{N}$, $i, j \in J$, $x \in X_i^n$ and $y \in X_j^n$ we have*

$$x \sim y \iff \phi_i^n(x) = \phi_j^n(y)$$

Proof. Theorem 2.21 gives us the implication to the right.
From theorem 2.20 it follows that $(Y, [-] \circ \varphi)$ is a co-cone of X. Therefore there exists a unique precubical map $q : L \to Y$ such that for all $n \in \mathbb{N}$, $i \in J$ and $x \in X_i^n$ we have $[-]^n \circ \varphi_i^n(x) = q^n \circ \phi_i^n(x)$.
Suppose that $\phi_i^n(x) = \phi_j^n(y)$ for certain $n \in \mathbb{N}$, $i, j \in J$, $x \in X_i^n$ and $y \in X_j^n$. Then we get

$$[-]^n \circ \varphi_i^n(x) = q^n \circ \phi_i^n(x) = q^n \circ \phi_j^n(y) = [-]^n \circ \varphi_j^n(y)$$

and therefore $x \sim y$, which gives us the implication to the left. $\square$

Theorem 2.23. *Let $X : J \to \mathbf{Set}^{\square^{op}}$ be a small diagram and let $(Y, [-] \circ \varphi)$ be the co-cone with the precubical set Y as defined in theorem 2.19 and in theorem 2.20. Then $(Y, [-] \circ \varphi)$ is a colimit of X.*

Proof. Suppose that (N, ψ) is a co-cone of X. We start by defining a precubical map $p : Y \to N$ as the following: for all $n \in \mathbb{N}$, $y \in Y^n$ and for all $i \in J$, $x \in X_i^n$ such that $[-]^n \circ \varphi_i^n(x) = y$ we have

$$p^n(y) = p^n \circ [-]^n \circ \varphi_i^n(x) = \psi_i^n(x)$$

Note that because of the construction of Y there will always exist such an x. Suppose that $i, j \in J$, $x \in X_i^n$ and $z \in X_j^n$ with $[-]^n \circ \varphi_i^n(x) = y = [-]^n \circ \varphi_j^n(z)$. This gives us $x \sim z$, which as a consequence of theorem 2.21 gives us that $\psi_i^n(x) = \psi_j^n(z)$. This makes p^n well-defined for all $n \in \mathbb{N}$. Suppose that $n \in \mathbb{N}_{\geq 1}$, $\nu \in \{0, 1\}$ and $m \in \mathbb{N}_{\geq 1}$ with $m \leq n$. Suppose that $y \in Y^n$. Then for all $i \in J$, $x \in X_i^n$ with $[-]^n \circ \varphi_i^n(x) = y$ we have

$$[-]^{n-1} \circ \varphi_i^{n-1} \circ \delta_{\nu,a}^n(x) = \delta_{\nu,a}^n \circ [-]^n \circ \varphi_i^n(x) = \delta_{\nu,a}^n(y)$$

This gives us

$$\begin{aligned} p^{n-1} \circ \delta_{\nu,a}^n(y) &= p^{n-1} \circ [-]^{n-1} \circ \varphi_i^{n-1} \circ \delta_{\nu,a}^n(x) = \psi_i^{n-1} \circ \delta_{\nu,a}^n(x) \\ &= \delta_{\nu,a}^n \circ \psi_i^n(x) = \delta_{\nu,a}^n \circ p^n \circ [-]^n \circ \varphi_i^n(x) = \delta_{\nu,a}^n \circ p^n(y) \end{aligned}$$

making $p : Y \to N$ a precubical map. Let $r : Y \to N$ be another precubical map with the property $r \circ [-] \circ \varphi_i = \psi_i$. Suppose that $y \in Y^n$ for a certain $n \in \mathbb{N}$. Then for all $i \in J$, $x \in X_i^n$ such that $[-]^n \circ \varphi_i^n(x) = y$ we have

$$r^n(y) = r^n \circ [-]^n \circ \varphi_i^n(x) = \varphi_i^n(x) = p^n \circ [-]^n \circ \varphi_i^n(x) = p^n(y)$$

Therefore for all $n \in \mathbb{N}$, $y \in Y^n$ we have $r^n(y) = p^n(y)$, which gives us that $r = p$. Therefore p is a unique precubical map.
Therefore for all co-cones (N, ψ) of X there exists a unique precubical map $p : Y \to N$ such that $p \circ [-] \circ \varphi_i = \psi_i$ for all $i \in J$, which makes $(Y, [-] \circ \varphi)$ a colimit of X. □

With this we have shown that we can construct our colimit using the coproduct and the quotient map.

Theorem 2.24. *Let $X : J \to \mathbf{Set}^{\square^{op}}$ be a small diagram with the colimit (L, ϕ). Then for all $n \in \mathbb{N}$, $x \in L^n$ there exists at least one $i \in J$, $y \in X_i^n$ such that $\phi_i^n(y) = x$.*

Proof. As a consequence of theorem 2.23 we have $\phi = [-] \circ \varphi$. Theorem 2.13 in combination with the properties of the quotient map $[-]^n$ then gives us the above result. □

Theorem 2.25. *Let $X : J \to \mathbf{Set}^{\square^{op}}$ be a small diagram. Suppose that (L, ϕ) is a co-cone of X such that for all $i, j \in J$, $n \in \mathbb{N}$, $x \in X_i^n$ and $y \in X_j^n$ we have*

$$x \sim y \iff \phi_i^n(x) = \phi_j^n(y)$$

and such that for all $n \in \mathbb{N}$, $x \in L^n$ there exists a $i \in J$ with $\phi_i^n(y) = x$. Then (L, ϕ) is a colimit of X.

Proof. Using theorem 2.19, theorem 2.20 and theorem 2.23 we get that $(Y, [-] \circ \varphi)$ is a colimit of X with $Y^n \cong \bigsqcup_{i \in J} X_i^n / \sim$ for all $n \in \mathbb{N}$. Since we have $x \sim y \iff \phi_i^n(x) = \phi_j^n(y)$ for all $i, j \in J$, $n \in \mathbb{N}$, $x \in X_i^n$ and $y \in X_j^n$ we also have

$$[-]^n \circ \varphi_i^n(x) = [-]^n \circ \varphi_j^n(y) \iff \phi_i^n(x) = \phi_j^n(y)$$

for all $i, j \in J$, $n \in \mathbb{N}$, $x \in X_i^n$ and $y \in X_j^n$. Because $(Y, [-] \circ \varphi)$ is a colimit of X and (L, ϕ) is a co-cone of X there exists a precubical map $q : Y \to L$ such that $q \circ [-] \circ \varphi_i = \phi_i$ for all $i \in J$. Using theorem 2.24 we get that for all $n \in \mathbb{N}$, $x \in Y^n$ there exists at least one $i \in J$, $y \in X_i^n$ such that $[-]^n \circ \varphi_i^n(y) = x$. Suppose that $x, y \in Y^n$ for a certain $n \in \mathbb{N}$. Because there exist $i, j \in J$, $x' \in X_i^n$, $y' \in X_j^n$ with $[-]^n \circ \varphi_i^n(x') = x$ and $[-]^n \circ \varphi_j^n(y') = y$ we get

$$q^n(x) = q^n(y) \implies q^n \circ [-]^n \circ \varphi_i^n(x') = q^n \circ [-]^n \circ \varphi_j^n(y')$$

$$\implies \phi_i^n(x') = \phi_j^n(y') \implies [-]^n \circ \varphi_i^n(x') = [-]^n \circ \varphi_j^n(y') \implies x = y$$

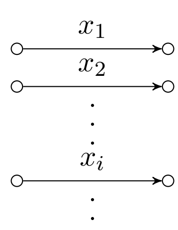
Because we obviously have $x = y \implies q^n(x) = q^n(y)$ we get

$$q^n(x) = q^n(y) \iff x = y$$

which means that $q : Y \to L$ is injective. Suppose that we have a $n \in \mathbb{N}$ and $x \in L^n$. Then there exists a $i \in J$, $y \in X_i^n$ with $\phi_i^n(y) = x$ which gives us that $q^n \circ [-]^n \circ \varphi_i^n(y) = x$, which makes q surjective as well. Using theorem 2.5 we then get that $q : Y \to L$ is an isomorphism, which means that $(Y, [-] \circ \varphi)$ is isomorphic with (L, ϕ), making (L, ϕ) a colimit of X. $\square$

2.3.1 Intuitive explanation of colimits of precubical sets

Now that we have proven some important structural theorems for the colimits of precubical sets, we want to stop and look at what this looks like in practice.
Let's start with a simple example of a coproduct. We take the small discrete category J with $\mathrm{obj}(J) = \mathbb{N}_{\geq 1}$ and $\mathrm{mor}(J) = \emptyset$. For each $i \in J$ let X_i be the precubical set containing two nodes which are connected by a single edge which we will label x_i. This gives us the coproduct

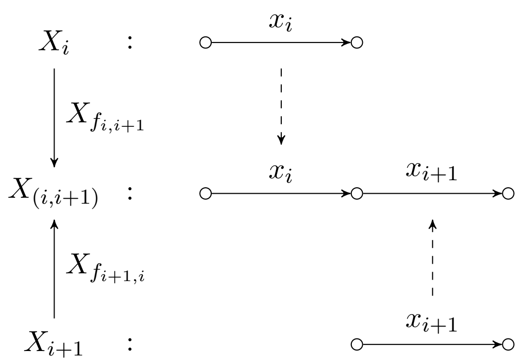


Now we want to look at a colimit that is neither a coproduct nor a filtered colimit. Let J be a small category with

$$\mathrm{obj}(J) = \mathbb{N}_{\geq 1} \cup \{(n, n+1) \mid n \in \mathbb{N}_{\geq 1}\}$$

$$\mathrm{mor}(J) = \begin{array}{c} \{\mathrm{id}_i : i \to i \mid i \in \mathrm{obj}(J)\} \cup \\ \{(f_{n,n+1} : n \to (n, n+1)) \mid n \in \mathbb{N}_{\geq 1}\} \cup \\ \{(f_{n+1,n} : n+1 \to (n, n+1)) \mid n \in \mathbb{N}_{\geq 1}\} \end{array}$$

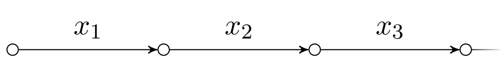
For each $i \in \mathbb{N}_{\geq 1}$ we let X_i again be the precubical set containing two nodes connected by a single edge labelled x_i. For each $(i, i+1) \in \{(n, n+1) \mid n \in \mathbb{N}_{\geq 1}\}$ let $X_{(i,i+1)}$ be the precubical set containing three nodes and two edges, where node 1 and 2 are connected by the edge labelled x_i and where node 2 and 3 are connected by the edge labelled x_{i+1}. For all $i \in \mathbb{N}_{\geq 1}$ we define the precubical maps $X_{f_{i,i+1}} : X_i \to X_{(i,i+1)}$ and $X_{f_{i+1,i}} : X_{i+1} \to X_{(i,i+1)}$ as mapping the edge labelled x_i in X_i to the edge x_i in $X_{(i,i+1)}$ and mapping the edge labelled x_{i+1} in X_{i+1} to the edge x_{i+1} in $X_{(i,i+1)}$. For all $i \in \mathbb{N}_{\geq 1}$ this gives us



With this we have defined the diagram $X : J \to \mathtt{Set}^{\square^{\mathrm{op}}}$. Let (L, ϕ) be the colimit of this diagram. Recall that theorem 2.22 gives us that for all $i, j \in J$, $n \in \mathbb{N}$, $x \in X_i^n$ and $y \in X_j^n$ we have

$$x \sim y \iff \phi_i^n(x) = \phi_j^n(y)$$

To understand what the colimit looks like we need to look at the nodes. Let $i \in \mathbb{N}_{\geq 1}$. In the precubical set X_i we have a node at the start of the edge x_i and one at the end. The end node of X_i, as shown in the figure above, is mapped to the same node as the start node of X_{i+1} in the precubical set $X_{i,i+1}$. Therefore these nodes are equivalent under the relation $\sim$, which means that they are also mapped to the same node in the colimit L by the precubical injection maps ϕ_i and ϕ_{i+1}. The same goes for the end node of X_{i+1} and the start node of X_{i+2}, and the same is true for all $i \in \mathbb{N}_{\geq 1}$. The colimit will therefore be as follows:



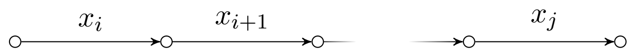
The diagram we just defined is not filtered however. The objects $i \in \mathbb{N}_{\geq 1}$ and $i + 2 \in \mathbb{N}_{\geq 1}$ are not mapped to a common object. We want to define a filtered diagram with the same colimit as above. We define J as the following small category:

$$\mathrm{obj}(J) = \{(i, j) \mid i, j \in \mathbb{N}_{\geq 1},\, i \leq j\}$$

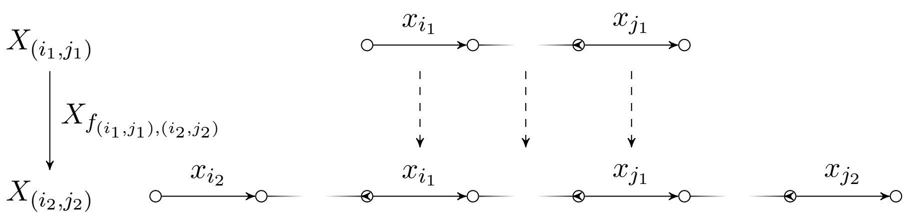
$$\mathrm{mor}(J) = \{(f : (i_1, j_1) \to (i_2, j_2)) \mid i_1, i_2, j_1, j_2 \in \mathbb{N}_{\geq 1},\, i_2 \leq i_1 \leq j_1 \leq j_2\}$$

It is clear that this category has all identity morphisms and all compositions of morphisms. It is not empty, it has no parallel morphisms and for all objects $(i_1, j_1), (i_2, j_2) \in \mathrm{obj}(J)$ there exists the object $(\max(i_1, i_2), \max(j_1, j_2))$ for which both objects are mapped onto. Therefore J is a filtered category.

We define the diagram $X : J \to \mathtt{Set}^{\square^{\mathrm{op}}}$ as follows: For all objects $(i, j) \in J$ $X_{(i,j)}$ is the precubical set with $j - i + 2$ nodes and $j - i + 1$ edges labelled x_i, x_{i+1},..., x_j. This gives us:



The precubical maps we define as inserting the smaller precubical set into the larger one. This gives us:



for all $(i_1, j_1), (i_2, j_2) \in \mathrm{obj}(J)$ with $i_2 \leq i_1 \leq j_1 \leq j_2$. Using the same arguments as for the non-filtered colimit it follows that the filtered colimit of this diagram is the same.

2.3.2 Filtered colimits

While we prove everything for all small colimits, we are specifically interested in filtered colimits. Something we will prove later is that all precubical sets (and all HDA) are filtered colimits of a small filtered diagram of finite precubical sets (or finite HDA). For starters we will prove the following property:

Theorem 2.26. *Let $X : J \to \mathtt{Set}^{\square^{op}}$ be a small filtered diagram. Let $n \in \mathbb{N}$ and let $I \subseteq obj(J)$ be a finite subset. For all $i \in I$ we identify the elements $x_i \in X^n_i$ such that for all $i_1, i_2 \in I$ there exist $j \in J$ and morphisms $f : i_1 \to j$ and $g : i_2 \to j$ such that $X^n_f(x_{i_1}) = X^n_g(x_{i_2})$. Then there exists a $k \in J$, $y \in X^n_k$ and morphisms $h_i : i \to k$ for all $i \in I$ such that $X^n_{h_i}(x_i) = y$.*

Proof. The statement is trivial for $|I| \in \{0, 1, 2\}$. Let $I = \{i_1, i_2, i_3\}$. For ease of notation we say that $x_{i_1} = x_1$, $x_{i_2} = x_2$ and $x_{i_3} = x_3$. We have the objects $j_1, j_3 \in J$ and the morphisms $f_1 : i_1 \to j_1$, $f_2 : i_2 \to j_1$, $g_2 : i_2 \to j_3$ and $g_3 : i_3 \to j_3$ such that $X^n_{f_1}(x_1) = X^n_{f_2}(x_2)$ and $X^n_{g_2}(x_2) = X^n_{g_3}(x_3)$. Because J is a filtered category we can use the second property which tells us there exists a $k \in \mathrm{obj}(J)$ and morphisms $h_1 : j_1 \to k$ and $h_3 : j_3 \to k$. Then because $h_1 \circ f_2 : i_2 \to k$ and $h_3 \circ g_2 : i_2 \to k$ are parallel morphisms the third property then gives us there exists a $k' \in \mathrm{obj}(J)$ and a morphism $h : k \to k'$ such that $h \circ h_1 \circ f_2 = h \circ h_3 \circ g_2$. Finally this gives us

$$X^n_{h\circ h_1\circ f_1}(x_1) = X^n_{h\circ h_1\circ f_2}(x_2) = X^n_{h\circ h_3\circ g_2}(x_2) = X^n_{h\circ h_3\circ g_3}(x_3)$$

which proves the statement for all I with $|I| = 3$.
Suppose that $|I| > 3$. We can apply the above for any three elements $\{i_1, i_2, i_3\}$ of I, which then gives us a $k \in J$ and a $y \in X^n_k$ as shown above. We can use this to construct a new set $I' (\{k\} \cup I) \setminus \{i_1, i_2, i_3\}$ with $|I'| = |I| - 2$. Since the statement is true for all $|I| \in \{0, 1, 2, 3\}$ it is therefore true for all $|I| \in \mathbb{N}$. $\square$

Let $X : J \to \mathtt{Set}^{\square^{op}}$ be a small filtered diagram. Recall that the relation $\sim$ is defined as the transitive closure of the set in the theorem below. In this theorem we will prove that this set is already transitive.

Theorem 2.27. *In the case that J is a filtered category the equivalence relation $\sim$ can be defined as*

$$\sim = \left\{ (x,y) \middle| \begin{array}{c} n \in \mathbb{N},\ i,j \in J,\ x \in X_i^n,\ y \in X_j^n \text{ such that} \\ \exists k \in J,\ (f : i \to k),\ (g : j \to k) \text{ with } X_f^n(x) = X_g^n(y) \end{array} \right\}$$

Proof. Suppose that for certain $n \in \mathbb{N}$, $i_1, i_2, i_3 \in J$ and $x_1 \in X_{i_1}^n$, $x_2 \in X_{i_2}^n$ and $x_3 \in X_{i_3}^n$ there exist $j_1, j_3 \in J$, $f_1 : i_1 \to j_1$, $f_2 : i_2 \to j_1$, $g_2 : i_2 \to j_3$ and $g_3 : i_3 \to j_3$ such that $X_{f_1}^n(x_1) = X_{f_2}^n(x_2)$ and $X_{g_2}^n(x_2) = X_{g_3}^n(x_3)$. Then theorem 2.26 gives us that there exists a $k \in J$ and maps $h_1 : j_2 \to k$ and $h_3 : j_3 \to k$ such that $h_1 \circ f_2 = h_3 \circ g_2$. This gives us

$$X_{h_1}^n \circ X_{f_1}^n(x_1) = X_{h_1}^n \circ X_{f_2}^n(x_2) = X_{h_3}^n \circ X_{g_2}^n(x_2) = X_{h_3}^n \circ X_{g_3}^n(x_3)$$

which shows that for filtered categories the above set is already transitive, making it the same as the set in definition 2.14. ☐

Theorem 2.28. *Let $X : J \to \mathtt{Set}^{\square^{op}}$ be a small filtered diagram. Let $n \in \mathbb{N}$ and let $I \subseteq obj(J)$ be a finite subset. For all $i \in I$ we identify the elements $x_i \in X_i^n$ such that for all $i_1, i_2 \in I$ we have $x_{i_1} \sim x_{i_2}$. Then there exists a $k \in J$, $y \in X_k^n$ and morphisms $h_i : i \to k$ for all $i \in I$ such that $X_{h_i}^n(x_i) = y$.*

Proof. This follows from theorem 2.26 and theorem 2.27. ☐

2.4 Images of precubical sets

In this subsection we introduce a new way of construction precubical sets. Namely through families of subsets of precubical sets that are closed under the face maps and later through images of precubical maps. We end this subsection by constructing a certain filtered diagram of certain precubical sets which will eventually be used to prove that every precubical set is the filtered colimit of finite precubical sets.
We start with this first theorem:

Theorem 2.29. *Let Y be a precubical set. Any family of sets $\{X^n\}_{n\in\mathbb{N}}$ with $X^n \subseteq Y^n$ for all $n \in \mathbb{N}$ that is closed under the face maps defines a precubical set X and there exists a canonical precubical map $f : X \to Y$.*

Proof. This trivially follows from the fact that Y is a precubical set and we take the definition of the face maps from there. We can define the precubical map $f : X \to Y$ as the one that sends every $x \in X^n$ to $f^n(x) = x \in Y^n$ for all $n \in \mathbb{N}$. ☐

The family of sets being closed under the face maps means that for all $y \in Y^n$ with $n \in \mathbb{N}$ if we have $y \in X^n$ then for all $\nu \in \{0,1\}$ and $a \in \mathbb{N}_{\geq 1}$ with $1 \leq a \leq n$ we have $\delta_{\nu,a}^n(y) \in X^{n-1}$.

Theorem 2.30. *Let X and Y be precubical sets and suppose that $X^n \subseteq Y^n$ for all $n \in \mathbb{N}$. If for all $n \in \mathbb{N}_{\geq 1}$, all $x \in X^n \subseteq Y^n$, $\nu \in \{0,1\}$ and $a \in \mathbb{N}_{\geq 1}$ with $1 \leq a \leq n$ we have $(\delta_X)_{\nu,a}^n(x) = (\delta_Y)_{\nu,a}^n(x)$ then there exists a canonical precubical map $f : X \to Y$.*

Proof. The precubical set X gives us a family of sets $\{X^n\}_{n\in\mathbb{N}}$ that satisfies the conditions in theorem 2.29. Since the precubical set that is constructed using this theorem has the same elements and the same face maps as X it is the same as X which therefore gives us the canonical precubical map $f : X \to Y$. $\square$

These first two theorems underline the relation between precubical sets and subsets of the components of precubical sets. As the image of a map is also a subset of the codomain we get the following theorem:

Theorem 2.31. *Let X and Y be precubical sets and let $f : X \to Y$ be a precubical map. Then the image of f, notation $f(X)$, is a precubical set and there exist canonical precubical maps $g : X \to f(X)$ and $h : f(X) \to Y$ with $f = h \circ g$.*

Proof. From theorem 2.29 it follows that $f(X)$ defines a precubical set. We can simply define a precubical map $g : X \to f(X)$ with $g^n(x) = f^n(x)$ for all $n \in \mathbb{N}$ and $x \in X^n$, which by definition of f preserves the face maps. We can define h as the canonical precubical defined in theorem 2.29, which by construction gives us $f = h \circ g$. $\square$

This shows that the image of a precubical map is a precubical set. The same is true for the union of images of precubical maps (with the same codomain).

Theorem 2.32. *Let $\{X_i\}_{i\in I}$ be a family of precubical sets, Y be a precubical set and let $\{f_i : X_i \to Y\}_{i\in I}$ be a family of precubical maps. Then the union of the images of $f_i : X_i \to Y$, notation $\bigcup_{i\in I} f_i(X_i)$, is a precubical set as well and there exist canonical precubical maps $X_i \to \bigcup_{i\in I} f_i(X_i)$ for all $i \in I$.*

Proof. Using theorem 2.29 we define $\bigcup_{i\in I} f_i(X_i)$ as the precubical set with $\left(\bigcup_{i\in I} f_i(X_i)\right)^n = \bigcup_{i\in I} (f_i(X_i))^n$ for all $n \in \mathbb{N}$. Then for all $i \in I$ we can define $f_i(X_i)$ using theorem 2.31. Theorem 2.30 then gives us the canonical precubical maps $g_i : f_i(X_i) \to \bigcup_{i\in I} f_i(X_i)$ for all $i \in I$, which combined with the canonical precubical maps $h_i : X_i \to f_i(X_i)$ gives us the precubical map $g_i \circ h_i : X_i \to \bigcup_{i\in I} f_i(X_i)$. $\square$

Theorem 2.33. *Let $\{X_i\}_{i\in I}$ be a family of precubical sets, Y be a precubical set and let $\{f_i : X_i \to Y\}_{i\in I}$ be a family of precubical maps. Then for all $S \subseteq T \subseteq I$ there exist canonical inclusion maps $\bigcup_{i\in S} f_i(X_i) \to \bigcup_{i\in T} f_i(X_i)$.*

Proof. This follows from theorem 2.30 and theorem 2.32. $\square$

In the following theorem we construct the small filtered diagram we mentioned in the beginning. The goal is that, from some precubical set Y, we can create a small filtered diagram in a specific way such that every precubical set in the diagram is finite (defined later) and the colimit is Y itself.

Theorem 2.34. *Let $\{X_i\}_{i\in I}$ be a non-empty family of precubical sets, Y be a precubical set and let $\{f_i : X_i \to Y\}_{i\in I}$ be a family of precubical maps. We define the small category J as follows:*

$$obj(J) = \{S \mid S \subseteq I,\ 1 \leq |S| < \infty\}$$

$$mor(J) = \{(f : S \to T) \mid S, T \in obj(J),\ S \subseteq T\}$$

We define the diagram $D : J \to \mathbf{Set}^{\square^{op}}$ as the following: For all $S \in obj(J)$ we define $D_S = \bigcup_{i\in S} f_i(X_i)$.

For all morphisms $f : S \to T$ if $S = T$ we define $D_f : D_S \to D_T$ as the identity map and if $S \subsetneq T$ we define $D_f : D_S \to D_T$ as the canonical inclusion map.
The category J is filtered. The filtered colimit of this diagram is $\left(\bigcup_{i\in I} f_i(X_i), \phi\right)$, with $\phi_S : D_S \to \bigcup_{i\in I} f_i(X_i)$ the canonical inclusion maps.

Proof. It is clear that J is not empty. For all $S, T \in J$ there exists a $S \cup T \in J$ and by definition morphisms $f : S \to S \cup T$ and $g : T \to S \cup T$. By definition the category J contains no parallel morphisms, which therefore makes it a filtered category.
For all $S \in \operatorname{obj}(J)$ we define $\phi_S : D_S \to \bigcup_{i\in I} f_i(X_i)$ as the canonical precubical maps as defined in theorem 2.33. We define the precubical maps $D_f : D_S \to D_T$ as the canonical inclusion maps from theorem 2.33, which means that for all $x \in D_S^n$, $n \in \mathbb{N}$ we have $D_f^n(x) = x \in D_T^n$. Since ϕ_S and ϕ_T are canonical inclusion maps as well this gives us $\phi_S = \phi_T \circ D_f$. This makes $\left(\bigcup_{i\in I} f_i(X_i), \phi\right)$ a co-cone of the diagram D. We clearly have for all $n \in \mathbb{N}$, $x \in \left(\bigcup_{i\in I} f_i(X_i)\right)^n$ that there exists a $S \in J$ and a $y \in D_S^n$ such that $\phi_S^n(y) = x$.
For all $S, T \in J$, $n \in \mathbb{N}$, $x \in D_S$ and $y \in D_T$ if we have $\phi_S^n(x) = \phi_T^n(y)$ then we have by definition that $x = y$. For all $S, T \in J$ we have $S \cup T \in J$ with morphisms $f : S \to S \cup T$ and $g : T \to S \cup T$ which gives us $X_f^n(x) = x = y = X_g^n(y)$ and therefore $x \sim y$. Combined with theorem 2.21 this shows that $\left(\bigcup_{i\in I} f_i(X_i), \phi\right)$ satisfies the conditions of theorem 2.25, making it a filtered colimit of D. □

2.5 Finite precubical sets

We now define what finiteness means for precubical sets, which works as one would expect.

Definition 2.35. *A precubical set $X : \square^{op} \to$ `Set` is called finite if it satisfies the following conditions:*

- *For all $n \in \mathbb{N}$ the set X^n is finite.*
- *There exists a $m \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq m$ we have $X^n = \emptyset$.*

Alternatively requiring that $\coprod_{n\in\mathbb{N}} X^n$ must be finite gives us an equivalent definition.

Theorem 2.36. *Let $\{X_i\}_{i\in I}$ be a family of precubical sets, Y be a precubical set and let $\{f_i : X_i \to Y\}_{i\in I}$ be a family of precubical maps. If $S \subseteq I$ is a finite subset and for all $i \in I$ the precubical sets X_i are finite then $\bigcup_{i\in S} f_i(X_i)$ is finite as well.*

Proof. This follows from the fact that for all $n \in \mathbb{N}$ we have $\left(\bigcup_{i\in S} f_i(X_i)\right)^n = \bigcup_{i\in S} (f_i(X_i))^n$ by definition, since the finite union of finite sets is again a finite set and since for all $n \geq \max_{i\in S}(m_i)$ with m_i being the dimension of $f_i(X_i)$ for all $i \in S$ we have $\left(\bigcup_{i\in S} f_i(X_i)\right)^n = \bigcup_{i\in S} (f_i(X_i))^n = \emptyset$. □

Theorem 2.37. *Let X be a precubical set. For all $n \in \mathbb{N}$, $x \in X^n$ we can construct a precubical set Y_x which contains only x and every element that can be reached with the face maps from x. There exists a canonical precubical map $\gamma_x : Y_x \to X$ with $\gamma_x^n(x) = x$.*

Proof. This follows from theorem 2.29. □

Theorem 2.38. *Every precubical set is the filtered colimit of finite precubical sets.*

Proof. Let X be a precubical set and let $\{Y_x\}_{x\in X^n,\, n\in\mathbb{N}}$ be the family of precubical sets as defined in theorem 2.37. Since by definition for all $x \in X^n$, $n \in \mathbb{N}$ we have $Y_x = \gamma_x(Y_x)$ the family with its precubical maps satisfies the conditions for theorem 2.34.
Theorem 2.36 gives us that for all finite subsets $S \subseteq I$ the precubical set D_S is finite. Since by definition we have $X = \bigcup_{x\in X^n,\, n\in\mathbb{N}} Y_x$ this makes X the filtered colimit of a diagram of finite precubical sets. $\square$

2.6 Compact precubical sets

Now we define what compact precubical sets are. In this subsection we want to prove that this condition is equivalent with finiteness, which means that every compact precubical set is finite and every finite precubical set is compact.

Definition 2.39. *A precubical set $X : \square^{op} \to \mathtt{Set}$ is compact if the corepresentable functor*

$$Hom(X, -) : X \to \mathtt{Set}$$

preserves filtered colimits. This means that for every filtered category J and every diagram $Y : J \to \mathtt{Set}^{\square^{op}}$ the canonical morphism

$$\varinjlim_{i\in J} Hom(X, Y_i) \xrightarrow{\simeq} Hom\left(X, \varinjlim_{i\in J} Y_i\right)$$

is an isomorphism.

This definition requires some explaining. What we first need to do is construct a filtered diagram $\operatorname{Hom}(X, Y)$ using the filtered diagram Y. For all $i \in J$ we have the objects $\operatorname{Hom}(X, Y_i)$ and for all $i, j \in \operatorname{obj}(J)$, $(f : i \to j) \in \operatorname{mor}(J)$ we have the maps

$$\begin{array}{cccc} \operatorname{Hom}(X, Y_f) : & \operatorname{Hom}(X, Y_i) & \longrightarrow & \operatorname{Hom}(X, Y_j) \\ & g & \longmapsto & Y_f \circ g \end{array}$$

Where $g : X \to Y_i$ is a precubical map which is mapped to the precubical map $Y_f \circ g : X \to Y_j$. Let $\left(\varinjlim_{i\in J} \operatorname{Hom}(X, Y_i), \Phi\right)$ be a filtered colimit of this diagram, and let $\left(\varinjlim_{i\in J} Y_i, \phi\right)$ be a filtered colimit of the diagram $Y : J \to \mathtt{Set}^{\square^{op}}$. Then $\left(\operatorname{Hom}\left(X, \varinjlim_{i\in J} Y_i\right), \operatorname{Hom}(X, \phi_i)\right)$ is a co-cone of the diagram $\operatorname{Hom}(X, Y)$ with

$$\begin{array}{cccc} \operatorname{Hom}(X, \phi_i) : & \operatorname{Hom}(X, Y_i) & \longrightarrow & \operatorname{Hom}\left(X, \varinjlim_{i\in J} Y_i\right) \\ & g & \longmapsto & \phi_i \circ g \end{array}$$

being the injection maps. Note that for all $i, j \in \operatorname{obj}(J)$ and $(f : i \to j) \in \operatorname{mor}(J)$ we have $\operatorname{Hom}(X, \phi_j) \circ \operatorname{Hom}(X, Y_f) = \operatorname{Hom}(X, \phi_i)$ since we have $\phi_j \circ Y_f = \phi_i$.
Because of the universal property there exists a unique morphism $U : \varinjlim_{i\in J} \operatorname{Hom}(X, Y_i) \to \operatorname{Hom}\left(X, \varinjlim_{i\in J} Y_i\right)$, which is the canonical morphism mentioned in definition 2.39. Furthermore we get the following commutative diagram:

$$\begin{tikzcd}
\varinjlim_{i\in J} \operatorname{Hom}(X,Y_i) \arrow[r, "U"] & \operatorname{Hom}\left(X, \varinjlim_{i\in J} Y_i\right) \\
\operatorname{Hom}(X,Y_i) \arrow[u, "\Phi_i"] \arrow[ur, "\operatorname{Hom}(X,\phi_i)"'] &
\end{tikzcd}$$

for all $i \in J$.

Theorem 2.40. *Every compact precubical set is finite.*

Proof. Suppose that $X : \square^{\text{op}} \to \texttt{Set}$ is a compact precubical set. Theorem 2.34 gives us that X is the filtered colimit of finite precubical sets. Let $D : J \to \texttt{Set}^{\square^{\text{op}}}$ be the diagram defined in this theorem with $X = \varinjlim_{i\in J} D_i$. We will use the maps shown above. Because X is compact we get that the canonical morphism

$$U : \varinjlim_{i\in J} \operatorname{Hom}(X,Y_i) \xrightarrow{\simeq} \operatorname{Hom}(X,X)$$

is an isomorphism. We have $\mathrm{id}_X \in \operatorname{Hom}(X,X)$ and therefore $U^{-1}(\mathrm{id}_X) \in \varinjlim_{i\in J} \operatorname{Hom}(X,Y_i)$. By definition of the colimit on sets the morphisms Φ_i are jointly surjective for all $i \in J$. Therefore there exists a $i \in J$ and a $g \in \operatorname{Hom}(X,Y_i)$ such that $\Phi_i(g) = U^{-1}(\mathrm{id}_X)$. This then gives us $\mathrm{id}_X = U \circ \Phi_i(g)$ which because the above diagram commutes gives us $\mathrm{id}_X = \operatorname{Hom}(X,\phi_i)(g) = \phi_i \circ g$. This means that the identity factors through a finite precubical set Y_i. Because id_X is surjective the map $\phi_i : Y_i \to X$ must be surjective as well and since Y_i is finite this means that X must be finite as well. $\square$

Definition 2.41. *A precubical set X is representable if the following statements are true:*

1. *There exists a $k \in \mathbb{N}$ such that $|X^k| = 1$ and for all $n \in \mathbb{N}$ with $n > k$ we have $X^n = \emptyset$.*

2. *Every element in X can be reached by the unique element $x \in X^k$ through the face maps.*

3. *Let $n, m \in \mathbb{N}$, $1 \leq m \leq n$ and $x \in X^n$. Then for all $\nu_i, \mu_i \in \{0,1\}$, $a_i, b_i \in \mathbb{N}$, $1 \leq a_i \leq n$ and $1 \leq b_i \leq n$ for all $i \in \mathbb{N}$, $1 \leq i \leq m$ such that for all $i, j \in \mathbb{N}$, $1 \leq i < j \leq m$ we have $a_i < a_j$ and $b_i < b_j$. Then we have*

   $$\delta^n_{(\nu_1,\ldots,\nu_m),(a_1,\ldots,a_m)}(x) = \delta^n_{(\mu_1,\ldots,\mu_m),(b_1,\ldots,b_m)}(x)$$

   if and only if $\nu_i = \mu_i$ and $a_i = b_i$ for all $i \in \mathbb{N}$ with $1 \leq i \leq m$.

This definition also gives us that every representable precubical set is finite as well. From theorem 2.8 it follows that every face map $X^n \to X^{n-m}$ with $n, m \in \mathbb{N}$, $1 \leq m \leq n$ can be expressed as a face map $\delta^n_{(\nu_1,\ldots,\nu_m),(a_1,\ldots,a_m)}$ with $(\nu_1, \ldots, \nu_m)$ and $(a_1, \ldots, a_m)$ as defined above. Therefore the third statement states that every pair of different face maps will map the same element to two different elements.
This definition might look complicated but the result is that the representable precubical sets are the most "trivial" precubical sets of their dimension. A representable precubical set of dimension 0 is simply a single node, one of dimension 1 is two nodes connected by a single edge and one of dimension 2 is a precubical set like in figure 2. The representable precubical sets are the largest precubical sets of their dimension that satisfy the first two statements of definition 2.41.

Remark 2.41.1. *Our definition of representability is the same as the precubical set being naturally isomorphic to Hom $(-, C)$ for some $C \in \square$. If $k \in \mathbb{N}$ is the number such that $|X^k| = 1$ then this object is $C = [k] = (1 \to 2 \to \dots \to k)$ as described in* [FJSZ21].

Note that if $\mathbb{Y} : \square \to \mathtt{Set}^{\square^{\mathrm{op}}}$ is the Yoneda embedding then for all $C \in \square$ we have $\mathrm{Hom}\,(-, C) = \mathbb{Y}(C)$. The k-representable precubical sets are also referred to as the standard k-cubes, analogous to the terminology in simplical sets.
The above remark would be a very useful fact as it would give us that all representable precubical sets are compact with little proof, but since we haven't properly defined the category $\square$ we can't really do that. Proving it manually will also give us more insight as to how precubical sets, precubical maps and colimits work.

Theorem 2.42. *Let X and Y be finite precubical sets of dimension $k \in \mathbb{N}$ that satisfy the first two properties of definition 2.41. Then there exists a precubical map $f : X \to Y$ if and only if for $x \in X^k$ and $y \in Y^k$ we have*

$$\begin{gathered} \delta^k_{(\nu_1,\dots,\nu_m),(a_1,\dots,a_m)}(x) = \delta^k_{(\mu_1,\dots,\mu_m),(b_1,\dots,b_m)}(x) \\ \Downarrow \\ \delta^k_{(\nu_1,\dots,\nu_m),(a_1,\dots,a_m)}(y) = \delta^k_{(\mu_1,\dots,\mu_m),(b_1,\dots,b_m)}(y) \end{gathered}$$

for all $\nu_i, \mu_i \in \{0, 1\}$, $a_i, b_i \in \mathbb{N}$, $1 \leq a_i \leq n$ and $1 \leq b_i \leq n$ for all $i \in \mathbb{N}$, $1 \leq i \leq m$ such that for all $i, j \in \mathbb{N}$, $1 \leq i < j \leq m$ we have $a_i < a_j$ and $b_i < b_j$.

Proof. Let $f : X \to Y$ be a precubical map and let $x \in X^k$ and $y \in Y^k$. Then we have to have $f^k(x) = y$ and because f commutes with the face maps we have

$$\begin{gathered} f^{k-m} \circ \delta^k_{(\nu_1,\dots,\nu_m),(a_1,\dots,a_m)}(x) = f^{k-m} \circ \delta^k_{(\mu_1,\dots,\mu_m),(b_1,\dots,b_m)}(x) \\ \Downarrow \\ \delta^k_{(\nu_1,\dots,\nu_m),(a_1,\dots,a_m)} \circ f^k(x) = \delta^k_{(\mu_1,\dots,\mu_m),(b_1,\dots,b_m)} \circ f^k(x) \end{gathered}$$

Suppose that the implication is true. We can define $f^n : X^n \to Y^n$ for all $n \in \mathbb{N}$ with $f^k(x) = y$ and

$$f^{k-m} \circ \delta^k_{(\nu_1,\dots,\nu_m),(a_1,\dots,a_m)}(x) = \delta^k_{(\nu_1,\dots,\nu_m),(a_1,\dots,a_m)}(y)$$

with $m = k - n$. Because of theorem 2.8 and the third property of representable precubical sets every element in X^n with $n \in \mathbb{N}$, $n < k$ can be described as $\delta^n_{(\nu_1,\dots,\nu_{k-n}),(a_1,\dots,a_{k-n})}(x)$ for certain $(\nu_1, \dots, \nu_{k-n})$ and $(a_1, \dots, a_{k-n})$. This describes the behaviour of f^n for all $n \in \mathbb{N}$ on every element of X^n and since the implication in the theorem statement is assumed to be true this makes f^n well-defined. Since the components commute with the face maps by definition the family of maps $\{f^n\}_{n\in\mathbb{N}}$ satisfies the properties for precubical maps and therefore f is a precubical map. $\square$

There are two things to note: The first is that the precubical map $f : X \to Y$ in the theorem above is always unique, since there is only one element $x \in X^k$ can be mapped to and every other mapping is then defined by the requirement that f commutes with the face maps. The other thing to note is that in the case that X is representable then the statement is true for every precubical set Y that has the required properties. This leads into the following theorem:

Theorem 2.43. *Let X and Y be precubical sets such that X is a representable precubical set of dimension $k \in \mathbb{N}$. Then for every $y \in Y^k$ there exists a unique precubical map $f_y : X \to Y$ such that for the unique element $x \in X^k$ we have $f_y^k(x) = y$. These are the only precubical maps $X \to Y$.*

Proof. Using theorem 2.29 we can create a precubical set Z with $Z^k = \{y\}$ which is closed under the face maps (Z^{k-1} contains every element that can be reached by an elementary face map from y, Z^{k-2} contains every element that can be reached by an elementary face map from an element in Z^{k-1} etc.). This gives us the precubical map $g : Z \to Y$ with $g^k(y) = y$. Theorem 2.42 then gives us that there exists a precubical map $h : X \to Z$, which then gives us the precubical map $f_y : X \to Y$ with $f_y = g \circ h$.
Since we must have $h^k(x) = y$ (with $x \in X^k$) this gives us $f_y^k(x) = y$ as required. Note that f_y must be unique, since the requirement $f_y^k(x) = y$ determines the mapping of the other elements because precubical maps must commute with face maps and every other element can be reached by x through the face maps.
Note that because $x \in X^k$ needs to be mapped to some $y \in Y^k$ these precubical maps are the only precubical maps $X \to Y$. $\square$

This means that any precubical map $f : X \to Y$ with X a k-dimensional representable precubical set can be uniquely identified with a $y \in Y^k$. We can now finally prove the following:

Theorem 2.44. *All representable precubical sets are compact.*

Proof. Let X be a representable precubical set of dimension n and let $D : J \to \mathtt{Set}^{\square^{\mathrm{op}}}$ be a small filtered diagram with the colimit $\left(\varinjlim_{i \in J} D_i, \phi\right)$. Recall the diagram below theorem 2.39. Suppose that $\varinjlim_{i \in J} D_i$ is not empty (in which case the statement would be trivial, since then both sets in the top of the diagram would be empty as well).
Let $f \in \mathrm{Hom}\left(X, \varinjlim_{i \in J} D_i\right)$. Because of theorem 2.43 there exists a unique $y \in \varinjlim_{i \in J} D_i^n$ such that for the unique element $x \in X^n$ we have $f^n(x) = y$. Also note that f is the only precubical map in $\mathrm{Hom}\left(X, \varinjlim_{i \in J} D_i\right)$ that sends x to y. Theorem 2.24 then gives us that there exists a $i \in J$ and a $x_i \in D_i^n$ such that $\phi_i^n(x_i) = y$. Using theorem 2.43 again gives us that there exists a unique precubical map $g \in \mathrm{Hom}(X, D_i)$ with $g^n(x) = x_i$ and therefore $\phi_i^n \circ g^n(x) = \phi_i^n(x_i) = y$. Therefore the morphism $\mathrm{Hom}(X, \phi_i)$ sends g to f, which also means that $U \circ \Phi_i \circ g = f$. This then gives us that U is surjective.
Let $f_1, f_2 \in \varinjlim_{i \in J} \mathrm{Hom}(X, D_i)$, $f \in \mathrm{Hom}\left(X, \varinjlim_{i \in J} D_i\right)$ such that $U \circ f_1 = U \circ f_2 = f$. Then there exist $i, j \in J$, $g_i \in \mathrm{Hom}(X, D_i)$ and $g_j \in \mathrm{Hom}(X, D_j)$ such that $\Phi_i \circ g_i = f_1$ and $\Phi_j \circ g_j = f_2$. This then also gives us that $U \circ \Phi_i \circ g_i = f$ and $U \circ \Phi_j \circ g_j = f$ which gives us that $\phi_i \circ g_i = \phi_j \circ g_j = f$. Because of theorem 2.43 there exist unique $x_i \in D_i^n$, $x_j \in D_j^n$ and $y \in \varinjlim_{i \in J} D_i^n$ such that $g_i^n(x) = x_i$, $g_j^n(x) = x_j$ and $f^n(x) = y$. This gives us $\phi_i^n(x_i) = \phi_j^n(x_j) = y$, which due to theorem 2.22 means that there exist $k \in J$, $h_i : D_i \to D_k$ and $h_j : D_j \to D_k$ such that $h_i \circ g_i(x_i) = h_j \circ g_j(x_j)$ and therefore $h_i \circ g_i = h_j \circ g_j$. This means that we have $\mathrm{Hom}(X, h_i)(g_i) = \mathrm{Hom}(X, h_j)(g_j)$ and therefore $f_1 = \Phi_i \circ g_i = \Phi_j \circ g_j = f_2$. This then gives us that U is injective.
Therefore the canonical morphism $U : \varinjlim_{i \in J} \mathrm{Hom}(X, D_i) \to \mathrm{Hom}\left(X, \varinjlim_{i \in J} D_i\right)$ is an isomorphism for every small filtered diagram $D : J \to \mathtt{Set}^{\square^{\mathrm{op}}}$, which means that X is a compact precubical set. $\square$

Theorem 2.45. *The finite colimit of compact precubical sets is again a compact precubical set.*

Proof. This follows from proposition 1.3 of [AR94]. □

Theorem 2.46. *Every finite precubical set is compact.*

Proof. Let X be a finite precubical set of dimension $k \in \mathbb{N}$. We define the small category J as

$$\mathrm{obj}(J) = \bigsqcup_{n \in \mathbb{N}} X^n$$

$$\mathrm{mor}(J) = \left\{ \left(f : \delta^n_{V,A}(x) \to x \right) \middle| n, m \in \mathbb{N},\ x \in X^n,\ V, A \text{ with } |V| = |A| = m \right\}$$

We define the diagram $D : J \to \mathtt{Set}^{\square^{\mathrm{op}}}$ where for all $n \in \mathbb{N}$, $x \in X^n$ the precubical set D_x is a representable precubical set of dimension n and for all $\left(f : \delta^n_{V,A}(x) \to x \right)$ we define $D_f : D_{\delta^n_{V,A}(x)} \to D_x$ as the precubical map that sends the unique element of $D^m_{\delta^n_{V,A}(x)}$ to $\delta^n_{V,A}(y)$, with y being the unique element of D^n_x.
We can then define (X, ϕ), with for all $n \in \mathbb{N}$, $x \in X^n$ the injection maps $\phi_x : D_x \to X$ being the canonical precubical maps that send the unique element of D^n_x to $x \in X^n$. Suppose that we have $i, j \in J$ and a morphism $f : i \to j$. Let $x \in X^n$ with $n \in \mathbb{N}$ be the element $x = j$. By definition we have $i = \delta^n_{V,A}(x)$. The precubical map ϕ_i sends the unique element of D^n_i to $\delta^n_{V,A}(x)$ and the precubical map ϕ_j sends the unique element of D^n_j to x. Because of theorem 2.43 there exists at most one precubical map from D_i to X. Since we have $\phi_i : D_i \to X$ and $\phi_j \circ D_f : D_i \to X$ this means that we must have $\phi_i = \phi_j \circ D_f$. This makes (X, ϕ) a co-cone of D.
Suppose that we have $n \in \mathbb{N}$, $i, j \in J$, $x \in D^n_i$ and $y \in D^n_j$ such that $x \sim y$. Theorem 2.21 then gives us that $\phi^n_i(x) = \phi^n_j(y)$.
Suppose that we have $n \in \mathbb{N}$, $i, j \in J$, $x \in D^n_i$ and $y \in D^n_j$ such that $\phi^n_i(x) = \phi^n_j(y) = z$. Then there exist precubical maps $D_z \to D_i$ and $D_z \to D_j$ which send the unique element of D^n_z to x and y respectively. This then gives us that $x \sim y$.
For all $n \in \mathbb{N}$, $x \in X^n$ we have $x \in J$ such that $\phi_i : D_x \to X$ sends the unique element of D^n_x to x. Therefore (X, ϕ) satisfies the requirements for theorem 2.25 which makes it a colimit of D.
Because X is a finite precubical set $\bigsqcup_{n \in \mathbb{N}} X^n$ is finite as well and therefore J is a finite category. The precubical sets D_i are representable for all $i \in J$ which because of theorem 2.44 makes them compact. Therefore X is the finite colimit of compact precubical sets which because of theorem 2.45 makes it compact. □

Corollary 2.46.1. *A precubical set is finite if and only if it is compact.*

Proof. This follows from theorem 2.40 and theorem 2.46. □

2.7 Category of precubical sets

We can now understand the following definition:

Definition 2.47. *A category $\mathcal{C}$ is locally finitely presentable if it satisfies the following condition:*

1. *$\mathcal{C}$ is cocomplete.*
2. *The full subcategory $\mathcal{C}_c$ of $\mathcal{C}$ consisting of the compact objects is essentially small.*

3. Any object in C is a filtered colimit of a diagram of compact objects.

As we have proven before of course we can replace "compact" with "finite" in the above definition.

Theorem 2.48. *For C any small category the category of presheaves $\mathtt{Set}^C$ is locally finitely presentable.*

Proof. This follows from remark 3 of [CRV0401]. $\square$

Theorem 2.49. *The category of precubical sets $\mathtt{Set}^{\square^{op}}$ is locally finitely presentable.*

Proof. Since $\square^{op}$ is small theorem 2.48 gives us that $\mathtt{Set}^{\square^{op}}$ is locally finitely presentable. $\square$

In later chapters we will keep checking how many of the conditions stated in definition 2.47 remain true for the category of event consistent precubical sets and the category of HDAs. While these categories might not turn out to be locally finitely presentable it's still interesting to see why not and what does turn out to be true. The same goes for the equivalence of finiteness and compactness.

3 Event consistent precubical sets

In this section we introduce event consistent precubical sets, which are precubical sets that satisfy a certain condition. This condition introduces the idea of events, which will become relevant when we move on to higher-dimensional automata. We will look at colimits of diagrams of event consistent precubical sets, we will introduce the category of event consistent precubical sets as a full subcategory of $\mathtt{Set}^{\square^{op}}$ and look at what of the conditions for local finite presentability it inherits.

3.1 Definition event consistency

Using lemma 18 from [FJSZ21] we get the following:

Definition 3.1. *A precubical set X is event consistent if and only if there exists an equivalence relation $\equiv$ on X^1 such that for all $x \in X^2$, $\nu, \mu \in \{0, 1\}$ and $a, b \in \{1, 2\}$ we have $\delta^2_{\nu,a}(x) \equiv \delta^2_{\mu,b}(x)$ if and only if $a = b$.*

Here we have defined an equivalence relation on the set of edges of a precubical set. To understand how this equivalence relation works it is best to look at the following example again.

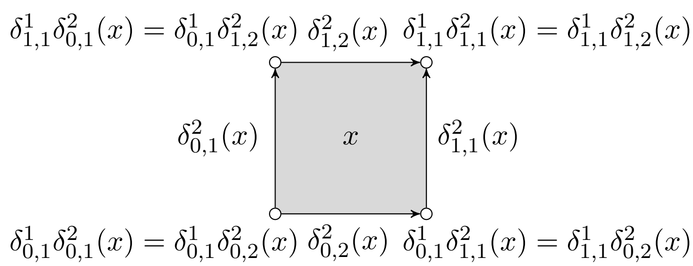
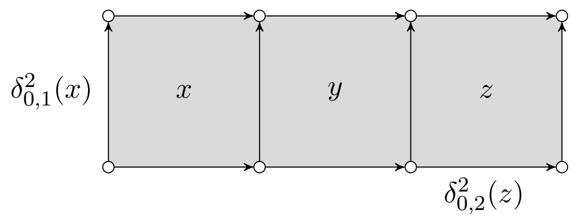


Figure 6: The square $x \in X^2$, its four elementary faces and their four corners.

The condition on the equivalence relation is that for all $x \in X^2$, $\nu, \mu \in \{0,1\}$ and $a, b \in \{1,2\}$ we have $\delta^2_{\nu,a}(x) \equiv \delta^2_{\mu,b}(x)$ if and only if $a = b$. This means that in the example above we must have $\delta^2_{0,1}(x) \equiv \delta^2_{1,1}(x)$ and $\delta^2_{0,2}(x) \equiv \delta^2_{1,2}(x)$ and we must also have $\delta^2_{0,1}(x) \not\equiv \delta^2_{0,2}(x)$, $\delta^2_{0,1}(x) \not\equiv \delta^2_{1,2}(x)$, $\delta^2_{1,1}(x) \not\equiv \delta^2_{0,2}(x)$ and $\delta^2_{1,1}(x) \not\equiv \delta^2_{1,2}(x)$. In other words: the parallel edges are equivalent under $\equiv$ and edges that have a common higher face (in this case x) but that are not parallel cannot be equivalent under $\equiv$. A precubical set is event consistent if such an equivalence relation can be realized without contradiction. The above precubical set is a very simple and clear example of an event consistent precubical set. For an example of a precubical set that is not event consistent we have the following: We start with a precubical set



which consists of three squares x, y and z glued together by identifying $\delta^2_{1,1}(x) = \delta^2_{0,1}(y)$ and $\delta^2_{1,1}(y) = \delta^2_{0,1}(z)$. This precubical set is still event consistent. We can define an equivalence relation with four different equivalence classes: one containing all of the vertical edges and three containing both horizontal edges for each of x, y and z. This is the smallest possible equivalence relation as defined in definition 3.1, which we will later refer to the event relation. With this it becomes clear what we can do to make the precubical set not event consistent. By identifying $\delta^2_{0,1}(x) = \delta^2_{0,2}(z)$ we get the following precubical set:

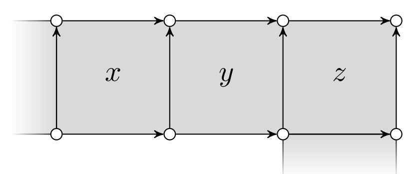


Figure 7: Example of a precubical set that is not event consistent (left and bottom right edges identified). Taken from [FJSZ21].

Here we have to have $\delta^2_{0,1}(z) \equiv \delta^2_{0,1}(x)$, but since we have identified $\delta^2_{0,1}(x) = \delta^2_{0,2}(z)$ this then also gives us $\delta^2_{0,1}(z) \equiv \delta^2_{0,2}(z)$. This means that there exists no equivalence relation that satisfies both conditions as defined in definition 3.1, which means that this precubical set is not event consistent. One might look at this example and think that a precubical set cannot be event consistent if for certain $x, y \in X^2$ and $\nu, \mu \in \{0,1\}$ we have $\delta^2_{\nu,1}(x) = \delta^2_{\mu,2}(y)$, but this is not the case. It is only a problem if we would also have $\delta^2_{\nu,2}(x) \equiv \delta^2_{\mu,2}(y)$, since this would by the transitive property give us $\delta^2_{\nu,1}(x) \equiv \delta^2_{\mu,2}(x)$. A precubical set is not event consistent if it contains a square of which all of the edges have somehow become parallel.
Event consistency of a precubical set depends on the existence of an equivalence relation that satisfies certain properties. We can instead using one of the properties define a unique equivalence relation which we will refer to as the event relation.

Definition 3.2. *Any event consistent precubical set X admits a smallest equivalence relation owing to lemma 18 from* [FJSZ21]. *It is given as the transitive closure of* $\{(\delta^2_{\nu,a}(x), \delta^2_{\mu,a}(x)) \mid x \in X^2,\ \nu, \mu \in \{0,1\},\ a \in \{1,2\}\}$. *We will refer to this smallest equivalence relation as the event relation, and we call its equivalence classes the universal events of X.*

We will now mostly refer to the event relation, instead of a possible different equivalence relation. It is by definition the smallest equivalence relation such that for all $x \in X^2$, $\nu, \mu \in \{0,1\}$ and $a, b \in \{1,2\}$ if $a = b$, and therefore if it doesn't satisfy the "only if" part of the condition then no equivalence relation will (since this requires certain element to not be equivalent). Note that the event relation exists for all precubical sets, but doesn't always satisfy the "only if" part of the condition.
It is now clear what we mean when we are talking about events of precubical sets. Edges correspond to single events, the squares correspond with two events executing at the same time and the cubes correspond with three events executing at the same time. This continues for higher dimensions. The event consistent precubical sets will eventually become the base for our higher-dimensional automata.
We have looked at some examples of event consistent precubical sets. In the next theorem we will prove that all representable precubical sets are event consistent, which because of the proof of theorem 2.46 can be used to construct any other finite precubical set through a colimit, which then can be used to construct any non-finite precubical set through a filtered colimit as proven in theorem 2.38.

Theorem 3.3. *All representable precubical sets are event consistent.*

Proof. Let X be a k-dimensional representable precubical set with $k \in \mathbb{N}_{\geq 2}$. Note that the statement is trivial for 0-dimensional and 1-dimensional precubical sets, since by definition they are always event consistent. From theorem 2.9 it follows that all face maps $X^k \to X^1$ can be represented as $\delta^k_{V,A^k_a}(x)$, with $x \in X^k$ the unique element, V a $k-1$-dimensional vector of elements $\nu_i \in \{0,1\}$ and A^k_a as defined in definition 2.10 with $a \in \mathbb{N}$, $1 \leq a \leq k$. Note that because of the definition of representable precubical sets the representation for every element X^1 is unique.
Let $a, b \in \mathbb{N}$ with $1 \leq a \leq k$ and $1 \leq b \leq k$ and let V and U be $k-1$-dimensional vector of elements $\nu_i, \mu_i \in \{0,1\}$. Suppose that there exists a $y \in X^2$, $c \in \{1,2\}$ such that $\delta^2_{0,c}(y) = \delta^k_{V,A^k_a}(x)$ and $\delta^2_{1,c}(y) = \delta^k_{U,A^k_b}(x)$. Then we have $a = b$ and the vectors V and U are the same except for a single element. This follows from the fact that y can also be uniquely represented as $y = \delta^k_{W,A}(x)$ with $k-2$-dimensional vectors W and A as in definition 2.7. Then we get $\delta^k_{V,A^k_a}(x) = \delta^2_{0,c} \circ \delta^k_{W,A}(x)$ and $\delta^k_{U,A^k_b}(x) = \delta^2_{1,c} \circ \delta^k_{W,A}(x)$ from which we get $\delta^k_{V,A^k_a}(x)$ and $\delta^k_{U,A^k_b}(x)$ through applying the same identities, which proves the statement.
Note that the opposite is true as well. Let $a \in \mathbb{N}$ with $1 \leq a \leq k$ and let V_0 and V_1 be $k-1$-dimensional vectors of elements $\nu_i, \mu_i \in \{0,1\}$. Suppose that there exists a $j \in \mathbb{N}$, $1 \leq j \leq k-1$ such that for all $i \in \mathbb{N}$, $1 \leq i \leq k-1$ we have $\nu_i = \mu_i$ if and only if $i \neq j$. This gives us the elements $\delta^k_{V_0,A^k_a}(x)$ and $\delta^k_{V_1,A^k_a}(x)$. There exists a $k-2$-dimensional vector A with every element of A^k_a except for a_j and a $k-2$-dimensional vector V with every element of V_0 and V_1 except for ν_j and ν_i such that $\delta^k_{V_0,A^k_a}(x) = \delta^2_{0,a_j-j+1} \circ \delta^k_{V,A}(x)$ and $\delta^k_{V_1,A^k_a}(x) = \delta^2_{1,a_j-j+1} \circ \delta^k_{V,A}(x)$.
We now define the equivalence relation $\equiv_X$ on X^1 as

$$\left\{\left(\delta^k_{V,A^k_a}(x), \delta^k_{U,A^k_a}(x)\right) \mid x \in X^k,\ a \in \mathbb{N},\ 1 \leq a \leq k,\ V, U \in \{0,1\}^k\right\}$$

It's clear that this relation $\equiv_X$ is reflexive, symmetric and transitive. Suppose that we have $y \in X^2$, $\nu, \mu \in \{0,1\}$, $b, c \in \{1,2\}$ with $\delta^2_{\nu,b}(y) \equiv_x \delta^2_{\mu,c}$. Then we have $\delta^2_{\nu,b}(y) = \delta^k_{V,A^k_a}(x)$ and $\delta^2_{\mu,c} = \delta^k_{U,A^k_a}(x)$ for certain vectors V, U and A^k_a which is true if and only if $b = c$. This shows that $\equiv_X$ satisfies the conditions of definition 3.1 which makes X an event consistent precubical set. $\square$

3.2 Preserving event consistency

Theorem 3.4. *Let X and Y be precubical sets, let $f : X \to Y$ be a precubical map and let $\equiv_X$ and $\equiv_Y$ be the event relations on X^1 and Y^1. For all $x_1, x_2 \in X^1$ we have*

$$x_1 \equiv_X x_2 \implies f^1(x_1) \equiv_Y f^1(x_2)$$

Proof. The equivalence relation $\equiv_Y$ is defined as the transitive closure of

$$\left\{\left(\delta^2_{\nu,a}(y), \delta^2_{\mu,a}(y)\right) \mid y \in Y^2,\, \nu, \mu \in \{0,1\},\, a \in \{1,2\}\right\}$$

and the equivalence relation $\equiv_X$ is defined as the transitive closure of

$$\left\{\left(\delta^2_{\nu,a}(x), \delta^2_{\mu,a}(x)\right) \mid x \in X^2,\, \nu, \mu \in \{0,1\},\, a \in \{1,2\}\right\}$$

We define $\equiv_{f(X)}$ as the transitive closure of

$$\left\{\left(\delta^2_{\nu,a} \circ f^2(x), \delta^2_{\mu,a} \circ f^2(x)\right) \mid x \in X^2,\, \nu, \mu \in \{0,1\},\, a \in \{1,2\}\right\}$$

which is equal to the transitive closure of

$$\left\{\left(\delta^2_{\nu,a}(y), \delta^2_{\mu,a}(y)\right) \mid y \in Y^2,\, \exists x \in X^2 \text{ s.t. } f^2(x) = y,\, \nu, \mu \in \{0,1\},\, a \in \{1,2\}\right\}$$

Here we clearly have for all $x_1, x_2 \in X^1$ that $x_1 \equiv_X x_2 \implies f^1(x_1) \equiv_{f(X)} f^1(x_2)$. Since $\equiv_{f(X)}$ must be contained within $\equiv_Y$ we get the result. $\square$

Theorem 3.5. *Let X and Y be precubical sets and let $f : X \to Y$ be a precubical map. If Y is event consistent then X is event consistent as well.*

Proof. Suppose that X is not event consistent. Then there exists a $x \in X^2$ and $\nu, \mu \in \{0,1\}$ such that $\delta^2_{\nu,1}(x) \equiv_X \delta^2_{\mu,2}(x)$. Theorem 3.4 then gives us that

$$\delta^2_{\nu,1} \circ f^2(x) = f^1 \circ \delta^2_{\nu,1}(x) \equiv_Y f_1 \circ \delta^2_{\mu,2}(x) = \delta^2_{\mu,2} \circ f^2(x)$$

which would make Y not event consistent. This means that no such $x \in X^2$ can exist and therefore X is event consistent as well. $\square$

Theorem 3.6. *Let $X : J \to \mathbf{Set}^{\square^{op}}$ be a small diagram. If X has an event consistent co-cone then X_i is event consistent for all $i \in J$.*

Proof. Let (N, ψ) be an event consistent co-cone of X. Then for all $i \in J$ there exists the precubical map $\psi_i : X_i \to N$ which using theorem 3.5 gives us that X_i is event consistent for all $i \in J$. $\square$

Theorem 3.7. *Let $X : J \to \mathbf{Set}^{\square^{op}}$ be a small diagram. If X has an event consistent co-cone then it has an event consistent colimit.*

Proof. Let (N, ψ) be an event consistent co-cone. From theorem 2.11 it follows that X has a colimit (L, ϕ). By definition there must exist a precubical map $q : L \to N$, which because of theorem 3.5 gives us that L is event consistent as well. $\square$

Theorem 3.8. *Let $X : J \to \mathbf{Set}^{\square^{op}}$ be a small diagram and let (L, ϕ) be the colimit of this diagram. For all $i \in J$ let $\equiv_i$ be the event relation on X_i^1.*
We define the relation $\equiv_L$ on L^1 as the following: for every two elements $y_1, y_2 \in L^1$ we have $y_1 \equiv_L y_2$ if and only if there exist $i \in J$, X_i with $x_1, x_2 \in X_i^1$ such that $\phi_i^1(x_1) = y_1$, $\phi_i^1(x_2) = y_2$ and $x_1 \equiv_i x_2$. If J is a discrete or filtered category then this relation $\equiv_L$ is well-defined and is equal to the event relation.

Proof. It is clear that the relation $\equiv_L$ is reflexive and symmetric. Suppose that we have $y_1, y_2, y_3 \in L^1$ with $y_1 \equiv_L y_2$ and $y_2 \equiv_L y_3$. Then there exist $i, j \in J$, $x_1, x_2 \in X_i^1$ and $x_2', x_3 \in X_j^1$ with $\phi_i^1(x_1) = y_1$, $\phi_i^1(x_2) = y_2 = \phi_j^1(x_2')$ and $\phi_j^1(x_3) = y_3$ such that $x_1 \equiv_i x_2$ and $x_2 \equiv_j x_3$.
In the case that J is discrete theorem 2.13 gives us that $x_2 = x_2'$ and $i = j$, which because $\equiv_i$ is transitive gives us $x_1 \equiv_i x_3$ and therefore $y_1 \equiv_L y_3$.
Let J be a filtered category. Theorem 2.22 gives us that since we have $\phi_i^n(x_2) = \phi_j^n(x_2')$ we therefore have $x_2 \sim x_2'$. Theorem 2.24 then gives us that there exists a $k \in J$ and morphisms $f : i \to k$ and $g : j \to k$ in J such that $X_f^n(x_2) = X_g^n(x_2')$. As a result of theorem 3.4 the event relation is preserved through precubical maps which gives us

$$X_f^n(x_1) \equiv_k X_f^n(x_2) = X_g^n(x_2') \equiv_k X_g^n(x_3)$$

which gives us $X_f^n(x_1) \equiv_k X_g^n(x_3)$ and therefore $y_1 \equiv_L y_3$.
In both cases the relation $\equiv_L$ is therefore an equivalence relation.
Let $y \in L^2$ and let $i \in J$, $x \in X_i^2$ such that $\phi_i^n(x) = y$. Then because for all $a \in \{1, 2\}$ and $\nu, \mu \in \{0, 1\}$ we have $\delta_{\nu,a}^2(x) \equiv_i \delta_{\mu,a}^2(x)$ we by definition also have $\delta_{\nu,a}^2(y) \equiv_L \delta_{\mu,a}^2(y)$. Since the event relation is the transitive closure of these relations and since $\equiv_L$ is transitive this gives us that if two elements are equivalent by the event relation then they are equivalent by $\equiv_L$ as well.
Let $y_1, y_2 \in Y^1$ such that $y_1 \equiv_L y_2$. By definition there must exist $i \in J$, $x_1, x_2 \in X_i^1$ with $\phi_i^1(x_1) = y_1$ and $\phi_i^1(x_2) = y_2$ such that $x_1 \equiv_i x_2$. Theorem 3.4 then gives us that y_1 and y_2 must also be equivalent by the event relation.
This gives us that two elements are equivalent by the event relation if and only if they are equivalent by $\equiv_L$, which gives us that $\equiv_L$ is the event relation. $\square$

Theorem 3.9. *Let $X : J \to \mathbf{Set}^{\square^{op}}$ be a small discrete diagram. The coproduct of this diagram is event consistent if and only if X_i is event consistent for all $i \in J$.*

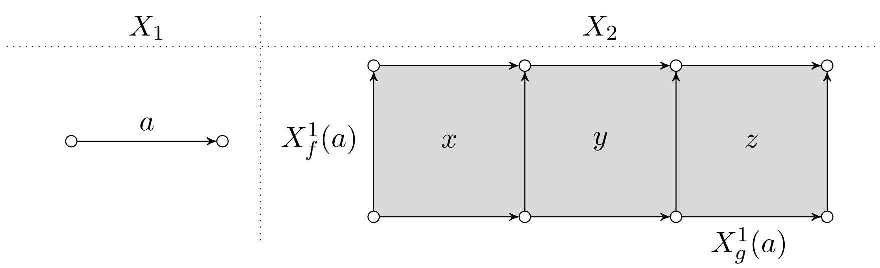
Proof. From theorem 3.6 it follows that if the coproduct is event consistent then for all $i \in J$, X_i is event consistent as well.
Let (L, φ) be a coproduct of the diagram, let $\equiv_L$ be the event relation on L^1 and for all $i \in J$ let $\equiv_i$ be the event relation on X_i^1. Suppose that there exists a $y \in L^2$ such that there exist $\nu, \mu \in \{0, 1\}$ for which we have $\delta_{\nu,1}^n(y) \equiv_L \delta_{\mu,2}^n(y)$. Theorem 2.13 gives us that there exists a unique $i \in J$, $x \in X_i^2$ such that $\varphi_i^2(x) = y$. We have to have $\varphi_i^1 \circ \delta_{\nu,1}^n(x) = \delta_{\nu,1}^n(y)$ and $\varphi_i^1 \circ \delta_{\mu,2}^n(x) = \delta_{\mu,2}^n(y)$ due to theorem 2.13 again. Then because of theorem 3.8 we have to have $\delta_{\nu,1}^n(x) \equiv_i \delta_{\mu,2}^n(x)$, which is in contradiction with X_i being event consistent. Therefore such an element $y \in L^2$ cannot exist, making L event consistent as well. $\square$

Theorem 3.10. *Let $X : J \to \mathbf{Set}^{\square^{op}}$ be a small filtered diagram. The filtered colimit of this diagram is event consistent if and only if X_i is event consistent for all $i \in J$.*

Proof. From theorem 3.6 it follows that if the filtered colimit is event consistent then for all $i \in J$, X_i is event consistent as well.
Let (L, φ) be a filtered colimit of the diagram, let $\equiv_L$ be the event relation on L^1 and for all $i \in J$ let $\equiv_i$ be the event relation on X_i^1. Suppose that there exists a $y \in L^2$ such that there exist $\nu, \mu \in \{0, 1\}$ for which we have $\delta^n_{\nu,1}(y) \equiv_L \delta^n_{\mu,2}(y)$.
Theorem 2.24 and theorem 3.8 gives us that there exists a $i, j \in J$, $x \in X_i^2$, $x_1 \in X_j^1$ and $x_2 \in X_j^2$ such that $\phi_i^2(x) = y$, $\phi_j^1(x_1) = \delta^n_{\nu,1}(y)$ and $\phi_j^1(x_2) = \delta^n_{\mu,2}(y)$ with $x_1 \equiv_j x_2$. Since J is a filtered category and by theorem 2.24 we have $x_1 \sim \delta^n_{\nu,1}(x)$ and $x_2 \sim \delta^n_{\mu,2}(x)$ there exists a $k \in J$ and maps $f : i \to k$, $g : j \to k$ such that $X_f^1 \circ \delta^n_{\nu,1}(x) = X_g^1(x_1)$ and $X_f^1 \circ \delta^n_{\mu,2}(x) = X_g^1(x_2)$. Theorem 3.5 then gives us that $X_f^1 \circ \delta^2_{\nu,1}(x) \equiv_k X_f^1 \circ \delta^2_{\mu,2}(x)$ and therefore $\delta^2_{\nu,1} \circ X_f^2(x) \equiv_k \delta^2_{\mu,2} \circ X_f^2(x)$, which is in contradiction with X_k being event consistent. □

Theorem 3.11. *Not every colimit of event consistent precubical sets is event consistent.*

Proof. We can show this using a simple example. We define J as the small category with $\mathrm{obj}(J) = \{1, 2\}$ and $\mathrm{mor}(J) = \{(f : 1 \to 2), (g : 1 \to 2)\}$. We define the diagram $X : J \to \mathtt{Set}^{\square^{op}}$ as the following:



Here the precubical maps $X_f : X_1 \to X_2$ and $X_g : X_1 \to X_2$ send the element $a \in X_1^1$ to the elements $\delta^2_{0,1}(x)$ and $\delta^2_{0,2}(z)$ respectively. This gives us that $\delta^2_{0,1}(x) \sim \delta^2_{0,2}(z)$, and therefore $\phi_2^1 \circ \delta^2_{0,1}(x) = \phi_2^1 \circ \delta^2_{0,2}(z)$ with ϕ the injection map for the colimit (L, ϕ) of the diagram X. If we take $\equiv_L$ as the event relation on L^1 then we get

$$\delta^2_{0,1} \circ \phi_2^2(x) \equiv_L \delta^2_{1,1} \circ \phi_2^2(x) = \delta^2_{0,1} \circ \phi_2^2(y) \equiv_L \delta^2_{1,1} \circ \phi_2^2(y) = \delta^2_{0,1} \circ \phi_2^2(z)$$

which gives us that $\delta^2_{0,1} \circ \phi_2^2(x) \equiv_L \delta^2_{0,1} \circ \phi_2^2(z)$. Because we have $\delta^2_{0,1} \circ \phi_2^2(x) = \delta^2_{0,2} \circ \phi_2^2(z)$ this then gives us that $\delta^2_{0,1} \circ \phi_2^2(z) \equiv_L \delta^2_{0,2} \circ \phi_2^2(z)$, which means that L is not event consistent. Moreover, the colimit L is actually also the precubical set shown in figure 7. □

3.3 The category ECPS

As we have defined the category of precubical sets $\mathtt{Set}^{\square^{op}}$, we now also want to define the category of event consistent precubical sets.

Definition 3.12. *The category **ECPS** is the full subcategory of the category of precubical sets $\mathbf{Set}^{\square^{op}}$ containing only the event consistent precubical sets and their morphisms.*

The category $\mathtt{ECPS}$ is a subcategory of $\mathtt{Set}^{\square^{op}}$, which means that it inherits many of the properties of the precubical sets. We are specifically interested in local finite presentability, but before that we first need to make sure colimits and compactness work properly.

Theorem 3.13. *The category of event consistent precubical sets $\mathtt{ECPS}$ is not cocomplete.*

Proof. As a result of theorem 3.11 the category of event consistent precubical sets is not cocomplete, since there exist diagrams of event consistent precubical sets with colimits that are not event consistent. $\square$

Definition 3.14. *We define $v : \mathtt{ECPS} \to \mathtt{Set}^{\square^{op}}$ as the canonical inclusion functor.*

The reason why we define something like this is because we need to make sure colimits and compactness are the same irregardless if we are working in the category $\mathtt{Set}^{\square^{op}}$ or $\mathtt{ECPS}$.

Theorem 3.15. *Any diagram in $\mathtt{ECPS}$ is an event consistent diagram in $\mathtt{Set}^{\square^{op}}$ and any event consistent diagram in $\mathtt{Set}^{\square^{op}}$ is a diagram in $\mathtt{ECPS}$.*

Proof. Let $X : J \to \mathtt{Set}^{\square^{op}}$ be an event consistent diagram in $\mathtt{Set}^{\square^{op}}$. The fact that it is a diagram in $\mathtt{ECPS}$ as well follows from the fact that $\mathtt{ECPS}$ is a full subcategory of $\mathtt{Set}^{\square^{op}}$, containing all event consistent precubical sets and all precubical maps between event consistent precubical sets. This means that for all objects $i \in J$ and all morphisms $f \in J$ the event consistent precubical sets X_i and the precubical maps X_f are contained in $\mathtt{ECPS}$ and have the same properties, which gives us that X is a diagram in $\mathtt{Set}^{\square^{op}}$ as well.
Similarly all of the objects and morphisms in $\mathtt{ECPS}$ are contained in $\mathtt{Set}^{\square^{op}}$, therefore if $X : J \to \mathtt{ECPS}$ is a diagram in $\mathtt{ECPS}$ then it must be an event consistent diagram in $\mathtt{Set}^{\square^{op}}$ as well. $\square$

Theorem 3.16. *Any co-cone in $\mathtt{ECPS}$ is an event consistent co-cone in $\mathtt{Set}^{\square^{op}}$ and any event consistent co-cone in $\mathtt{Set}^{\square^{op}}$ is a co-cone in $\mathtt{ECPS}$.*

Proof. This is due to the same reasons as stated in theorem 3.15. Both the precubical sets and the precubical maps are in both categories, and since the diagrams are as well and the conditions on the injection maps on the co-cones remain untouched it follows that the statement is true. $\square$

Theorem 3.17. *The functor $v : \mathtt{ECPS} \to \mathtt{Set}^{\square^{op}}$ reflects colimits.*

Proof. Let $X : J \to \mathtt{ECPS}$ be a small diagram and let (L, ϕ) be a co-cone of this diagram in $\mathtt{ECPS}$. We want to prove that if (L, ϕ) is a colimit of X in $\mathtt{Set}^{\square^{op}}$, then it is a colimit of X in $\mathtt{ECPS}$ as well. Suppose that (L, ϕ) is a colimit of X in $\mathtt{Set}^{\square^{op}}$, but not a colimit in $\mathtt{ECPS}$. Because of theorem 3.16 it is still a co-cone, which means that there must exists a co-cone (N, ψ) in $\mathtt{ECPS}$ such that there exists no unique precubical map $q : L \to N$ such that $q \circ \phi_i = \psi_i$ for all $i \in J$. This cannot be true since the co-cone (N, ψ) is a co-cone in $\mathtt{Set}^{\square^{op}}$ as well which means that this unique precubical map $q : L \to N$ does exist. $\square$

Note that the above is also a consequence of v being fully faithful (see [Rie17]).

Theorem 3.18. *The functor $v : \mathtt{ECPS} \to \mathtt{Set}^{\square^{op}}$ preserves colimits.*

Proof. Let $X : J \to \texttt{ECPS}$ be a small diagram and let (L, ϕ) be a co-cone of this diagram in ECPS. We want to prove that if (L, ϕ) is a colimit of X in ECPS, then it is a colimit of X in $\texttt{Set}^{\square^{\text{op}}}$ as well. From theorem 3.16 it follows that (L, ϕ) is an event consistent co-cone of X in $\texttt{Set}^{\square^{\text{op}}}$. From theorem 3.7 it follows that there exists an event consistent colimit (N, ψ) of X in $\texttt{Set}^{\square^{\text{op}}}$. From theorem 3.17 it then follows that (N, ψ) is also a colimit of X in ECPS. This gives us that the unique precubical map $q : L \to N$ is an isomorphism, which because of theorem 2.12 means that L is a colimit of X in $\texttt{Set}^{\square^{\text{op}}}$ as well. □

This shows that diagrams, co-cones and colimits work the same on ECPS as they do on $\texttt{Set}^{\square^{\text{op}}}$.

Theorem 3.19. *Every event consistent precubical set is the filtered colimit of a diagram of finite event consistent precubical sets.*

Proof. This follows from theorem 2.38 and theorem 3.10. □

Theorem 3.20. *An object in* ***ECPS*** *is compact in* ***ECPS*** *if and only if it is compact in* $\texttt{Set}^{\square^{\text{op}}}$.

Proof. Let X be an event consistent precubical set and suppose X is compact in $\texttt{Set}^{\square^{\text{op}}}$. Then for every filtered category J and every diagram $Y : J \to \texttt{Set}^{\square^{\text{op}}}$ the canonical morphism

$$\varinjlim_{i \in J} \operatorname{Hom}(X, Y_i) \xrightarrow{\simeq} \operatorname{Hom}\left(X, \varinjlim_{i \in J} Y_i\right)$$

is an isomorphism. This refers to all event consistent filtered diagrams which because of theorem 3.15 means it refers to every filtered diagram in ECPS. Therefore X is compact in ECPS as well. Suppose that X is compact in ECPS. Theorem 3.19 gives us that X is a colimit of a filtered diagram of finite event consistent precubical sets, which following the proof of theorem 2.40 gives us that X is finite as well. Because of theorem 2.46 this then means that X is compact in $\texttt{Set}^{\square^{\text{op}}}$. □

Recall definition 2.47 for local finite presentability.

Theorem 3.21. *The following statements about the category* ***ECPS*** *are true:*

1. *The category* ***ECPS*** *is not cocomplete, but does have all small coproducts and filtered colimits.*
2. *The full subcategory of* ***ECPS*** *consisting of the compact objects is essentially small.*
3. *Any object in* ***ECPS*** *is a filtered colimit of a diagram of compact objects.*

Proof. Statement 1 follows from theorem 3.9, theorem 3.10 and theorem 3.11.
Because of theorem 3.20 an object is compact in ECPS if and only if it is compact in $\texttt{Set}^{\square^{\text{op}}}$. Statement 2 then follows from the fact that ECPS is a full subcategory of $\texttt{Set}^{\square^{\text{op}}}$, which means that the same is true for the full subcategories containing the compact objects.
Statement 3 follows from theorem 3.19. □

This means that ECPS is not locally finitely presentable, because it does not satisfy the first condition of cocompleteness. It does however satisfy the other conditions. A property that is weaker than local finite presentability is the following:

Definition 3.22. *A category C is finitely accessible if*

1. *C is locally small.*

2. *C has all small filtered colimits.*

3. *There is a set of compact objects that generate C under small filtered colimits.*

Theorem 3.23. *The category* `ECPS` *of event consistent precubical sets is finitely accessible.*

Proof. The category `ECPS` is locally small since it is a full subcategory of $\texttt{Set}^{\square^{\mathrm{op}}}$ which is essentially small.
Theorem 3.10 gives us that `ECPS` has all small filtered colimits.
The third statement follows from the second and third statements of theorem 3.21. □

4 Higher-Dimensional Automata

4.1 Labelled precubical sets

The last thing we need to define before we can introduce higher-dimensional automata are the labelled precubical sets, which are event consistent precubical sets combined with something called the labelling function. Unlike with event consistent precubical sets we are not going to prove any properties for them. This is because they are not sufficiently different to the higher-dimensional automata themselves, and it is a lot easier to just prove the properties for HDA directly. Before we define labelled precubical sets we first need to define the labelling object.

Definition 4.1. *Let Σ be a non-empty set. The labelling object on Σ is the precubical set $!\Sigma$ with $(!\Sigma)^0 = !\Sigma^0 = \{\varepsilon\}$ and $(!\Sigma)^n = !\Sigma^n = \prod_{i\geq 1}^n \Sigma$ for all $n \in \mathbb{N}_{\geq 1}$. Here we define ε as a unique element such that $\varepsilon \notin \Sigma$. The face maps are defined by*

$$\delta^n_{\nu,a}\left((x_1, ..., x_n)\right) = (x_1, ..., x_{a-1}, x_{a+1}, ..., x_n)$$

for all $n \in \mathbb{N}$, $\nu \in \{0, 1\}$, $a \in \{1, 2, ..., n\}$ and $x = (x_1, ..., x_n) \in !\Sigma^n$ with $x_i \in \Sigma$ for all $i \in \mathbb{N}_{\geq 1}$, $i \leq n$.

In other words the delta map $\delta^n_{\nu,a}$ removes the a'th element from the vector. It's easy to see that these elementary face maps satisfy the condition for precubical sets, since for all $a, b \in \mathbb{N}_{\geq 1}$ with $a < b$ removing the b'th element first and the a'th element second is the same as removing the a'th element first and the $b-1$'th element second. Also note that the precubical set $!\Sigma$ is never event consistent, since the face maps $\delta^n_{0,a}$ and $\delta^n_{1,a}$ do exactly the same for all $n, a \in \mathbb{N}$, $1 \leq a \leq n$.

Theorem 4.2. *Let X be a precubical set and let Σ be a set. Any function $\lambda^1 : X^1 \to \Sigma$ for which $\lambda^1 \circ \delta^2_{0,1}(x) = \lambda^1 \circ \delta^2_{1,1}(x)$ and $\lambda^1 \circ \delta^2_{0,2}(x) = \lambda^1 \circ \delta^2_{1,2}(x)$ for all $x \in X^2$ extends uniquely to a precubical map $\lambda : X \to !\Sigma$.*

Proof. For all $x \in X^1$ we have $\lambda^1(x)$ defined. For all $x \in X^0$ we have $\lambda^0(x) = \epsilon = (-)$, as in the empty tuple. For all $n \in \mathbb{N}_{\geq 2}$, $x \in X^n$ we define

$$\lambda^n(x) = \left(\lambda^1 \circ \delta^n_{\nu,A^n_1}(x), ..., \lambda^1 \circ \delta^n_{\nu,A^n_n}(x)\right)$$

for a certain $\nu \in \{0,1\}$. Because of theorem A.8 it does not matter if we have $\nu = 0$ or $\nu = 1$ since $\lambda^1 \circ \delta^n_{0,A^n_t}(x) = \lambda^1 \circ \delta^n_{1,A^n_t}(x)$ for all $1 \leq t \leq n$. Suppose that we have $n \in \mathbb{N}_{\geq 2}$, $x \in X^{n-1}$, $\nu \in \{0,1\}$ and $a \in \mathbb{N}_{\geq 1}$, $a \leq n$. Then we have

$$\delta^n_{\nu,a} \circ \lambda^n(x) =$$

$$\left(\lambda^1 \circ \delta^n_{\nu,A^n_1}(x), ..., \lambda^1 \circ \delta^n_{\nu,A^n_{a-1}}(x), \lambda^1 \circ \delta^n_{\nu,A^n_{a+1}}(x), ..., \lambda^1 \circ \delta^n_{\nu,A^n_n}(x)\right)$$

$$\lambda^{n-1} \circ \delta^n_{\nu,a}(x) =$$

$$\Big(\lambda^1 \circ \delta^{n-1}_{\nu,A^{n-1}_1} \circ \delta^n_{\nu,a}(x), ..., \lambda^1 \circ \delta^{n-1}_{\nu,A^{n-1}_{a-1}} \circ \delta^n_{\nu,a}(x),$$

$$\lambda^1 \circ \delta^{n-1}_{\nu,A^{n-1}_{a+1}} \circ \delta^n_{\nu,a}(x), ..., \lambda^1 \circ \delta^{n-1}_{\nu,A^{n-1}_{n-1}} \circ \delta^n_{\nu,a}(x)\Big)$$

Using theorem A.6 we get that

$$\delta^{n-1}_{\nu,A^{n-1}_t} \circ \delta^n_{\nu,a}(x) = \begin{cases} \delta^n_{\nu,A^n_t}(x) & \text{for all } a > t \\ \delta^n_{\nu,A^n_{t+1}}(x) & \text{for all } a \leq t \end{cases}$$

for all $1 \leq t \leq n-2$. This gives us

$$\lambda^{n-1} \circ \delta^n_{\nu,a}(x) =$$

$$\left(\lambda^1 \circ \delta^n_{\nu,A^n_1}(x), ..., \lambda^1 \circ \delta^n_{\nu,A^n_{a-1}}(x), \lambda^1 \circ \delta^n_{\nu,A^n_{a+1}}(x), ..., \lambda^1 \circ \delta^n_{\nu,A^n_n}(x)\right)$$

and therefore $\lambda^{n-1} \circ \delta^n_{\nu,a}(x) = \delta^n_{\nu,a} \circ \lambda^n(x)$ for all $n \in \mathbb{N}_{\geq 1}$, $\nu \in \{0,1\}$, $a \in \mathbb{N}_{\geq 1}$ with $a \leq n$ and $x \in X^n$. This makes $\lambda : X \to !\Sigma$ into a unique precubical map. □

Note that because of the way the elementary face maps on $!\Sigma$ are defined the condition $\lambda^1 \circ \delta^2_{0,1}(x) = \lambda^1 \circ \delta^2_{1,1}(x)$ and $\lambda^1 \circ \delta^2_{0,2}(x) = \lambda^1 \circ \delta^2_{1,2}(x)$ for all $x \in X^2$ is required for every precubical map $\lambda : X \to !\Sigma$.

Theorem 4.3. *For every non-empty set Σ and every precubical set X there exists a precubical map $\lambda : X \to !\Sigma$.*

Proof. We can simply define $\lambda^1 : X^1 \to \Sigma$ as the morphism with $\lambda^1(x_1) = \lambda^1(x_2)$ for all $x_1, x_2 \in X^1$. This morphism then clearly satisfies $\lambda^1 \circ \delta^2_{0,1}(x) = \lambda^1 \circ \delta^2_{1,1}(x)$ and $\lambda^1 \circ \delta^2_{0,2}(x) = \lambda^1 \circ \delta^2_{1,2}(x)$ for all $x \in X^2$. Because of theorem 4.2 this then extends uniquely to a precubical map $\lambda : X \to !\Sigma$. □

Definition 4.4. *A labelled precubical set is a pair (X, λ) with X an event consistent precubical set and $\lambda : X \to !\Sigma$ a precubical map which we call the labelling or labelling function.*

Because of theorem 4.3 every event consistent precubical set has a labelling function which means that every event consistent precubical set can be converted to a labelled precubical set.
In section 6 we will introduce what are called the event object and the event identifications, which work similarly to the labelling object and labelling functions and can be seen as an alternative definition for event consistency. The labelling of any event consistent precubical set will factor through this event object.

4.2 Definition of HDA

We can now finally define higher-dimensional automata.

Definition 4.5. *A Higher-Dimensional Automata or HDA is a tuple* (X, I, F, λ) *consisting of*

- X *an event consistent precubical set,*
- $I = \{I^n\}_{n\in\mathbb{N}}$ *the initial cells with* $I^n \subseteq X^n$ *for all* $n \in \mathbb{N}$*,*
- $F = \{F^n\}_{n\in\mathbb{N}}$ *the accepting cells with* $F^n \subseteq X^n$ *for all* $n \in \mathbb{N}$ *and*
- $\lambda : X \to !\Sigma$ *the labelling function.*

Like with ordinary automata we will not pay much attention to the alphabet Σ and simply assume it is the same for all HDA mentioned. When talking about a HDA (X, I, F, λ) we will often only mention X instead of the entire tuple.
Let's take a look at an example:

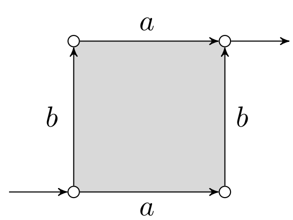


Figure 8: The HDA where the underlying event consistent precubical set is the one shown in figure 2, the labelling is defined by $\lambda^1 \circ \delta^1_{0,1}(x) = \lambda^1 \circ \delta^1_{1,1}(x) = b$ and $\lambda^1 \circ \delta^1_{0,2}(x) = \lambda^1 \circ \delta^1_{1,2}(x) = a$, the node in the bottom left being the sole initial cell and the node in the top right being the sole final cell.

We denote initial and final cells with incoming and outgoing arrows. When a higher-dimensional element is an initial/final cell we will denote it as the following:

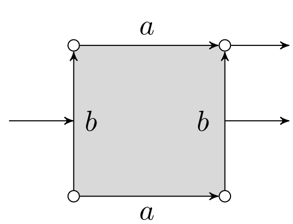


Figure 9: The HDA as in figure 8 where the node in the bottom left is not an initial cell but the left-most edge is and where the node in the top right and the right-most edge are final cells.

In the next section we will actually use these initial and final cells in what will be called (accepting) tracks. For now we will just leave it at this. While HDA with initial or final cells that are of higher dimension than 0 or 1 do exist, we don't actually have a good way to denote these kind of cells in a figure. We will therefore not really bother with examples that have these initial or final cells, but know that they do exist.

Definition 4.6. *Suppose that* (X, I_X, F_X, λ_X) *and* (Y, I_Y, F_Y, λ_Y) *are HDAs. A HDA-map is a precubical map* $f : X \to Y$ *such that for all* $n \in \mathbb{N}$*,* $x \in X^n$ *the following statements are true:*

- *If* $x \in I_X$ *we have* $f^n(x) \in I_Y$*,*
- *if* $x \in F_X$ *we have* $f^n(x) \in F_Y$*,*
- $\lambda_X^n(x) = \lambda_Y^n \circ f^n(x)$*.*

In other words: HDA maps are precubical maps that preserve the labelling and the initial and final cells.

Definition 4.7. *Let* $\mathcal{X} = (X, I_X, F_X, \lambda_X)$ *and* $\mathcal{Y} = (Y, I_Y, F_Y, \lambda_Y)$ *be HDA and let* $f : \mathcal{X} \to \mathcal{Y}$ *be a HDA map. Then* f *is a HDA isomorphism if there exists a HDA map* $g : \mathcal{Y} \to \mathcal{X}$ *such that* $g \circ f = id_{\mathcal{X}}$ *and* $f \circ g = id_{\mathcal{Y}}$*.*

4.3 Category of HDA

The main goal of this subsection is to show the connection between colimits of HDA and colimits of (event consistent) precubical sets.

Definition 4.8. *We define the forgetful functor* $u : \texttt{HDA} \to \texttt{ECPS}$ *which sends HDAs* (X, I, F, λ) *to the event consistent precubical sets* X *and HDA maps* f *to precubical maps* f*.*

The below simply follows from u being a functor:

Remark 4.8.1. *The functor* $u : \texttt{HDA} \to \texttt{ECPS}$ *maps small diagrams* $\mathcal{X} : J \to \texttt{HDA}$ *with* $\mathcal{X}_i = (X_i, I_i, F_i, \lambda_i)$ *onto small diagrams* $X : J \to \texttt{ECPS}$ *and co-cones* $(\mathcal{N}, \psi)$ *of* $\mathcal{X}$ *onto co-cones* (N, ψ) *of* X*.*

This means that every small diagram of HDA forms a small diagram of event consistent precubical sets and every co-cone of HDA also forms a co-cone of event consistent precubical sets. What we now want to show is that we can canonically construct a HDA from an event consistent precubical set with a precubical map where the codomain is a HDA.

Theorem 4.9. *Suppose that* X *and* Y *are event consistent precubical sets,* $\mathcal{Y} = (Y, I_Y, F_Y, \lambda_Y)$ *is a HDA and* $f : X \to Y$ *is a precubical map. We can construct a HDA* $\mathcal{X} = (X, I_X, F_X, \lambda_X)$ *from* X *in the following way:*

- *For all* $n \in \mathbb{N}$*,* $x \in X^n$ *we take* $x \in I_X$ *if* $f^n(x) \in I_Y$*.*
- *For all* $n \in \mathbb{N}$*,* $x \in X^n$ *we take* $x \in F_X$ *if* $f^n(x) \in F_Y$*.*
- $\lambda_X = \lambda_Y \circ f$*.*

This gives us the HDA $\mathcal{X}$ *and by construction makes* f *a HDA-map.*

Proof. It's clear that λ_X^1 satisfies the conditions of theorem 4.2 as it inherits them from λ_Y^1, making it a labelling function. The initial and final cells are clearly well-defined. This makes $\mathcal{X}$ a HDA, and due to the way it was constructed the precubical map f automatically becomes a HDA map. $\square$

We can apply this theorem to small diagrams and co-cones as well.

Theorem 4.10. *Let $X : J \to$ **ECPS** be a small diagram of event consistent precubical sets and let (N, ψ) be a co-cone. Suppose that $\mathcal{N} = (N, I_N, F_N, \lambda_N)$ is a HDA. Then we can define the small diagram $\mathcal{X} : J \to$ **HDA** of HDA with $\mathcal{X}_i = (X_i, I_i, F_i, \lambda_i)$ for all $i \in J$ such that $(\mathcal{N}, \psi)$ is a co-cone of this diagram as well.*

Proof. Since for all X_i with $i \in J$ we have the precubical maps $\psi_i : X_i \to N$ we can use theorem 4.9. This gives us the HDA $\mathcal{X}_i = (X_i, I_i, F_i, \lambda_i)$ and HDA maps $\psi_i : \mathcal{X}_i \to \mathcal{N}$ for all $i \in J$.
Now we need to prove that for all $i, j \in J$, $f : i \to j$ the precubical maps $X_f : X_i \to X_j$ are HDA maps. For all $n \in \mathbb{N}$, $x \in X_i^n$ we have $\psi_i^n(x) = \psi_j^n \circ X_f^n(x)$ and by definition we have $\lambda_i^n(x) = \lambda_N^n \circ \psi_i^n(x)$ and $\lambda_j^n \circ X_f^n(x) = \lambda_N^n \circ \psi_j^n \circ X_f^n(x) = \lambda_N^n \circ \psi_i^n(x)$ which therefore gives us that $\lambda_i^n(x) = \lambda_j^n \circ X_f^n(x)$. We have $x \in I_i$ if and only if $\psi_i^n(x) \in I_N$ and since $\psi_j^n \circ X_f^n(x) = \psi_i^n(x)$ this gives us $X_f^n(x) \in I_j$. Analogously the same is true for the final cells. This makes X_f a HDA map which we will refer to as $\mathcal{X}_f$, which therefore makes $\mathcal{X} : J \to \mathtt{HDA}$ a small diagram of HDA. For all $i \in J$ we have the HDA maps $\psi_i : \mathcal{X}_i \to \mathcal{N}$ such that for all $j \in J$, $f : i \to j$ we have $\psi_i = \psi_j \circ \mathcal{X}_f$, which therefore makes $(\mathcal{N}, \psi)$ a co-cone of the diagram H. □

This gives us a nice connection between diagrams of HDA and diagrams of event consistent precubical sets. Do note however that generally speaking the above diagram is not the only diagram of HDA that one can generate using the specified co-cone. It's possible for some HDA in the diagram to have smaller sets of initial and final cells, or even empty ones, which doesn't matter for the co-cone of the diagram. It does matter for the colimit of the diagram, which is proven in the following theorem:

Theorem 4.11. *Let $X : J \to$ **ECPS** be a small diagram of event consistent precubical sets and let (L, ϕ) be a colimit. Suppose that $\mathcal{L} = (L, I_L, F_L, \lambda_L)$ is a HDA. Then we can define the small diagram $\mathcal{X} : J \to$ **HDA** of HDA with $\mathcal{X}_i = (X_i, I_i, F_i, \lambda_i)$ for all $i \in J$ such that $(\mathcal{L}, \phi)$ is a colimit of this diagram as well.*

Proof. Theorem 4.10 gives us the diagram $\mathcal{X} : J \to \mathtt{HDA}$ and that $(\mathcal{L}, \phi)$ is a co-cone of this diagram. Suppose that $(\mathcal{N}, \psi)$ with $\mathcal{N} = (N, I_N, F_N, \lambda_N)$ is a co-cone of the diagram $\mathcal{X}$. Then remark 4.8.1 gives us that (N, ψ) is a co-cone of $X : J \to \mathtt{ECPS}$ as well. Because (L, ϕ) is a colimit we get the unique precubical map $q : L \to N$ such that for all $i \in J$ we have $q \circ \phi_i = \psi_i$. Let $n \in \mathbb{N}$ and $y \in L^n$. Then because of theorem 2.24 there exists a $i \in J$ and a $x \in X_i^n$ such that $\phi_i^n(x) = y$. Then by definition we have

$$\lambda_L^n(y) = \lambda_L^n \circ \phi_i^n(x) = \lambda_i^n(x) = \lambda_N^n \circ \psi_i^n(x) = \lambda_N^n \circ q^n \circ \phi_i^n(x) = \lambda_N^n \circ q^n(y)$$

which shows that $q : L \to N$ preserves the labelling functions. Similarly if $y \in I_L$ then by construction we have $x \in I_i$ and therefore $\psi_i^n(x) \in I_N$ and $q^n(y) \in I_N$ since $q^n(y) = q^n \circ \phi_i^n(x) = \psi_i^n(x)$. Analogously the same is true for the final cells. This makes $q : \mathcal{L} \to \mathcal{N}$ a HDA map which is unique because it is unique as a precubical map. Therefore $(\mathcal{L}, \phi)$ is a colimit of the diagram $\mathcal{X}$. □

Like previously this isn't the only diagram that has the colimit $(\mathcal{L}, \phi)$. We don't always need every initial and final cell in the diagram. We only need the following:

Theorem 4.12. *Let $\mathcal{X} : J \to \texttt{HDA}$ be a small diagram of HDA with for all $i \in J$ we have $\mathcal{X}_i = (X_i, I_i, F_i, \lambda_i)$ and let $\mathcal{L} = (L, I_L, F_L, \lambda_L)$ be a HDA and $(\mathcal{L}, \phi)$ be a colimit of the diagram $\mathcal{X}$. Then for all $n \in \mathbb{N}$, $y \in L^n$ if we have $y \in I_L$ or $y \in F_L$ then there exists a $i \in J$ and a $x \in X_i^n$ with $\phi_i^n(x) = y$ and $x \in I_i$ or $x \in F_i$.*

Proof. Suppose that the statement is false, which means that there exists a $n \in \mathbb{N}$, $y \in L^n$ with $y \in I_L$ (or $y \in F_L$, it does not matter) such that for all $i \in J$, $x \in X_i^n$ with $\phi_i^n(x) = y$ we have $x \notin I_i$. We define the HDA $\mathcal{L}' = (L, I_L, F_L, \lambda_L)$ with $I_L'^n = I_L^n \backslash \{y\}$ and $I_L'^m = I_L^m$ for all $m \in \mathbb{N}$, $m \neq n$. We define $(\mathcal{L}', \psi)$ as the co-cone where $\psi_i : \mathcal{X}_i \to \mathcal{L}'$ is identical to $\phi_i : \mathcal{X}_i \to \mathcal{L}$. These precubical maps are HDA maps since by definition of I_L' they still preserve the initial cells which is the only thing different from $\mathcal{L}$. Since $(\mathcal{L}', \psi)$ is a co-cone there exists a unique HDA map $q : \mathcal{L} \to \mathcal{L}'$ such that for all $i \in J$ we have $q \circ \phi_i = \psi_i$. Because of theorem 2.24 there exists at least one $i \in J$, $x \in X_i^n$ with $\phi_i^n(x) = y$. This gives us $q^n(y) = q^n \circ \phi_i^n(x) = \psi_i^n(x) = y$ however since we have $y \in I_L$ but also $y \notin I_L'$ this means that $q : \mathcal{L} \to \mathcal{L}'$ does not preserve the initial cells, which is in contradiction with $(\mathcal{L}, \phi)$ being a colimit. Therefore there exists no such $y \in L^n$, $n \in \mathbb{N}$ with $y \in I_L$ but where for all $i \in J$, $x \in X_i^n$ with $\phi_i^n(x) = y$ we have $x \notin I_i$. Analogously the same is true for $y \in F_L$. □

Before we can finally properly connect colimits of HDA and of event consistent precubical sets we first prove the following theorem, which is an application of theorem 2.12 to HDA. While it should also follow from abstract principles we have written out the proof just to be sure.

Theorem 4.13. *Let $\mathcal{X} : J \to \texttt{HDA}$ be a small diagram of HDA with for all $i \in J$ we have $\mathcal{X}_i = (X_i, I_i, F_i, \lambda_i)$ and let $\mathcal{L} = (L, I_L, F_L, \lambda_L)$ be a HDA and $(\mathcal{L}, \phi)$ be a colimit of the diagram $\mathcal{X}$. Let $(\mathcal{N}, \psi)$ be a co-cone of $\mathcal{X}$ with $\mathcal{N} = (N, I_N, F_N, \lambda_N)$. Then $(\mathcal{N}, \psi)$ is a colimit of $\mathcal{X}$ as well if and only if the unique HDA map $q : \mathcal{L} \to \mathcal{N}$ such that $q \circ \phi_i = \psi_i$ for all $i \in J$ is an isomorphism.*

Proof. Let $q : \mathcal{L} \to \mathcal{N}$ be the unique HDA map such that $q \circ \phi_i = \psi_i$ for all $i \in J$.
Suppose that q is an isomorphism and let $p : \mathcal{N} \to \mathcal{L}$ be its inverse. Let $(\mathcal{M}, \psi)$ be a co-cone of $\mathcal{X}$ with $\mathcal{M} = (M, I_M, F_M, \lambda_M)$ and let $f : \mathcal{L} \to \mathcal{M}$ be the unique HDA map such that $f \circ \phi_i = \theta_i$ for all $i \in J$. Then $f \circ p : \mathcal{N} \to \mathcal{M}$ is a HDA map with $f \circ p \circ \psi_i = \theta_i$ for all $i \in J$. Suppose that $g : \mathcal{N} \to \mathcal{M}$ is a different HDA map that satisfies the property with $g \neq f \circ p$. Then $g \circ q : \mathcal{L} \to \mathcal{M}$ is another HDA map that satisfies the property with $g \circ q \neq f \circ p \circ q = f$ which is in contradiction with $(\mathcal{L}, \phi)$ being a colimit. Therefore no such g exists which means that $f \circ p : \mathcal{N} \to \mathcal{M}$ gives us an unique HDA map which therefore makes $(\mathcal{N}, \psi)$ a colimit of X.
Suppose that $(\mathcal{N}, \psi)$ is a colimit of X. There exists a unique HDA map $p : \mathcal{N} \to \mathcal{L}$ and we have $q \circ \phi_i = \psi_i$ and $p \circ \psi_i = \phi_i$ for all $i \in J$. Therefore $p \circ q \circ \phi_i = \phi_i$ and $q \circ p \circ \psi_i = \psi_i$ for all $i \in J$. Theorem 2.24 gives us that for every $n \in \mathbb{N}$, $y \in L^n$ there exists a $i \in J$, $x \in X_i^n$ such that $\phi_i^n(x) = y$, and the same is true for $(\mathcal{N}, \psi)$. For all $n \in \mathbb{N}$ every element $y \in L^n$ and $z \in N^n$ can therefore be expressed as $\phi_i^n(x)$ or $\psi_i^n(x)$ for certain $i \in J$ and $x \in X_i^n$. This gives us

$$p^n \circ q^n(y) = p^n \circ q^n \circ \phi_i^n(x) = \phi_i^n(x) = y$$

$$q^n \circ p^n(z) = q^n \circ p^n \circ \psi_i^n(x) = \psi_i^n(x) = z$$

and therefore $p \circ q = \mathrm{id}_L$ and $q \circ p = \mathrm{id}_N$. This shows that p and q are bijective as precubical maps and therefore because of theorem 2.5 this means that p and q are isomorphisms as precubical maps. Because they are also both HDA maps they are both HDA isomorphisms by definition 4.7. □

Now we can finally prove the following theorem:

Theorem 4.14. *Let $\mathcal{X} : J \to$ **HDA** be a small diagram of HDA with for all $i \in J$ we have $\mathcal{X}_i = (X_i, I_i, F_i, \lambda_i)$ and let $\mathcal{L} = (L, I_L, F_L, \lambda_L)$ be a HDA and $(\mathcal{L}, \phi)$ be a co-cone of the diagram $\mathcal{X}$. Then $(\mathcal{L}, \phi)$ is a colimit of $\mathcal{X}$ if and only if the following conditions are true:*

1. *For all $n \in \mathbb{N}$, $y \in L^n$ if we have $y \in I_L$ or $y \in F_L$ then there exists a $i \in J$ and a $x \in X_i^n$ with $\phi_i^n(x) = y$ and $x \in I_i$ or $x \in F_i$.*

2. *(L, ϕ) is a colimit of the diagram $X : J \to$ **HDA**.*

Proof. Suppose that the two conditions are true and let $(\mathcal{N}, \psi)$ with $\mathcal{N} = (N, I_N, F_N, \lambda_N)$ be a co-cone of $\mathcal{X}$. Then because of remark 4.8.1 (N, ψ) is a co-cone of the diagram $X : J \to$ ECPS. This gives us a unique precubical map $q : L \to N$ such that for all $i \in J$ we have $q \circ \phi_i = \psi_i$. Because of the first condition for all $y \in L^n$, $n \in \mathbb{N}$ if $y \in I_L$ or $y \in F_L$ there exists a $i \in J$ and $x \in X_i^n$ with $\phi_i^n(x) = y$ such that $x \in I_i$ and $x \in F_i$ respectively. Because ψ_i are HDA maps for all $i \in J$ they preserve initial and final cells which then gives us that if $y \in I_L$ or $y \in F_L$ we must have $q^n(y) \in I_N$ and $q_n(y) \in I_L$ respectively. Similarly because of theorem 2.24 every element in $y \in L^n$, $n \in \mathbb{N}$ is injected through at least one $x \in X_i^n$, $i \in J$ which gives us

$$\lambda_L^n(y) = \lambda_i^n(x) = \lambda_N^n \circ \psi_i^n(x) = \lambda_N^n \circ q^n \circ \phi_i^n(x) = \lambda_N^n \circ q^n(y)$$

which shows that $q : L \to N$ preserves the labelling function, making it a HDA map. Because q is unique as a precubical map it is unique as a HDA map as well, since there is only one way to map each element of L onto elements of N. This shows that for all co-cones $(\mathcal{N}, \psi)$ of $\mathcal{X}$ there exists a unique HDA map $q : \mathcal{L} \to \mathcal{N}$ such that for all $i \in J$ we have $q \circ \phi_i = \psi_i$, therefore making $(\mathcal{L}, \phi)$ a colimit of $\mathcal{X}$.
Theorem 4.12 gives us that the first condition needs to be true for $(\mathcal{L}, \phi)$ to be a colimit.
Finally suppose that the second condition is false, which means that (L, ϕ) is not a colimit of $X : J \to$ ECPS. Because of remark 4.8.1 it is still a co-cone, which because of theorem 3.7 gives us that there exists an event consistent colimit (N, ψ) of X. Then there exists a unique precubical map $q : N \to L$, with which through theorem 4.9 we can construct the HDA $\mathcal{N} = (N, I_N, F_N, \lambda_N)$ and the HDA map $q : \mathcal{N} \to \mathcal{L}$. We know that the first condition is true. Therefore because of the construction of $\mathcal{N}$ we also get that the first condition is true for $(\mathcal{N}, \phi)$, which because of our previous proof gives us that $(\mathcal{N}, \phi)$ is a colimit of $\mathcal{X}$. Because of theorem 4.13 this gives us that the HDA map $q : \mathcal{N} \to \mathcal{L}$ is an isomorphism, which is the unique HDA map from the definition of the colimit, since it satisfies the requirements. Then that means that the precubical map $q : N \to L$ is an isomorphism as well, which because of theorem 2.12 gives us that (L, ϕ) is a colimit of X. ☐

This then gives us the following theorem:

Theorem 4.15. *The functor $u :$ **HDA** $\to$ **ECPS** preserves colimits, but does not reflect them.*

Proof. The first statement follows from theorem 4.14. For the second statement we can take a diagram of HDA with a colimit such that not every element of this colimit is an initial and final cell. Then we can create a HDA based on this colimit for which it is true that every element is an initial and final cell. This is clearly still a co-cone. Then because of theorem 4.14 its underlying event consistent precubical set is the colimit of the underlying diagram of event consistent precubical sets, but it isn't a colimit of the diagram of HDA since it doesn't satisfy the second property. ☐

Having covered the relation between the category of event consistent precubical sets and the category of higher-dimensional automata we will prove that HDA does not have all small colimits, but does have all small coproducts and filtered colimits. The proof of this will build on the fact that this is the case for ECPS as well.

Theorem 4.16. *Suppose that $\mathcal{X} : J \to$ **HDA** is a diagram in **HDA** with J a small category. For all $i \in J$ we have $\mathcal{X}_i = (X_i, I_i, F_i, \lambda_i)$. Then we have for all $n \in \mathbb{N}$, $i, j \in J$, $x \in X_i^n$ and $y \in X_j^n$*

$$x \sim y \implies \lambda_i(x) = \lambda_j(y)$$

Proof. Suppose that we have $i, j \in J$, $f : i \to j$. Then $X_f : \mathcal{X}_i \to \mathcal{X}_j$ is a HDA map and for all $n \in \mathbb{N}$, $x \in X_i^n$ and $y \in X_j^n$ such that $X_f^n(x) = y$ we have $\lambda_i(x) = \lambda_j \circ X_f^n(x) = \lambda_j(y)$. In other words we have $X_f^n(x) = y \implies \lambda_i(x) = \lambda_j(y)$. Through the construction of the equivalence relation $\sim$ we then get $x \sim y \implies \lambda_i(x) = \lambda_j(y)$. □

Theorem 4.17. *Suppose that $\mathcal{X} : J \to$ **HDA** is a diagram in **HDA** with J a small category. Then $X : J \to$ **ECPS** is a diagram of event consistent precubical sets with $X_i = u\mathcal{X}_i$ (with $\mathcal{X}_i = (X_i, I_i, F_i, \lambda_i)$). If X has an event consistent colimit then $\mathcal{X}$ has a colimit as well.*

Proof. Suppose that (L, ϕ) is an event consistent colimit of X. We define the HDA $\mathcal{L} = (L, I_L, F_L, \lambda_L)$ as the following:

- For all $n \in \mathbb{N}$, $x \in X^n$ if there exists a $i \in J$, $y \in X_i^n$ with $\phi_i^n(y) = x$ and $y \in I_i$ then $x \in I_L$.
- For all $n \in \mathbb{N}$, $x \in X^n$ if there exists a $i \in J$, $y \in X_i^n$ with $\phi_i^n(y) = x$ and $y \in F_i$ then $x \in F_L$.
- For all $n \in \mathbb{N}$, $x \in X^n$ we define $\lambda_L(x) = \lambda_i(y)$ for all $i \in J$, $y \in X_i^n$ such that $\phi_i^n(y) = x$.

Theorem 2.24 gives us that for all $n \in \mathbb{N}$, $x \in X^n$ there exists at least one $i \in J$, $y \in X_i^n$ with $\phi_i^n(y) = x$.
We still need to show that this labelling on $\mathcal{L}$ is well-defined. Suppose that for certain $n \in \mathbb{N}$, $x \in X^n$ there exists $i, j \in J$, $y \in X_i^n$ and $z \in X_j^n$ such that $\phi_i^n(y) = x$ and $\phi_j^n(z) = x$. Therefore $\phi_i^n(y) = \phi_j^n(z)$, which as a consequence of theorem 2.22 gives us that $y \sim z$. Using theorem 4.16 we then get $\lambda_i(y) = \lambda_j(z)$.
Therefore the above defines the labelling function and the initial and final cells for all $n \in \mathbb{N}$, $x \in X^n$. This makes $\mathcal{L}$ a HDA and by definition the precubical maps ϕ_i for all $i \in J$ HDA maps. Because for all $i, j \in J$ and $(f : i \to j) \in J$ we by definition have $\phi_j \circ \mathcal{X}_f = \phi_i$ this makes $(\mathcal{L}, \phi)$ a co-cone. Theorem 4.14 then gives us that $(\mathcal{L}, \phi)$ is a colimit of $\mathcal{X}$ since the requirements are met by construction. □

Theorem 4.18. *The category of higher-dimensional automata* **HDA** *is not cocomplete but does have all small coproducts and small filtered colimits.*

Proof. Recall the diagram given in theorem 3.11, which does not have an event consistent colimit. Because of theorem 4.3 there exists a trivial labelling for the event consistent precubical sets in the diagram. Taking the I_1, I_2, F_1 and F_2 as completely empty then gives us a diagram of HDA. Then because of theorem 4.14 any colimit of this diagram must correspond to a colimit of the underlying diagram of event consistent precubical sets. Since this doesn't exist there cannot exist a colimit of the diagram of HDA either. This shows that the category HDA is not cocomplete.

Suppose that $\mathcal{X} : J \to \texttt{HDA}$ is a diagram of HDAs with J a small category that is either discrete or filtered. Since as a consequence of theorem 3.21 the category of event consistent precubical sets has all small coproducts and small filtered colimits. Therefore because of theorem 4.17 any small discrete or filtered diagram of HDA has a coproduct or filtered colimit. □

4.4 Finite/compact HDA

In this subsection we will cover finite/compact HDA. We define finite HDA as the following:

Definition 4.19. *A HDA $\mathcal{X} = (X, I_X, F_X, \lambda_X)$ is finite if and only if the event consistent precubical set X is finite.*

In the rest of this subsection we will work towards proving that a HDA is compact if and only if it is finite. Because of the way that finite HDA are defined this will also make it so that a HDA is compact if and only if its underlying event consistent precubical set is compact.

Theorem 4.20. *Every HDA is the filtered colimit of finite HDAs.*

Proof. Let $\mathcal{Y} = (Y, I_Y, F_Y, \lambda_Y)$ be a HDA. Let $X : J \to \texttt{ECPS}$ be the diagram defined in theorem 3.19 of which (Y, ϕ) is the colimit. Then theorem 4.11 gives us a filtered diagram $\mathcal{X} : J \to \texttt{HDA}$ of which $(\mathcal{L}, \phi)$ is the colimit. Since for all $i \in J$ the event consistent precubical set X_i is finite the HDA $\mathcal{X}_i = (X_i, I_i, F_i, \lambda_i)$ are finite as well. □

Unlike with the event consistent precubical sets we can't reuse most of the proofs of compact precubical sets. Therefore we define the following:

Definition 4.21. *A representable HDA $\mathcal{X} = (X, I_X, F_X, \lambda_X)$ is a HDA where X is a representable precubical set.*

Note that unlike with precubical sets two n-dimensional representable HDA can be completely different due to the labelling function and the initial and final cells not being the same for all representable HDA. The following theorem is therefore also slightly different than the corresponding theorem 2.43 for precubical sets.

Theorem 4.22. *Let $\mathcal{X} = (X, I_X, F_X, \lambda_X)$ and $\mathcal{Y} = (Y, I_Y, F_Y, \lambda_Y)$ be HDA such that $\mathcal{X}$ is a representable HDA of dimension $k \in \mathbb{N}$. Then for every $y \in Y^k$ there exists at most one HDA map $f_y : \mathcal{X} \to \mathcal{Y}$ such that for the unique element $x \in X^k$ we have $f_y^k(x) = y$. These are the only possible HDA maps $\mathcal{X} \to \mathcal{Y}$.*

Proof. For all $y \in Y^k$ theorem 2.43 gives us that there exists a unique precubical map $f_y : \mathcal{X} \to \mathcal{Y}$ with $f_y^k(x) = y$ for the unique element $x \in X^k$, and that these make up the only precubical maps $X \to Y$. Because HDA maps are precubical maps that preserve the labelling and initial and final cells, these precubical maps also define the possible HDA maps. However since not all precubical maps between HDA are HDA maps we get that there exists at most one HDA map $f_y : \mathcal{X} \to \mathcal{Y}$ such that for the unique element $x \in X^k$ we have $f_y^k(x) = y$. □

Theorem 4.23. *Let $\mathcal{X} : J \to \texttt{HDA}$ be a small filtered diagram of HDA with for all $i \in J$ we have $\mathcal{X}_i = (X_i, I_i, F_i, \lambda_i)$ and let $\mathcal{L} = (L, I_L, F_L, \lambda_L)$ be a HDA and $(\mathcal{L}, \phi)$ be a filtered colimit of the diagram $\mathcal{X}$. For all $k \in \mathbb{N}$, $y \in L^k$ there exists at least one $i \in J$ such that there exists a $x_i \in X_i^k$ with $\phi_i^k(x_i) = y$ such that any element in $\mathcal{X}_i$ that can be reached by x_i through the face maps is an initial or final cell if and only if it is mapped to an initial or final cell in $\mathcal{L}$.*

Proof. Theorem 2.24 gives us a HDA $\mathcal{X}_i$ and an element $x_i \in X_i^k$ with $\phi_i^k(x_i) = y$. It also gives us that for every element $z \in L^n$ that can be reached by y through the face maps there exists at least one $j \in J$, $x_j \in X_j^n$ with $\phi_j^n(x_j) = z$ such that $x_j \in I_i$ and $x_j \in F_i$ if $z \in I_L$ and $z \in F_L$ respectively.
Since $y \in X^k$ can only reach a finite amount of elements through its face maps this gives us a finite set of HDA in the diagram $\mathcal{X}$. Then theorem 2.26 gives us that there exists a $i \in J$ such that there exist precubical maps (and therefore HDA maps) from the mentioned HDA to $\mathcal{X}_i$ such that all the mentioned elements overlap. □

The above theorem can be a little confusing. Theorem 4.12 gives us that for any element in $\mathcal{L}$ that is an initial or final cell there must exist a $i \in J$ and an element in $\mathcal{X}_i$ that is mapped to this element in $\mathcal{L}$ that is also an initial and final cell respectively. The above theorem is essentially the same but applied to a specific finite set of elements. The reason why we choose the elements that can be reached by a certain top element through the face maps becomes clear in the next theorem:

Theorem 4.24. *All representable HDA are compact.*

Proof. For this proof we will mostly follow the proof of theorem 2.44. We will repeat this proof but change things where necessary.
Let $\mathcal{X} = (X, I_X, F_X, \lambda_X)$ be a representable HDA of dimension n and let $\mathcal{X} : J \to \mathtt{HDA}$ be a small filtered diagram with the colimit $(\mathcal{L}, \phi)$ with $\mathcal{X}_i = (D_i, I_i, F_i, \lambda_i)$ for all $i \in J$ and $\mathcal{L} = (L, I_L, F_L, \lambda_L)$. Recall the diagram below theorem 2.39, which works the same for HDA instead of precubical sets. Suppose that $\mathrm{Hom}(\mathcal{X}, \mathcal{L})$ is not empty (in which case the statement would be trivial, since then both sets in the top of the diagram would be empty).
Let $f \in \mathrm{Hom}(\mathcal{X}, \mathcal{L})$. Because of theorem 4.22 there exists a unique $y \in L$ such that for the unique element $x \in X^n$ we have $f^n(x) = y$. Also note that f is the only HDA map in $\mathrm{Hom}(\mathcal{X}, \mathcal{L})$ that sends x to y.
From theorem 4.23 it follows that there exists a $i \in J$ such that there exists a $x_i \in X_i^n$ with $\phi_i^n(x_i) = y$ such that any element in $\mathcal{X}_i$ that can be reached by x_i through the face maps is an initial or final cell if and only if it is mapped to an initial or final cell in $\mathcal{L}$ by $\phi_i : \mathcal{X}_i \to \mathcal{L}$. Using theorem 2.43 gives us that there exists a unique precubical map $g \in \mathrm{Hom}(X, D_i)$ with $g^n(x) = x_i$ and therefore $\phi_i^n \circ g^n(x) = \phi_i^n(x_i) = y$ and therefore $\phi_i \circ g = f$. This precubical map g therefore preserves the labelling function and because of our choice of X_i it must also preserve the initial and final cells. This makes $g : \mathcal{X} \to \mathcal{X}_i$ a HDA map and therefore the morphism $\mathrm{Hom}(\mathcal{X}, \phi_i)$ sends g to f, which also means that $U \circ \Phi_i \circ g = f$. This then gives us that U is surjective.
Let $f_1, f_2 \in \varinjlim_{i \in J} \mathrm{Hom}(\mathcal{X}, \mathcal{X}_i)$, $f \in \mathrm{Hom}(\mathcal{X}, \mathcal{L})$ such that $U \circ f_1 = U \circ f_2 = f$. Then there exist $i, j \in J$, $g_i \in \mathrm{Hom}(X, D_i)$ and $g_j \in \mathrm{Hom}(X, D_j)$ such that $\Phi_i \circ g_i = f_1$ and $\Phi_j \circ g_j = f_2$. This then also gives us that $U \circ \Phi_i \circ g_i = f$ and $U \circ \Phi_j \circ g_j = f$ which gives us that $\phi_i \circ g_i = \phi_j \circ g_j = f$. Because of theorem 4.22 there exist unique $x_i \in D_i^n$, $x_j \in D_j^n$ and $y \in L^n$ such that $g_i^n(x) = x_i$, $g_j^n(x) = x_j$ and $f^n(x) = y$.
This gives us $\phi_i^n(x_i) = \phi_j^n(x_j) = y$, which due to theorem 2.22 means that there exist $k \in J$, $h_i : \mathcal{X}_i \to \mathcal{X}_k$ and $h_j : \mathcal{X}_j \to \mathcal{X}_k$ such that $h_i \circ g_i(x_i) = h_j \circ g_j(x_j)$ and therefore $h_i \circ g_i = h_j \circ g_j$. This means that we have $\mathrm{Hom}(\mathcal{X}, h_i)(g_i) = \mathrm{Hom}(\mathcal{X}, h_j)(g_j)$ and therefore $f_1 = \Phi_i \circ g_i = \Phi_j \circ g_j = f_2$. This then gives us that U is injective.
Therefore the canonical morphism $U : \varinjlim_{i \in J} \mathrm{Hom}(\mathcal{X}, \mathcal{X}_i) \to \mathrm{Hom}(\mathcal{X}, \mathcal{L})$ is an isomorphism for every small filtered diagram $\mathcal{X} : J \to \mathtt{HDA}$, which means that $\mathcal{X}$ is a compact HDA. □

Theorem 4.25. *The finite colimit of compact HDA is compact.*

Proof. This again follows from proposition 1.3 of [AR94]. ☐

Theorem 4.26. *A HDA $\mathcal{X} = (X, I_X, F_X, \lambda_X)$ is compact if and only if the event consistent precubical set X is compact. Equivalently every HDA is compact if and only if it is finite.*

Proof. Note that from theorem 2.40, theorem 2.46 and theorem 3.20 it follows that the event consistent precubical set X is compact if and only if it is finite, making the two statements equivalent.
Because of theorem 4.20 we can replace the precubical sets in the proof of theorem 2.40 with HDA which gives us that every compact HDA is finite.
The proof of theorem 2.46 and theorem 4.11 give us that every finite HDA is the finite colimit of representable HDA. Then theorem 4.24 and theorem 4.25 give us that every finite HDA is compact. ☐

As promised we will check what conditions of local finite presentability (definition 2.47) the category of `HDA` satisfies.

Theorem 4.27. *The following statements about the category **`HDA`** are true:*

1. *The category **`HDA`** is not cocomplete, but does have all small coproducts and filtered colimits.*

2. *The full subcategory of **`HDA`** consisting of the compact objects is essentially small.*

3. *Any object in **`HDA`** is a filtered colimit of a diagram of compact objects.*

Proof. Statement 1 follows from theorem 4.18 and statement 3 follows from theorem 4.20.
Statement 2 is somewhat more complicated. Note that because of theorem 4.26 a HDA is compact if and only if it is finite. Let Σ be a set and let $\mathcal{X} = (X, I, F, \lambda)$ be a finite HDA. Using this HDA we can then construct the following finite set:

$$\left\{ \left(x, n, \left(\delta^n_{\nu,a}(x)\right)_{\nu\in\{0,1\},\ a\in\mathbb{N}_{\geq 1},\ a\leq n}, \lambda^n(x), (x \in I), (x \in F) \right) \mid n \in \mathbb{N},\ x \in X^n \right\}$$

Here $x \in X^n$ is an element, $n \in \mathbb{N}$ represents the dimension of this element, $\left(\delta^n_{\nu,a}(x)\right)_{\nu\in\{0,1\},\ a\in\mathbb{N}_{\geq 1},\ a\leq n}$ is the sequence of elements that can be reached by x through the elementary face maps for which one can decide any canonical order, $\lambda^n(x)$ is the labelling of of x and $(x \in I) \in \{0, 1\}$ and $(x \in F) \in \{0, 1\}$ state whether x is an initial and/or a final cell. With this the category of HDA is equivalent to a subcategory of the category of finite sets. Since the category of finite sets is essentially small the category of HDA must be essentially small as well. ☐

Theorem 4.28. *The category of HDA is finitely accessible.*

Proof. For two HDA $\mathcal{X}$ and $\mathcal{Y}$ every HDA map $\mathcal{X} \to \mathcal{Y}$ is simply a precubical map that preserves the labelling function and the initial and final cells. Theorem 3.23 gives us that the category `ECPS` is locally small which gives us that $\mathrm{Hom}(X, Y)$ is a set. Therefore $\mathrm{Hom}(\mathcal{X}, \mathcal{Y})$ must be a set as well. Theorem 4.18 gives us that the category of HDA has all small filtered colimits.
The third statement follows from the second and third statements of theorem 4.27. ☐

5 Ipomsets

5.1 Posets and pomsets

Definition 5.1. *A partially ordered set or poset is a pair* $(P, \dashrightarrow)$*, with* P *a set and* $\dashrightarrow$ *a strict partial order on* P*.*

We will assume that the set P in $(P, \dashrightarrow)$ is finite.
Let $(P, \dashrightarrow)$ be a poset and let $p, q \in P$. If we have $p = q$, $p \dashrightarrow q$ or $q \dashrightarrow p$ then the elements p and q are *comparable*. If we have $p \neq q$, $p \not\dashrightarrow q$ and $q \not\dashrightarrow p$ then the elements p and q are *incomparable*, notation $p \parallel q$. Note that because $\dashrightarrow$ is a strict partial order exactly one of $p \parallel q$, $p = q$, $p \dashrightarrow q$ and $q \dashrightarrow p$ is true.

Definition 5.2. *Let* $(P, \dashrightarrow_P)$ *and* $(Q, \dashrightarrow_Q)$ *be posets such that* $P \subseteq Q$ *and such that for all* $p_1, p_2 \in P$ *we have* $p_1 \dashrightarrow_P p_2$ *if and only if* $p_1 \dashrightarrow_Q p_2$*. Then* $(P, \dashrightarrow_P)$ *is a subposet of* $(Q, \dashrightarrow_Q)$*.*

Definition 5.3. *For any set* Σ *a partially ordered multiset or pomset is a triple* $(P, \dashrightarrow, \lambda)$ *with* $(P, \dashrightarrow)$ *a poset and* $\lambda : P \to \Sigma$ *a labelling function.*

As one can see pomsets are just posets with a labelling function.

Definition 5.4. *Let* $(P, \dashrightarrow_P, \lambda_P)$ *and* $(Q, \dashrightarrow_Q, \lambda_Q)$ *be pomsets such that the poset* $(P, \dashrightarrow_P)$ *is a subposet of* $(Q, \dashrightarrow_Q)$*. If for all* $p \in P$ *we have* $\lambda_P(p) = \lambda_Q(p)$ *then* $(P, \dashrightarrow_P, \lambda_P)$ *is a subpomset* $(Q, \dashrightarrow_Q, \lambda_Q)$*.*

When talking about a poset $(P, \dashrightarrow)$ or a pomset $(P, \dashrightarrow, \lambda)$ we will often refer to it by just the set P. In this case one can assume that the relation on P is denoted with $\dashrightarrow$ or in the case of multiple posets with $\dashrightarrow_P$. Similarly the labelling function is λ or λ_P. For subposets or subpomsets we might also just use the notation $P \subseteq Q$, as long as the meaning is clear from context.
The following definitions apply to both posets and pomsets in the same way.

Definition 5.5. *Let* P *be a poset or a pomset. An element* $p \in P$ *is called* $\dashrightarrow$*-minimal of* P *if there exists no* $q \in P$ *such that* $q \dashrightarrow p$*. An element* $p \in P$ *is called* $\dashrightarrow$*-maximal of* P *if there exists no* $q \in P$ *such that* $p \dashrightarrow q$*.*

Definition 5.6. *Let* $(P, \dashrightarrow_P)$ *be a subposet of* $(Q, \dashrightarrow_Q)$*. Then* P *is called a* $\dashrightarrow$*-antichain of* Q *if for all* $p_1, p_2 \in P$ *we have* $p_1 = p_2$ *or* $p_1 \parallel p_2$*.*

Definition 5.7. *A pomset* $(P, \dashrightarrow, \lambda)$ *or a poset* $(P, \dashrightarrow)$ *is linear if for all* $p, q \in P$ *we have* $p = q$*,* $p \dashrightarrow q$ *or* $q \dashrightarrow p$*.*

In other words: if no two elements are incomparable. Equivalently this means that P has no non-trivial $\dashrightarrow$-antichains (a subposet or subpomset that is empty or has only one element is always a $\dashrightarrow$-antichain).

Theorem 5.8. *Let* $\mathcal{P} = (P, \dashrightarrow_P)$ *and* $\mathcal{Q} = (Q, \dashrightarrow_Q)$ *be two linear posets. Then there exists a bijection* $f : P \to Q$ *that preserves the* $\dashrightarrow$*-relation if and only if* $|P| = |Q|$*. This bijection if it exists is unique.*

Proof. We can define $f : P \to Q$ as the map that sends the $\dashrightarrow$-smallest element of P to the $\dashrightarrow$-smallest element of Q, the second smallest element of P to the second smallest element of Q etc. If $|P| = |Q|$ then it is clear that this map exists and it is also clear that there exists no other way to map the elements of P onto the elements of Q that also preserves the $\dashrightarrow$-relation. $\square$

Theorem 5.9. *Let $\mathcal{P} = (P, \dashrightarrow_P, \lambda_P)$ and $\mathcal{Q} = (Q, \dashrightarrow_Q, \lambda_Q)$ be two linear pomsets. If there exists a bijection $f : P \to Q$ that preserves the relation $\dashrightarrow$ and the labelling function then this bijection is unique.*

Proof. Reducing $\mathcal{P}$ and $\mathcal{Q}$ to posets then theorem 5.8 gives us that there exists a unique bijection $f : P \to Q$ that preserves the relation $\dashrightarrow$ if and only if $|P| = |Q|$ (if $|P| \neq |Q|$ then there exists no bijection $f : P \to Q$). Whether $f : P \to Q$ preserves the labelling function or not it is the only bijection that preserves the relation $\dashrightarrow$. $\square$

5.2 Definition of ipomsets

We can now define partially ordered multisets with interfaces or ipomsets for short. These ipomsets are build on pomsets, where cells can be marked as source and/or target cells.

Definition 5.10. *An ipomset is a tuple $(P, \prec, \dashrightarrow, \lambda, S, T)$ where*

- *P is a finite set,*
- *$\prec$ and $\dashrightarrow$ are strict partial orders on P such that $\dashrightarrow$ is linear on $\prec$-antichains.*
- *$\lambda : P \to \Sigma$ is the labelling function.*
- *$S \subseteq P$ is a subset of the $\prec$-minimal elements of P called the source set.*
- *$T \subseteq P$ is a subset of the $\prec$-maximal elements of P called the target set.*

The condition that $\dashrightarrow$ is linear on $\prec$-antichains implies that $\dashrightarrow$ and $\prec$ together form a total order.

Definition 5.11. *Let $\mathcal{P} = (P, \prec_P, \dashrightarrow_P, \lambda_P, S_P, T_P)$ and $\mathcal{Q} = (Q, \prec_Q, \dashrightarrow_Q, \lambda_Q, S_Q, T_Q)$ be ipomsets and Σ be a set with $\lambda_P : P \to \Sigma$ and $\lambda_Q : Q \to \Sigma$. We say that $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic if there exists a bijective map $f : P \to Q$ such that for all $p_1, p_2 \in P$ we have*

$$f(p_1) \prec_Q f(p_2) \iff p_1 \prec_P p_2$$

$$f(p_1) \dashrightarrow_Q f(p_2) \iff p_1 \dashrightarrow_P p_2$$

and for all $p \in P$ we have $\lambda_P(p) = \lambda_Q \circ f(p)$, $f(S_P) = S_Q$ and $f(T_P) = T_Q$. We refer to this bijective map as the ipomset isomorphism.

Definition 5.12. *Let $\mathcal{P} = (P, <, \dashrightarrow, \lambda, S, T)$ be an ipomset. Let $(P_i)_{0 \leq i \leq n}$ be a finite sequence of subsets of P such that P_0 is the subset of the $\prec$-minimal elements of P, P_1 is the subset of the $\prec$-minimal elements of $P \backslash P_0$ and let P_i be the subset of $\prec$-minimal elements of $P_{i-2} \backslash P_{i-1}$. In other words we split P up based on $\prec$, which gives us that $\bigcup_{i \geq 0}^{n} P_i = P$.*
We then define the strict linear order $<$ on P as the following: For all $i, j \in \mathbb{N}$ with $0 \leq s \leq t \leq n$, $x \in P_i$ and $y \in P_j$ we have

$$x < y \iff s < t \text{ or } s = t \text{ and } x \dashrightarrow y$$

Theorem 5.13. *The relation $<$ as defined above is a strict linear order.*

Proof. Let $p \in P$. By definition there exists a unique $t \in \mathbb{N}$ such that $p \in P_t$. It is clear that we cannot have $p < p$, since that would mean $p \dashrightarrow p$ which is in contradiction with $\dashrightarrow$ being a strict partial order. Therefore $\prec$ is irreflexive.
Let $p_1 \in P_s$ and $p_2 \in P_t$ for certain $s, t \in \mathbb{N}$. Suppose that we have $p_1 < p_2$. If we have $s < t$ then we cannot have $p_2 < p_1$ by definition. If we have $s = t$ then $p_1 < p_2$ implies $p_1 \dashrightarrow p_2$, which means that $p_2 \not\dashrightarrow p_1$ and therefore $p_2 \not< p_1$. This shows that $<$ is asymmetric.
Let $p_1 \in P_r$, $p_2 \in P_s$ and $p_3 \in P_t$ for certain $r, s, t \in \mathbb{N}$. Suppose that we have $p_1 < p_2$ and $p_2 < p_3$. Then we have $r < s < t$, $r = s < t$, $r < s = t$ or $r = s = t$. In the first three cases we get $r < t$, which by definition gives us $p_1 < p_3$. Suppose that $r = s = t$. Then $p_1 < p_2$ and $p_2 < p_3$ implies $p_1 \dashrightarrow p_2$ and $p_2 \dashrightarrow p_3$, which gives us $p_1 \dashrightarrow p_3$ and therefore $p_1 < p_3$. This shows that $<$ is transitive.
Let $p_1 \in P_s$ and $p_2 \in P_t$ for certain $s, t \in \mathbb{N}$. If we have $s < t$ or $t < s$ then we have $p_1 < p_2$ or $p_2 < p_1$. If we have $s = t$ then because $P_s = P_t$ is linearly ordered by $\dashrightarrow$ we have $p_1 \dashrightarrow p_2$ or $p_2 \dashrightarrow p_1$, which gives us $p_1 < p_2$ or $p_2 < p_1$. Therefore all elements in P are comparable by $<$.
This proves that $<$ is a strict linear order. □

Theorem 5.14. *Let $\mathcal{P} = (P, \prec_P, \dashrightarrow_P, \lambda_P, S_P, T_P)$ and $\mathcal{Q} = (Q, \prec_Q, \dashrightarrow_Q, \lambda_Q, S_P, T_P)$ be ipomsets and Σ be a set with $\lambda_P : P \to \Sigma$ and $\lambda_Q : Q \to \Sigma$. Suppose that there exists a bijective map $f : P \to Q$ that satisfies the conditions in definition 5.11. Then this map is unique.*

Proof. For this we define the strict linear orders $<_P$ on P and $<_Q$ on Q as in definition 5.12 and theorem 5.13. Let $g : Q \to P$ be the inverse of $f : P \to Q$. Because the bijective maps f and g respect the strict partial orders $\prec_P$ and $\prec_Q$ we have that f sends all $\prec_P$-minimal elements of P to $\prec_Q$-minimal elements of Q and g sends all $\prec_Q$-minimal elements of Q to $\prec_P$-minimal elements of P. This gives us that $P_0 = g(Q_0)$ and $Q_0 = f(P_0)$. The same goes for $P_t = g(Q_t)$ and $Q_t = f(P_t)$ for all $t \in \mathbb{N}$. Because f and g respect the strict partial orders $\dashrightarrow_P$ and $\dashrightarrow_Q$ as well we have that for all $t \in \mathbb{N}$, $p, p' \in P_t$ that if $p \dashrightarrow_P p'$ then $f(p) \dashrightarrow_Q f(p')$ (analogously the same is true for g). Therefore for all $p_1 \in P_s$, $p_2 \in P_t$, $q_1 \in Q_s$, $q_2 \in Q_t$, $s, t \in \mathbb{N}$ we have

$$p_1 <_P p_2 \iff f(p_1) <_Q f(p_2)$$

$$g(q_1) <_P g(q_2) \iff q_1 <_Q q_2$$

Therefore every bijective map $f : P \to Q$ that satisfies the conditions in definition 5.11 must preserve the strict linear order $<_P$ in the sense that for all $p, p' \in P$ we have $p <_P p' \iff f(p) <_Q f(p')$. Since P and Q are the same size there is only one way to injectively map the elements of P onto Q while preserving the linear order (since the smallest element in P needs to be mapped to the smallest element of Q, the second smallest element of P needs to be mapped to the second smallest element of Q etc.). Since the preservation of $<_P$ is a requirement this means that for all ipomsets $\mathcal{P}$ and $\mathcal{Q}$ there exists at most one bijective map that satisfies the conditions in definition 5.11. □

We have now defined ipomset isomorphisms, and shown that if two ipomsets are isomorphic then their isomorphism is unique. Let $(P, \prec, \dashrightarrow, \lambda, S, T)$ be an ipomset. Due to the labelling function it does not really matter what the elements in P are exactly. Isomorphic ipomsets are therefore functionally the same, however enforcing them being different will cause some of the later definitions and theorems to become extremely complicated. Therefore we will define ipomsets as isomorphism classes of ipomsets instead. We therefore make the following assumption:

Assumption 5.15. *Let $(P, \prec, \dashrightarrow, \lambda, S, T)$ be an ipomset. There exists a $m \in \mathbb{N}$ for which we have $P = [0, m] = \{n \mid n \in \mathbb{N},\, n < m\}$ is a finite set. We also require that relation $<$ on $\mathbb{N}$ is the same as the relation $<$ as defined in definition 5.12 and theorem 5.13.*

The reason why we need this is due to the gluing composition defined in the next subsection, and due to the maps of ipomset languages as defined in section 7.

5.3 Gluing composition

In this subsection we want to define a gluing composition on the ipomsets which works similarly to the serial composition of strings. However where the composition of ab and cd clearly results in $abcd$ it isn't that straightforward for ipomsets. For ipomsets we will glue the target elements of the first onto the source element of the second ipomset. This is only possible if these source and target sets are isomorphic, as defined below:

Definition 5.16. *Let $\mathcal{P} = (P, <_P, \dashrightarrow_P, \lambda_P, S_P, T_P)$ and $\mathcal{Q} = (Q, <_Q, \dashrightarrow_Q, \lambda_Q, S_Q, T_Q)$ be ipomsets. We say that T_P is isomorphic to S_Q, notation $T_P \cong S_Q$, if there exists a bijective map $\kappa : T_P \to S_Q$ such that for all $p, p_1, p_2 \in T_P$ we have*

$$\lambda_P(p) = \lambda_Q \circ \kappa(p)$$

$$p_1 \dashrightarrow_P p_2 \iff \kappa(p_1) \dashrightarrow_Q \kappa(p_2)$$

Note that the relations $\prec_P$ and $\prec_Q$ are not relevant here since T_P is a subset of the set of $\prec_P$-maximal elements and S_Q is a subset of the set of $\prec_Q$-minimal elements.

The isomorphism $\kappa : T_P \to S_Q$ is unique for the same reasons as given in theorem 5.14. We can now define the gluing composition on isomorphisms (as defined in assumption 5.15).

Definition 5.17. *Let $\mathcal{P} = (P, \prec_P, \dashrightarrow_P, \lambda_P, S_P, T_P)$ and $\mathcal{Q} = (Q, \prec_Q, \dashrightarrow_Q, \lambda_Q, S_Q, T_Q)$ be ipomsets with $P = [0, m]$ and $Q = [0, n]$ such that $T_P \cong S_Q$ as defined in definition 5.16. For ipomsets with $P \neq T_P$ and $Q \neq S_Q$ we define the gluing composition as the following:*

$$\mathcal{P} * \mathcal{Q} = (P \cup [m+1, m+n+1-|S_Q|], \prec, \dashrightarrow, \lambda, S_P, T)$$

$$\mathcal{P} * \mathcal{Q} = (R, \prec, \dashrightarrow, \lambda, S, T)$$

with

$$R = P \cup [m+1, m+n+1-|S_Q|] = [0, m+n+1-|S_Q|]$$

Let $\varkappa : S_Q \to T_P$ be the inverse of the unique isomorphism $\kappa : T_P \to S_Q$. We define the following maps:

$$f : P \to R,\ g : S_Q \to R \text{ and } h : Q \backslash S_Q \to R$$

For all $p \in P$ we define $f(p) = p$. For all $q \in S_Q$ we define $g(q) = f \circ \varkappa(q)$.
Defining the map $h : Q \backslash S_Q \to R$ is more complicated. We can define $(Q \backslash S_Q, <_Q)$ as a linear poset for which there exists a unique map to $[m+1, m+n+1-|S_Q|] \subseteq R$. We define $h : Q \backslash S_Q \to R$ as this map.
We can now define the relation $\prec$. Let $x, y \in R$ with $x < y$. If $x < y < m$ then

$$x \prec y \iff f^{-1}(x) \prec_P f^{-1}(y)$$

If $x < m \leq y$ and $f^{-1}(x) \in T_P$ then

$$x \prec y \iff g^{-1}(x) \prec_Q h^{-1}(y)$$

Note that if $f^{-1}(x) \in T_P$ then S_Q is not empty and $g^{-1}(x)$ exists. If $x < m \leq y$ and $f^{-1}(x) \notin T_P$ then we always have $x \prec y$. Lastly if $m \leq x < y$ then we have

$$x \prec y \iff h^{-1}(x) \prec_Q h^{-1}(y)$$

To define the relation $\dashrightarrow$ we will first define the relation $\dashrightarrow^*$, which is done in similar fashion to the relation $\prec$: Let $x, y \in R$ with $x < y$. If $x < y < m$ then

$$x \dashrightarrow^* y \iff f^{-1}(x) \dashrightarrow_P f^{-1}(y)$$

If $x < m \leq y$ and $f^{-1}(x) \in T_P$ then

$$x \dashrightarrow^* y \iff g^{-1}(x) \dashrightarrow_Q h^{-1}(y)$$

however here if $x < m \leq y$ and $f^{-1}(x) \notin T_P$ then we never have $x \dashrightarrow^* y$. Lastly if $m \leq x < y$ then we have

$$x \dashrightarrow^* y \iff h^{-1}(x) \dashrightarrow_Q h^{-1}(y)$$

We then define $\dashrightarrow$ as the transitive closure of $\dashrightarrow^*$. Having defined the relations we can now define the labelling function $\lambda : R \to \Sigma$. For all $r \in R$ with $r < m$ we define $\lambda(r) = \lambda_P \circ f^{-1}(r)$ and for all $r \in R$ with $m \leq r$ we define $\lambda(r) = \lambda_Q \circ h^{-1}(r)$.
Lastly we can define the source and target sets. For the source set we define $S = S_P$. For the target set we define $T = g(T_Q \cap S_Q) \cup h(T_Q \backslash S_Q)$.

We decided to treat the cases of $P \neq T_P$ and $Q \neq S_Q$ and $P = T_P$ and/or $Q = S_Q$ separately. This distinction is not entirely necessary, however since we will use the case of $P = T_P$ and/or $Q = S_Q$ quite often and since the definition below is a lot simpler and easier to use then the definition above we decided to use this distinction anyways.

Definition 5.18. *Let $\mathcal{P} = (P, \prec_P, \dashrightarrow_P, \lambda_P, S_P, T_P)$ and $\mathcal{Q} = (Q, \prec_Q, \dashrightarrow_Q, \lambda_Q, S_Q, T_Q)$ be ipomsets with $P = [0, m]$ and $Q = [0, n]$ such that $T_P \cong S_Q$ as defined in definition 5.16. For ipomsets with $P = T_P$ we define the gluing composition as the following:*

$$\mathcal{P} * \mathcal{Q} = (Q, \prec_Q, \dashrightarrow_Q, \lambda_Q, S_P, T_Q)$$

For ipomsets with $Q = S_Q$ we define the gluing composition as the following:

$$\mathcal{P} * \mathcal{Q} = (P, \prec_P, \dashrightarrow_P, \lambda_P, S_P, T_Q)$$

Note that if $P = T_P$ and $Q = S_Q$ then both $\prec_P$ and $\prec_Q$ must be empty and since $P = T_P \cong S_Q = Q$ as pomsets we have $P = Q$ (since they must be the same size). In this case it therefore does not matter which of the above definitions we use, since the result is the same.

6 Tracks and labelling

In the previous section we defined higher-dimensional automata. In the next section we will define the languages of higher-dimensional automata. This section bridges the two by defining tracks and their labelling, which will eventually form the elements of the languages.

6.1 Definition of tracks

In this subsection we will define tracks of HDA. For this we first need to define (elementary) upper/lower faces.

Definition 6.1. *Let X be an event consistent precubical set, $n, s \in \mathbb{N}$ with $s \geq 1$ and let $x \in X^n$ and $y \in X^{n+s}$.*
Suppose that there exists a s-dimensional vector A as defined in definition 2.7 such that $x = \delta_{0,A}^{n+s}(y)$. Then we say that x is a s-lower face of y, notation $x \lhd^s y$. The upper faces work analogously. Suppose that there exists a s-dimensional vector A such that $x = \delta_{1,A}^{n+s}(y)$. Then we say that x is a s-upper face of y, notation $y \rhd^s x$.
If we have $s = 1$ then we say that x is an elementary lower/upper face of y, notation $x \lhd y$ and $y \rhd x$. For any $s \in \mathbb{N}_{\geq 1}$ we can also just say that x is a lower/upper face of y, notation $x \lhd^* y$ and $y \rhd^* x$.

The element x being a face of the element y simply means that x can be reached by y through the face maps. The element x being an lower or upper face of y then means that the vectors V of elements in $\{0, 1\}$ are all exclusively 0 or 1 respectively. Note that this means that all elementary faces are lower/upper faces. We will give an intuitive explanation with the following example:

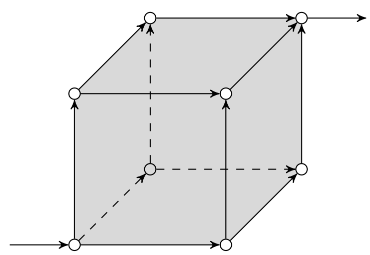


Figure 10: The 3-dimensional representable HDA $\mathcal{X} = (X, I_X, F_X, \lambda_X)$ with a certain labelling.

Let $x \in X^3$ be the unique 3-dimensional element. Then the elements $\delta_{0,1}^3(x)$, $\delta_{0,2}^3(x)$ and $\delta_{0,3}^3(x)$ are its elementary lower faces and the elements $\delta_{0,1}^3(x)$, $\delta_{0,2}^3(x)$ and $\delta_{0,3}^3(x)$ are its elementary upper faces. The initial state and the three edges coming from it are lower faces as well and the final state and the three edges going to it are upper faces. The six nodes other than the initial and final nodes are faces of x, but not lower or upper faces. The same is true for the six edges between these nodes. We can now define tracks:

Definition 6.2. *A track in an event consistent precubical set X is a non-empty finite sequence $\rho = (x_1, ..., x_m)$, $m \in \mathbb{N}_{\geq 1}$ of elements of X, with $x_t \in X^{n_t}$, $n_t \in \mathbb{N}$ for all $1 \leq t \leq m$ such that $x_t \lhd^* x_{t+1}$ or $x_t \rhd^* x_{t+1}$ for all $1 \leq t \leq m - 1$.*

The tracks of size 1 and 2 are special cases which we will refer to as:

Definition 6.3. *Let X be a precubical set and let ρ be a track in X. We say that ρ is a single track if $\rho = (x)$ for some $x \in X^n$, $n \in \mathbb{N}$.*

Definition 6.4. *Let X be a precubical set and let ρ be a track in X. We say that ρ is a basic track if $\rho = (x_1, x_2)$ for some $x_1 \in X^n$, $x_2 \in X^m$, $n, m \in \mathbb{N}$.*

The reason for these tracks being noteworthy is because we can express any track as a composition of single and basic tracks:

Definition 6.5. *Let X be a precubical set and let $\rho = (x_1, ..., x_m)$ and $\tau = (y_1, ..., y_l)$ be tracks in X. If we have $x_m = y_1$ then we define the composition of these tracks as*

$$\rho * \tau = (x_1, ..., x_m, y_2, ..., y_l)$$

Theorem 6.6. *Let X be a precubical set and let $\rho = (x_1, ..., x_m)$ be a track in X. Then ρ is the composition of single and basic tracks with*

$$\rho = (x_1, ..., x_m) = (x_1) * (x_1, x_2) * (x_2, x_3) * ... * (x_{m-2}, x_{m-1}) * (x_{m-1}, x_m)$$

Proof. This follows from definition 6.4 and definition 6.5. □

Note that we included the single track (x_1) in the above theorem just to make it applicable to all tracks. In the case that ρ is of size two or longer we get $(x_1) * (x_1, x_2) * ... = (x_1, x_2) * ...$ where the single track works as a sort of identity. We can only compose tracks of which the last element of the first is the same as the first element of the second.
We will use the following figure as an example:

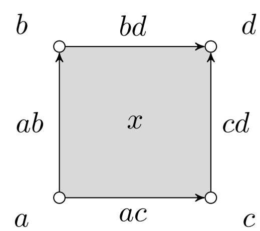


Figure 11: A 2-dimensional representable HDA. Here the letters do not denote the labelling but are just names for clarity.

First we have 9 single tracks of one element each. Suppose that we have a basic track of two elements of which the first is the element a. Since a is a node it has no faces, which means that the second element needs to be one of which a is a lower face. This gives us the possible tracks (a, ab), (a, ac) and (a, x). Suppose that we have a basic track of two elements of which the first is the element ab. Because ab is a lower face of x and because b is an upper face of ab we get the tracks (ab, x) and (ab, b). Similarly we get the basic tracks (ac, x) and (ac, c). The nodes b and c are only the lower face of the edges bd and cd respectively, and these edges only have the lower face d which gives us the basic tracks (b, bd), (bd, d), (c, cd) and (cd, d). If our first element is x we get the basic tracks (x, bd), (x, d) and (x, cd).
With this we have defined all of the single and basic tracks of the above HDA. Now using theorem 6.6 we can get all of the tracks of size 3 or greater. We for example get the tracks (a, ab, x, d), (a, ab, b, bd, d) and (ab, x, cd).
In the example above we have defined no initial or final cells. Moving forward what we are most interested in are accepting tracks.

Definition 6.7. *A track $\rho = (x_1, ..., x_m)$ with $m \in \mathbb{N}$, $m \geq 1$ in a HDA $\mathcal{X} = (X, I_X, F_X, \lambda_X)$ is called an accepting track if $x_1 \in I_X$ and $x_m \in F_X$.*

With this it becomes clear why we want to look at tracks in the first place. The accepting tracks of a HDA are its possible execution paths, where a HDA can be defined to describe something like a computer program or algorithm. Eventually we will, using the labelling function, define labellings of accepting tracks which will form the languages of higher-dimensional automata.

6.2 Properties of tracks

Theorem 6.8. *Let X be an event consistent precubical set and let (x, y) be a track with $x \in X^n$ and $y \in X^m$, $n, m \in \mathbb{N}$. We have $x \lhd^* y$ or $x \rhd^* y$ which means that there exists a vector A as described in definition 2.7 such that $x = \delta^m_{0,A}(y)$ or $\delta^n_{1,A}(x) = y$. This vector is unique.*

The following theorem shows that tracks are preserved by precubical maps.

Proof. Suppose that $n < m$. Then we have to have $x = \delta^m_{0,A}(y)$. Suppose that there exists a vector B such that $x = \delta^m_{0,B}(y)$. Then we have $\delta^m_{0,A}(y) = \delta^m_{0,B}(y)$. In the case that $n > m$ instead we analogously get $\delta^n_{1,A}(x) = \delta^n_{1,B}(x)$. Then from theorem A.1 it follows that $A = B$. □

Theorem 6.9. *Let X and Y be event consistent precubical sets and let $f : X \to Y$ be a precubical map. Let $\rho = (x_1, ..., x_m)$ be a track in X with $x_t \in X^{n_t}$, $n_t \in \mathbb{N}$ for all $1 \leq t \leq m$. Then $f(\rho) = (f^{n_1}(x_1), ..., f^{n_m}(x_m))$ is a track in Y and for all $1 \leq t \leq m-1$ we have*

$$x_t \lhd^s x_{t+1} \iff f^{n_t}(x_t) \lhd^s f^{n_{t+1}}(x_{t+1})$$
$$x_t \rhd^s x_{t+1} \iff f^{n_t}(x_t) \rhd^s f^{n_{t+1}}(x_{t+1})$$

for a certain $s \in \mathbb{N}$.

Proof. Let $t \in \mathbb{N}$ with $1 \leq t \leq m-1$. Suppose that we have $x_t \lhd^* x_{t+1}$. Then for $s = n_{t+1} - n_t$ we have $x_t \lhd^s x_{t+1}$ and therefore there exists a vector A such that $x_t = \delta^{n_{t+1}}_{0,A}(x_{t+1})$. This gives us $f^{n_t}(x_t) = \delta^{n_{t+1}}_{0,A} \circ f^{n_{t+1}}(x_{t+1})$ which means that we have $f^{n_t}(x_t) \lhd^s f^{n_{t+1}}(x_{t+1})$. Analogously the same is true for $x_t \rhd^* x_{t+1}$.
Now suppose that $f^{n_t}(x_t) \lhd^s f^{n_{t+1}}(x_{t+1})$. This means that $s = n_{t+1} - n_t$ and $n_{t+1} > n_t$. Since ρ is a track we therefore must have $x_t \lhd^s x_{t+1}$. Analogously the same is true for $f^{n_t}(x_t) \rhd^s f^{n_{t+1}}(x_{t+1})$. □

Remark 6.9.1. *Because of theorem 6.8 there can only be one face map such that $x_t = \delta^{n_{t+1}}_{0,A}(x_{t+1})$ or $\delta^{n_t}_{0,A}(x_t) = x_{t+1}$, and since all precubical maps commute with face maps this face map is the same for x_t, x_{t+1} and $f^{n_t}(x_t), f^{n_{t+1}}(x_{t+1})$.*

Theorem 6.10. *Let X and Y be event consistent precubical sets, let $f : X \to Y$ be a precubical map and let ρ_1 and ρ_2 be tracks in X such that $\rho_1 * \rho_2$ is a track in X. Then we have*

$$f(\rho_1 * \rho_2) = f(\rho_1) * f(\rho_2)$$

Proof. From theorem 6.9 it follows that $f(\rho_1 * \rho_2)$, $f(\rho_1)$ and $f(\rho_2)$ are tracks in Y.
Let $\rho_1 = (x_1, x_2, ...x_m)$ and $\rho_2 = (x_m, x_{m+1}, ..., x_n)$. Then we have $\rho_1 * \rho_2 = (x_1, x_2, ..., x_n)$ and

$$f(\rho_1 * \rho_2) = (f(x_1), f(x_2), ..., f(x_n))$$

and theorem 6.6 gives us that

$$(f(x_1), f(x_2), ..., f(x_m)) * (f(x_m), f(x_{m+1}), ..., f(x_n)) = f(\rho_1) * f(\rho_2)$$

□

As precubical maps preserve tracks, HDA maps preserve accepting tracks.

Theorem 6.11. *Let $\mathcal{X} = (X, I_X, F_X, \lambda_X)$ and $\mathcal{Y} = (Y, I_Y, F_Y, \lambda_Y)$ be HDAs and let $f : \mathcal{X} \to \mathcal{Y}$ be a HDA map. For all accepting tracks $\rho = (x_1, ..., x_m)$ in $\mathcal{X}$ we have that $f(\rho) = (f(x_1), ..., f(x_m))$ is an accepting track in $\mathcal{Y}$.*

Proof. From theorem 6.9 it follows that for every accepting track $\rho = (x_1, ..., x_m)$ in $\mathcal{X}$ we have that $f(\rho) = (f(x_1), ..., f(x_m))$ is a track in $\mathcal{Y}$. Since $x_1 \in I_X$ and $x_m \in F_X$ imply $f(x_1) \in I_Y$ and $f(x_m) \in F_Y$ we have that $f(\rho)$ is an accepting track in $\mathcal{Y}$. ☐

As we are mostly interested in accepting tracks and HDA we will at times only prove things for HDA. The following theorem is true for diagrams of event consistent precubical sets as well:

Theorem 6.12. *Let $\mathcal{X} : J \to \texttt{HDA}$ be a small diagram and let $(\mathcal{N}, \psi)$ be a co-cone. Let τ be a track in $\mathcal{X}_i$ for a certain $i \in J$. Suppose that there exist $j \in J$, $f : i \to j$. Then we have*

$$\psi_i(\tau) = \psi_j \circ \mathcal{X}_f(\tau)$$

Proof. Let $\tau = (x_1, ..., x_m)$. Then for all $1 \leq t \leq m$ we have $\psi_i(x_t) = \psi_j \circ \mathcal{X}_f(x_t)$. This gives us

$$\psi_i(\tau) = (\psi_i(x_1), ..., \psi_i(x_m)) = (\psi_j \circ \mathcal{X}_f(x_1), ..., \psi_j \circ \mathcal{X}_f(x_m)) = \psi_j \circ \mathcal{X}_f(\tau)$$

which proves the statement. ☐

Theorem 6.13. *Let $\mathcal{X} : J \to \texttt{HDA}$ be a small discrete diagram, let $(\mathcal{L}, \phi)$ be a coproduct of $\mathcal{X}$ and let ρ be an accepting track in $\mathcal{X}$. Then there exists a unique $i \in J$ and a unique accepting track τ in $\mathcal{X}_i$ such that $\phi_i(\tau) = \rho$.*

Proof. Let $\mathcal{L} = (X_L, I_X, F_X, \lambda_L)$, let $\rho = (x_1, x_2, ..., x_m)$ be the accepting track and let $\mathcal{X}_i = (X_i, I_i, F_i, \lambda_i)$ for all $i \in J$.
From theorem 2.13 it follows that for all $1 \leq t \leq m$ there exists a unique $i_t \in J$ such that there exists a unique $y_t \in X_{i_t}$ with $\phi_{i_t}(y_t) = x_t$.
For all $1 \leq t \leq m-1$ we have $x_t \lhd^* x_{t+1}$ or $x_t \rhd^* x_{t+1}$, which gives us that there exists a $n \in \mathbb{N}$ and a vector A as described in definition 2.7 such that $x_t = \delta^n_{0,A}(x_{t+1})$ or $\delta^n_{1,A}(x_t) = x_{t+1}$.
This gives us that we have $\phi_i(y_t) = x_t = \phi_{t+1} \circ \delta^n_{0,A}(y_{t+1})$ or $\phi_t \circ \delta^n_{1,A}(y_t) = x_{t+1} = \phi_{t+1}(y_{t+1})$ which gives us $i_t = i_{t+1}$ and $y_t = \delta^n_{0,A}(y_{t+1})$ or $\delta^n_{1,A}(y_t) = y_{t+1}$ for all $1 \leq t \leq m-1$.
Therefore we have $i_1 = i_2 = ... = i_m$, which means that there exists a unique $i \in J$ such that for all $1 \leq t \leq m$ there exists a $y_t \in X_i$ such that $\phi_i(y_t) = x_t$. This also gives us that $x_t \lhd^* x_{t+1} \iff y_t \lhd^* y_{t+1}$ and $x_t \rhd^* x_{t+1} \iff y_t \rhd^* y_{t+1}$ for all $1 \leq t \leq m-1$. Therefore $\tau = (y_1, y_2, ..., y_m)$ is a unique track in $\mathcal{X}_i$ such that $\phi_i(\tau) = \rho$. Theorem 2.13 gives us that $i \in J$ is unique and $y_1, y_m \in X_i$ are the only elements with $\phi_i(y_1) = x_1$ and $\phi_i(y_m) = x_m$ and theorem 4.12 then gives us that $y_1 \in I_i$ and $y_m \in F_i$. This makes τ an accepting track in $\mathcal{X}_i$ with $\phi_i(\tau) = \rho$. ☐

Theorem 6.14. *Let $\mathcal{X} : J \to \texttt{HDA}$ be a small filtered diagram and let $(\mathcal{L}, \phi)$ be a filtered colimit of this diagram. Suppose that we have $i \in J$ with τ a track in $\mathcal{X}_i$ such that $\phi_i(\tau)$ is an accepting track in $\mathcal{L}$. Then there exist a $j \in J$ and an accepting track ρ in $\mathcal{X}_j$ such that $\phi_i(\tau) = \phi_j(\rho)$.*

Proof. Let $\tau = (y_1, ..., y_m)$ be a track in $\mathcal{X}_i$ for a certain $i \in J$ and let $\phi_i(\tau) = (x_1, ..., x_m)$ be an accepting track in $\mathcal{L}$.
Theorem 4.12 gives us that there exist $k, l \in J$, $z_1 \in X_k$ and $z_m \in X_l$ with $z_1 \in I_k$ and $z_m \in F_l$ such that $\phi_k(z_1) = x_1$ and $\phi_l(z_m) = x_m$. From theorem 2.22 and theorem 2.28 it then follows that since $\phi_i(y_1) = \phi_k(z_1)$ and $\phi_i(y_m) = \phi_l(z_m)$ there exists a $j \in J$ and morphisms $f : i \to j$, $g : k \to j$ and $h : l \to j$ such that $X_f(y_1) = X_g(z_1)$ and $X_f(y_m) = X_h(z_m)$.
Because τ is a track in $\mathcal{X}_i$ we get that $X_f(\tau)$ is a track in $\mathcal{X}_j$ and since we have $z_1 \in I_k$ and $z_m \in F_l$ we have $X_f(y_1) = X_g(z_1) \in I_j$ and $X_f(y_m) = X_h(z_m) \in F_j$. Therefore $X_f(\tau)$ is an accepting track in $\mathcal{X}_j$ and we have $\phi_j \circ X_f(\tau) = \phi_i(\tau)$. □

Theorem 6.15. *Let $\mathcal{X} : J \to$ HDA be a small filtered diagram, let $(\mathcal{L}, \phi)$ be the filtered colimit of this diagram and let $\rho_1 = (x_1, x_2, ..., x_m)$ and $\rho_2 = (x_m, x_{m+1}, ..., x_n)$ be tracks in $\mathcal{L}$ such that $\rho_1 * \rho_2 = (x_1, x_2, ..., x_n)$ is defined and a track in $\mathcal{L}$.*
Suppose that $\tau_1 = (y_1, y_2, ..., y_m)$ is a track in $\mathcal{X}_i$ for $i \in J$ and $\tau_2 = (z_m, z_{m+1}, ..., z_n)$ is a track in $\mathcal{X}_j$ for $j \in J$ such that $\phi_i(\tau_1) = \rho_1$ and $\phi_j(\tau_2) = \rho_2$.
Then there exists a $k \in J$ and morphisms $f : i \to k$ and $g : j \to k$ such that $\mathcal{X}_f(y_m) = \mathcal{X}_g(z_m)$ and $\tau_3 = \mathcal{X}_f(\tau_1) * \mathcal{X}_g(\tau_2)$ is a track in $\mathcal{X}_k$ with $\phi_k(\tau_3) = \rho_1 * \rho_2$.

Proof. Since we have $\phi_i(\tau_1) = \rho_1$ and $\phi_j(\tau_2) = \rho_2$ we have $\phi_i(y_m) = x_m = \phi_j(z_m)$. Theorem 2.22 therefore gives us that $y_m \sim z_m$ which as a result of theorem 2.27 gives us that there exists a $k \in J$ and morphisms $f : i \to k$ and $g : j \to k$ such that $\mathcal{X}_f(y_m) = \mathcal{X}_g(z_m)$. Theorem 6.10 then gives us that $\tau_3 = \mathcal{X}_f(\tau_1) * \mathcal{X}_g(\tau_2)$ is a track in $\mathcal{X}_k$ with $\phi_k(\tau_3) = \phi_k(\mathcal{X}_f(\tau_1) * \mathcal{X}_g(\tau_2)) = \phi_k(\mathcal{X}_f(\tau_1)) * \phi_k(\mathcal{X}_f(\tau_2)) = \rho_1 * \rho_2$. □

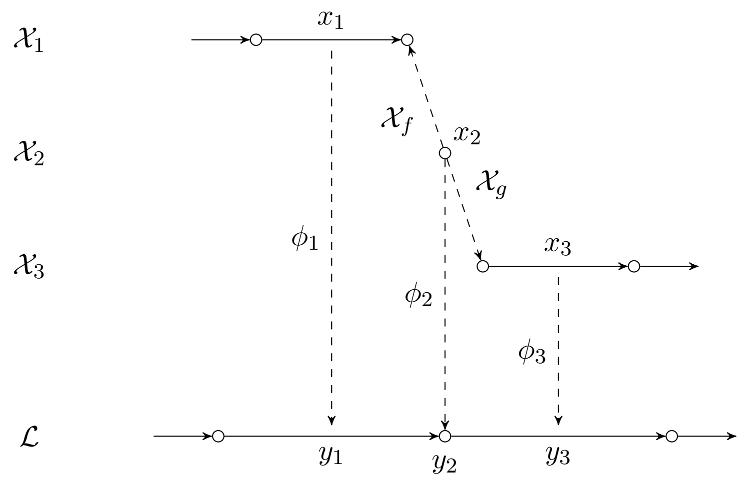
Theorem 6.16. *Let $\mathcal{X} : J \to$ HDA be a small filtered diagram and let $(\mathcal{L}, \phi)$ be the filtered colimit of this diagram. Suppose that ρ is an accepting track of $\mathcal{L}$. Then there exists a $i \in J$ and an accepting track τ in $\mathcal{X}_i$ such that $\phi_i(\tau) = \rho$.*

Proof. First suppose that $\rho = (x_1)$. Then from theorem 2.24 we get that there exists a $i \in J$ and a $y_1 \in X_i$ such that $\phi_i(y_1) = x_1$. Then $\tau = (y_1)$ is a track in $\mathcal{X}_i$ such that $\phi_i(\tau) = \rho$.
Suppose that $\rho = (x_1, x_2)$. We have $x_1 = \delta_{0,A}(x_2)$ or $\delta_{1,A}(x_1) = x_2$ for a certain vector A as described in definition 2.7. Therefore from theorem 2.24 it follows that there exists a $i \in J$ such that there exist $y_1, y_2 \in X_i$ with $y_1 = \delta_{0,A}(y_2)$ or $\delta_{1,A}(y_1) = y_2$ and $\phi_i(y_1) = x_1$ and $\phi_i(y_2) = x_2$.
Suppose that $\rho = (x_1, x_2, ..., x_m)$ for a certain $m \geq 3$. We have $\rho = (x_1, x_2) * (x_2, x_3) * .... * (x_{m-1}, x_m) = \rho_1 * \rho_2 * ... * \rho_{m-1}$. Then for every track $\rho_t = (x_t, x_{t+1})$ with $1 \leq t \leq m-1$ there exists a $i_t \in J$ and a track $\tau_t = (y_t, z_{t+1})$ in $\mathcal{X}_i$ with $\phi_{i_t}(\tau_t) = \rho_t$ as we have proven previously.
From theorem 6.15 it follows that there exists a $j_1 \in J$ and a track σ_1 in $\mathcal{X}_{j_1}$ with $\phi_{j_1}(\sigma_1) = \rho_1 * \rho_2$. Applying theorem 6.15 again gives us that there exists a $j_2 \in J$ and a track σ_2 in $\mathcal{X}_{j_2}$ with $\phi_{j_2}(\sigma_2) = \rho_1 * \rho_2 * \rho_3$. Repeating this step $m-4$ more times (for a total of $m-2$ times) gives us a $j_{m-2} \in J$ and a track σ_{m-2} in $\mathcal{X}_{j_{m-2}}$ with $\phi_{j_{m-2}}(\sigma_{m-2}) = \rho_1 * \rho_2 * ... * \rho_{m-1} = \rho$.
This means that for every track ρ in $\mathcal{L}$ there exists a $i \in J$ and a track τ in $\mathcal{X}_i$ such that $\phi_i(\tau) = \rho$.
Then theorem 6.14 gives us that for every accepting track ρ in $\mathcal{L}$ there exists a $i \in J$ and an accepting track τ in $\mathcal{X}_i$ such that $\phi_i(\tau) = \rho$. □

For discrete and filtered diagrams we have proven that if ρ is an accepting track of $\mathcal{L}$ there exists a $i \in J$ and an accepting track τ in $\mathcal{X}_i$ such that $\phi_i(\tau) = \rho$. This is not true for every small diagram, even if we don't require ρ and τ to be accepting tracks.

Theorem 6.17. *There exists a small diagram $\mathcal{X} : J \to$ `HDA` with the colimit $(\mathcal{L}, \phi)$ such that there exists a track ρ in $\mathcal{L}$ such that for all $i \in J$ there exists no track τ in $\mathcal{X}_i$ with $\phi_i(\tau) = \rho$.*

Proof. We can prove this with a very simple example: Let J be the small category with $\mathrm{obj}(J) = \{1, 2, 3\}$ and $\mathrm{mor}(J) = \{f : 2 \to 1, g : 2 \to 3\}$. Let $\mathcal{X} : J \to$ `HDA` be the diagram with:



Here we have three HDA in the diagram and their colimit $\mathcal{L}$. The HDA $\mathcal{X}_1 = \left(X_1, \left\{\delta^1_{0,1}(x_1)\right\}, \emptyset, \lambda_1\right)$ consists of a single edge x_1 and where the node $\delta^1_{0,1}(x_1)$ is the only initial cell. The HDA $\mathcal{X}_2 = (X_2, \emptyset, \emptyset, \lambda_2)$ is only a single node x_2 with no initial or final cells and the HDA $\mathcal{X}_3 = \left(X_3, \emptyset, \left\{\delta^1_{2,1}(x_3)\right\}, \lambda_3\right)$ consists of a single edge x_3 and where the node $\delta^1_{1,1}(x_3)$ is the only final cell. There are two HDA maps: $\mathcal{X}_f : \mathcal{X}_2 \to \mathcal{X}_1$ and $\mathcal{X}_g : \mathcal{X}_2 \to \mathcal{X}_3$ which are defined as $\mathcal{X}^0_f(x_2) = \delta^1_{0,1}(x_1)$ and $\mathcal{X}^0_g(x_2) = \delta^1_{1,1}(x_3)$. This means that $\delta^1_{0,1}(x_1) \sim x_2 \sim \delta^1_{1,1}(x_3)$ which means that for the colimit we have to have $\phi^0_1 \circ \delta^1_{0,1}(x_1) = \phi^0_2(x_2) = \phi_3 \circ \delta^1_{1,1}(x_3)$. This then gives us the colimit $\mathcal{L}$ as depicted above.
It is clear that $\rho = (y_1, y_2, y_3)$ is a track in $\mathcal{L}$ however there exists no track τ in $\mathcal{X}_1$, $\mathcal{X}_2$ or $\mathcal{X}_3$ such that $\phi_i(\tau) = \rho$. We also have that $\rho = \left(\delta^1_{0,1}(y_1), y_1, y_2, y_3, \delta^1_{1,1}(y_3)\right)$ is an accepting track in $\mathcal{L}$ but $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$ have no accepting tracks themselves. This proves the theorem. ☐

These results about tracks and accepting tracks for diagrams and colimits will be useful again in section 7, where we will define languages of higher-dimensional automata.

6.3 Event identification

Before we can move on to the labelling of tracks we first need to define event identification. This works similar to the labelling function in subsection 4.1 and this subsection will have approximately the same structure. We will also show that the event identification is equivalent with the event relation we defined in section 3.

Definition 6.18. *Let Σ be a set. The event object on Σ is the precubical subset $!!\Sigma \subseteq !\Sigma$ with*

$$!!\Sigma^n = \{(x_1, ..., x_n) \mid (x_1, ..., x_n) \in !\Sigma^n,\ x_s \neq x_t \textit{ whenever } s \neq t\}$$

Definition 6.19. *Let X be a precubical set and Σ a set. An event identification on X is a precubical map $\mathtt{ev} : X \to !!\Sigma$.*

The following theorem proves that event identifications generate equivalence relations as defined in definition 3.1 uniquely.

Theorem 6.20. *Let X be a precubical set and let Σ be a set. Let $\boldsymbol{ev}^1 : X^1 \to \Sigma$ be a function for which $\boldsymbol{ev}^1 \circ \delta^2_{\nu,a}(x) = \boldsymbol{ev}^1 \circ \delta^2_{\mu,b}(x)$ for all $\nu, \mu \in \{0,1\}$ and $a, b \in \{1,2\}$ if and only if $a = b$. Then $\boldsymbol{ev}^1$ extends to an equivalence relation $\equiv_X$ as defined in definition 3.1 with*

$$\boldsymbol{ev}^1(x) = \boldsymbol{ev}^1(y) \iff x \equiv_X y$$

for all $x, y \in X^1$.

Proof. By definition we have $\delta^2_{\nu,a}(x) \equiv_X \delta^2_{\mu,b}(x)$ if $a = b$. Since $\mathtt{ev}^1 \circ \delta^2_{1,1}(x) = \mathtt{ev}^1 \circ \delta^2_{0,1}(x) \neq \mathtt{ev}^1 \circ \delta^2_{0,2}(x) = \mathtt{ev}^1 \circ \delta^2_{1,2}(x)$ this also automatically gives us that for all $x \in X^2$, $\nu, \mu \in \{0,1\}$ and $a, b \in \{1,2\}$ we have $\delta^2_{\nu,a}(x) \equiv_X \delta^2_{\mu,b}(x)$ if and only if $a = b$.
It is clear that $\equiv_X$ is reflexive and symmetric. Since the relation $=$ on Σ is transitive $\equiv_X$ is transitive as well. Therefore $\equiv_X$ is an equivalence relation on X^1 as defined in definition 3.1. □

Note that the equivalence relation described here does not have to be the event relation as defined in definition 3.2. We can still have $\mathtt{ev}^1(x) = \mathtt{ev}^1(y)$ for unrelated $x, y \in X^1$ for example.

Theorem 6.21. *Let X be a precubical set and let Σ be a set. Any function $\boldsymbol{ev}^1 : X^1 \to \Sigma$ for which $\boldsymbol{ev}^1 \circ \delta^2_{0,1}(x) = \boldsymbol{ev}^1 \circ \delta^2_{1,1}(x)$, $\boldsymbol{ev}^1 \circ \delta^2_{0,2}(x) = \boldsymbol{ev}^1 \circ \delta^2_{1,2}(x)$ and $\boldsymbol{ev}^1 \circ \delta^2_{0,1}(x) \neq \boldsymbol{ev}^1 \circ \delta^2_{0,2}(x)$ for all $x \in X^2$ extends uniquely to a precubical map $\boldsymbol{ev} : X \to !!\Sigma$.*

Proof. Using theorem 4.2 we define the precubical map $\mathtt{ev} : X \to !\Sigma$. This gives us for all $n \in \mathbb{N}$, $x \in X^n$

$$\mathtt{ev}^n(x) = \left(\mathtt{ev}^1 \circ \delta^n_{\nu,A^n_1}(x), ..., \mathtt{ev}^1 \circ \delta^n_{\nu,A^n_n}(x)\right)$$

for all $\nu \in \{0,1\}$. As a result of theorem A.7 and theorem 6.20 we get that for all $1 \leq s, t \leq n$ we have $\mathtt{ev}^1 \circ \delta^n_{\nu,A^n_s}(x) = \mathtt{ev}^1 \circ \delta^n_{\nu,A^n_t}(x)$ if and only if $s = t$. This gives us that all elements in $\mathtt{ev}^n(x)$ are unique. Therefore $\mathtt{ev}^n(x) \in !!\Sigma$ for all $n \in \mathbb{N}$, $x \in X^n$ which makes $\mathtt{ev}$ a unique precubical map $\mathtt{ev} : X \to !!\Sigma$. □

We will now move on to new theorems that will be used in the next subsection for the labelling of tracks.

Definition 6.22. *Let X be an event consistent precubical set and let $\equiv_X$ be the event relation on X^1. We define $E_X = X^1 / \equiv_X$ which we call the set of universal events of X.*

Theorem 6.23. *Let X be an event consistent precubical set and let E_X be the set of universal events of X as given in definition 6.22. Then the quotient map $X \to E_X$ extends uniquely to an event identification $\boldsymbol{ev} : X \to !!E_X$.*

Proof. Suppose that $q_X : X \to E_X$ is the quotient map. By definition we then have $q_X \circ \delta^2_{0,1}(x) = q_X \circ \delta^2_{1,1}(x)$, $q_X \circ \delta^2_{0,2}(x) = q_X \circ \delta^2_{1,2}(x)$ and $q_X \circ \delta^2_{0,1}(x) \neq q_X \circ \delta^2_{0,2}(x)$ for all $x \in X^2$. Using theorem 6.21 we then get that this extends uniquely to an event identification $\mathtt{ev} : X \to !!E_X$ with $\mathtt{ev}^1 = q_X$. □

Theorem 6.24. *Let (X, λ) be a labelled precubical set and let $\equiv_X$ be the event relation. Then for all $x, y \in X^1$ we have*

$$x \equiv_X y \implies \lambda^1(x) = \lambda^1(y)$$

Proof. This follows from the fact that $\lambda^1 \circ \delta^2_{0,1}(x) = \lambda^1 \circ \delta^2_{1,1}(x)$ and $\lambda^1 \circ \delta^2_{0,2}(x) = \lambda^1 \circ \delta^2_{1,2}(x)$ for all $x \in X^2$ and the fact that $\equiv_X$ is generated by the transitive closure of $\big\{\big(\delta^2_{\nu,a}(x), \delta^2_{\mu,a}(x)\big) \mid \nu, \mu \in \{0,1\},\, x \in X^2,\, a \in \{1,2\}\big\}$. $\square$

Theorem 6.25. *Let (X, λ) be a labelled precubical set. Then there exists a unique precubical map $\lambda_{ev} :!!E_X \to !\Sigma$ such that $\lambda = \lambda_{ev} \circ ev : X \to !!E_X \to !\Sigma$, with E_X and ev as defined in definition 6.22 and theorem 6.23.*

Proof. From theorem 6.24 it follows that for all $e \in E_X$, $x, y \in X^1$ such that $\mathtt{ev}^1(x) = e$, $\mathtt{ev}^1(y) = e$ we have $\lambda^1(x) = \lambda^1(y)$ since $x \equiv_X y$. We define the function $\lambda^1_{\mathtt{ev}} : E_X \to \Sigma$ (or equivalently $\lambda^1_{\mathtt{ev}} :!!E^1_X \to \Sigma$) as

$$\lambda^1_{\mathtt{ev}}(e) = \lambda^1_{\mathtt{ev}}\left(\mathtt{ev}^1(x)\right) = \lambda^1(x)$$

for all $x \in X^1$ such that $\mathtt{ev}^1(x) = e$. Since for all $e \in E_X$ there exists such $x \in X^1$ and all of these $x \in X^1$ have the same labelling $\lambda^1(x)$ we have that $\lambda^1_{\mathtt{ev}}$ is well-defined.
We have that $!!E_X$ is a precubical set. For all $e \in !!E^2_X$ we have by definition that $\delta^2_{\nu,a}(e) = \delta^2_{\mu,b}(e)$ if and only if $a = b$. This gives us $\lambda^1_{\mathtt{ev}} \circ \delta^2_{\nu,a}(e) = \lambda^1_{\mathtt{ev}} \circ \delta^2_{\mu,b}(e)$ for all $e \in !!E^2_X$ if $a = b$. Therefore we can use theorem 4.2 which gives us the precubical map $\lambda_{\mathtt{ev}} :!!E_X \to !\Sigma$.
Since we have $\lambda^1_{\mathtt{ev}} \circ \mathtt{ev}^1 = \lambda^1$ and because λ is uniquely generated using λ^1 and theorem 4.2 we get $\lambda_{\mathtt{ev}} \circ \mathtt{ev} = \lambda$, which proves the theorem. $\square$

In other words we have that by definition of the labelling function two elements in X^1 that are equivalent by the event relation must have the same label. This is then extended to elements X^n for all $n \in \mathbb{N}$ since $\mathtt{ev} : X \to !!E_X$ and $\lambda : X \to !\Sigma$ are uniquely defined by $\mathtt{ev}^1$ and λ^1.

6.4 Labelling

For this subsection it is important to recall the definitions and theorems of section 5. Here we will define a labelling function ℓ which for a track gives us an ipomset. Before we do this we first define a labelling function on individual elements.

Definition 6.26. *Let (X, λ) be a labelled precubical set. The label of an element $x \in X^n$ for a certain $n \in \mathbb{N}$ is the linear pomset*

$$\ell(x) = (EV(x), \dashrightarrow, \lambda_{ev})$$

with $EV(x) = \left(ev^1 \circ \delta^n_{\nu,A^n_1}(x) \dashrightarrow \ldots \dashrightarrow ev^1 \circ \delta^n_{\nu,A^n_n}(x)\right)$ and $\lambda_{ev} :!!E_X \to !\Sigma$ as defined in theorem 6.25.

Note here that $\mathtt{EV}(x)$ is equivalent to $\mathtt{ev}(x)$ but instead of a vector it is a linear poset. Recall that as a result of theorem A.7 and theorem 6.20 we get that for all $1 \leq s, t \leq n$ we have $\mathtt{ev}^1 \circ \delta^n_{\nu,A^n_s}(x) = \mathtt{ev}^1 \circ \delta^n_{\nu,A^n_t}(x)$ if and only if $s = t$. This means that every element in $\mathtt{EV}(x)$ as described above is unique.

Theorem 6.27. *Let (X, λ) be a labelled precubical set. For all $n, m \in \mathbb{N}$, $x \in X^n$ and $y \in X^m$ we have $\ell(x) \subseteq \ell(y)$ if $x \lhd^* y$ or $y \rhd^* x$.*

Proof. Suppose that $x \lhd^* y$ or $y \rhd^* x$. Then there exist a $\nu \in \{0,1\}$ and a vector A such that $x = \delta^m_{\nu,A}(y)$. We have

$$\ell(y) = \left(\left(\mathtt{ev}^1 \circ \delta^m_{\nu,A^m_1}(y) \dashrightarrow \dots \dashrightarrow \mathtt{ev}^1 \circ \delta^m_{\nu,A^m_m}(y)\right), \dashrightarrow_y, \lambda_{\mathtt{ev}}\right)$$

$$\ell(x) = \left(\left(\mathtt{ev}^1 \circ \delta^n_{\nu,A^n_1}(x) \dashrightarrow \dots \dashrightarrow \mathtt{ev}^1 \circ \delta^n_{\nu,A^n_n}(x)\right), \dashrightarrow_x, \lambda_{\mathtt{ev}}\right)$$

Every element in $\left(\mathtt{ev}^1 \circ \delta^n_{\nu,A^n_1}(x) \dashrightarrow \dots \dashrightarrow \mathtt{ev}^1 \circ \delta^n_{\nu,A^n_n}(x)\right)$ is by definition unique. Since $x = \delta^m_{\nu,A}(y)$ for all $e \in \mathtt{EV}(x)$ we have $e \in \mathtt{EV}(y)$ (replace x in $\ell(x)$ with $\delta^m_{\nu,A}(y)$ and note that all $(m-1)$-dimensional vectors A are of the form A^m_a). Suppose that we have $s, t \in \mathbb{N}$, $1 \leq s < t \leq n$. Then we have

$$\mathtt{ev}^1 \circ \delta^n_{\nu,A^n_s} \circ \delta^m_{\nu,A}(y) \dashrightarrow \mathtt{ev}^1 \circ \delta^n_{\nu,A^n_t} \circ \delta^m_{\nu,A}(y)$$

Let $A = (a_1, \dots, a_l)$. Then we get

$$\mathtt{ev}^1 \circ \delta^n_{\nu,A^n_s} \circ \delta^m_{\nu,A}(y) = \mathtt{ev}^1 \circ \delta^n_{\nu,A^n_s} \circ \delta^{n+1}_{\nu,a_1} \circ \dots \circ \delta^m_{\nu,a_l}(y)$$

Using theorem A.6 we get

$$\delta^n_{\nu,A^n_s} \circ \delta^{n+1}_{\nu,a_1} = \begin{cases} \delta^{n+1}_{\nu,A^{n+1}_s} & \text{for all } a_1 > s \\ \delta^{n+1}_{\nu,A^{n+1}_{s+1}} & \text{for all } a_1 \leq s \end{cases}$$

If $a_1 > s$, then we have $a_i > s$ for all $2 \leq i \leq l$. If $a_1 \leq s$ then we need to compare a_2 with $s+1$ instead. Let r_s be the smallest integer $1 \leq r_s \leq l$ such that $a_{r_s} > s + r_s - 1$. Then we get

$$\mathtt{ev}^1 \circ \delta^n_{\nu,A^n_s} \circ \delta^{n+1}_{\nu,a_1} \circ \dots \circ \delta^m_{\nu,a_l}(y) = \delta^m_{\nu,A^m_{s+r_s-1}}(y)$$

Let r_t be defined analogously. Since for all $1 \leq i \leq l$ if $a_i > t + i - 1$ we have $a_i > s + i - 1$, since $s < t$. Therefore $r_s \leq r_t$, which gives us $s + r_s - 1 < t + r_t - 1$ and therefore

$$\mathtt{ev}^1 \circ \delta^n_{\nu,A^n_s} \circ \delta^m_{\nu,A}(y) = \mathtt{ev}^1 \circ \delta^m_{\nu,A^m_{s+r_s}}(y)$$

$$\dashrightarrow_y \mathtt{ev}^1 \circ \delta^m_{\nu,A^m_{t+r_t}}(y) = \mathtt{ev}^1 \circ \delta^n_{\nu,A^n_t} \circ \delta^m_{\nu,A}(y)$$

This shows that $\ell(x) \subseteq \ell(y)$ as pomsets, since $\mathtt{EV}(x) \subseteq \mathtt{EV}(y)$ as posets and since $\lambda_{\mathtt{ev}}$ is defined the same on both $\ell(x)$ and $\ell(y)$. $\square$

To understand how the labeling works, let's look at an example:

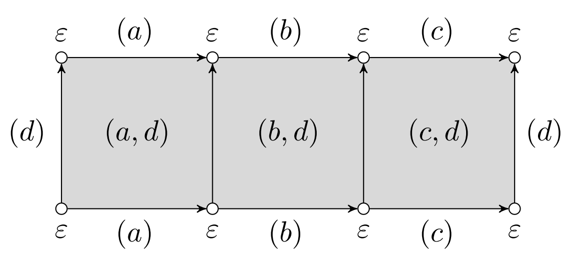


Figure 12: A 2-dimensional rectangular HDA with labels for the elements. The two unlabeled vertical arrows have the labels (d).

Here we have used the vector notation for the labels. The vertical arrows are all labeled (d), and the pairs of parallel horizontal arrows are labeled (a), (b) and (c). The labels of the three 2-dimensional elements are then determined by the labels of their adjacent arrows, giving us (a, d), (b, d) and (c, d). The nodes all have the empty labeling $\varepsilon = ()$. This figure also intuitively explains why theorem 6.27 is true.
For an intuitive explanation for HDA with dimension greater than 2 we can use the idea of direction as introduced in subsection 2.1 again. If $x \in X^n$ is a higher-dimensional element then $\mathtt{EV}(x)$ is a linear poset containing the equivalence classes of the elements of X^1 that can be reached by x in canonical order. Applying the unique labelling function $\lambda_{\mathtt{ev}}$ as defined in theorem 6.25 on each of these equivalence classes then gives us the labelling of x.
For the next definition we will make use of the theorems and definitions in section 5 to define labelling on tracks. We will add an intuitive explanation with some examples after the definition.

Theorem 6.28. *Let (X, λ) be a labelled precubical set, let $n \in \mathbb{N}$ and $x \in X^n$. Then there exists a unique bijection $\xi : \mathtt{EV}(x) \to [0, n-1]$ such that the for all $a, b \in \mathtt{EV}(x)$ we have*

$$a \dashrightarrow b \iff \xi(a) < \xi(b)$$

note that if $n = 0$ then we define $[0, n-1] = [0, -1] = \emptyset$.

Proof. This follows from the fact that $\mathtt{EV}(x)$ and $[0, n-1]$ are both linear posets of the same size, since $|\mathtt{EV}(x)| = n$ by definition. If $\mathtt{EV}(x) = \emptyset$ then the bijection is the identity on $\emptyset$. $\square$

Definition 6.29. *Let (X, λ) be a labelled precubical set. We define the label $\ell(\rho)$ of a track $\rho = (x_1, ..., x_m,)$ as follows:*

- *If we have $m = 1$ and therefore $\rho = (x_1)$ we define $\ell(\rho)$ as the following: Let $n = |\mathtt{EV}(x_1)|$ and let $\xi : \mathtt{EV}(x_1) \to [0, n-1]$ be the bijection as defined in theorem 6.28. Then we have*

$$\ell(\rho) = (\xi \circ \mathtt{EV}(x_1), \emptyset, \dashrightarrow, \lambda, \xi \circ \mathtt{EV}(x_1), \xi \circ \mathtt{EV}(x_1))$$

  where the relation $\dashrightarrow$ is equal to the $<$-relation on $\mathbb{N}$. For all $a \in [0, n-1]$ we define $\lambda(a) = \lambda_{\mathtt{ev}} \circ \xi^{-1}(a)$.

- *If we have $m = 2$ and therefore $\rho = (x_1, x_2)$ we define $\ell(\rho)$ as the following: If $x_1 \rhd^* x_2$ we define $a = 1$ and if $x_1 \lhd^* x_2$ we define $a = 2$. Let $n = |\mathtt{EV}(x_a)|$ and let $\xi : \mathtt{EV}(x_a) \to [0, n-1]$ be the bijection as defined in theorem 6.28. Then we have*

$$\ell(\rho) = (\xi \circ \mathtt{EV}(x_a), \emptyset, \dashrightarrow, \lambda, \xi \circ \mathtt{EV}(x_1), \xi \circ \mathtt{EV}(x_2))$$

  where the relation $\dashrightarrow$ is equal to the $<$-relation on $\mathbb{N}$. For all $b \in [0, n-1]$ we define $\lambda(b) = \lambda_{\mathtt{ev}} \circ \xi^{-1}(b)$. It's important to note here that as a result of theorem 6.27 we have $\ell(x_1) \subseteq \ell(x_a) \supseteq \ell(x_2)$, which means that $\lambda_{\mathtt{ev}}$ works the same on $\mathtt{EV}(x_1)$ and $\mathtt{EV}(x_2)$.

- *If we have $m \geq 3$ we can split ρ as in theorem 6.6. Then we define*

$$\ell(\rho) = \ell((x_1, ..., x_m,)) = \ell((x_1, x_2) * ... * (x_{m-1}, x_m))$$
$$= \ell((x_1, x_2)) * ... * \ell((x_{m-1}, x_m))$$

  It's important to note here that $\xi_1 \circ \mathtt{EV}(x_t) \cong \xi_2 \circ \mathtt{EV}(x_t)$ for bijections ξ_1 and ξ_2 as in theorem 6.28.

Theorem 6.30. *Let $\mathcal{X}$ and $\mathcal{Y}$ be HDA and let $f : \mathcal{X} \to \mathcal{Y}$ be a HDA map. For all $x \in X^n$, $n \in \mathbb{N}$ we have $\ell(x) \cong \ell \circ f^n(x)$.*

Proof. Let $\mathcal{X} = (X, I_X, F_X, \lambda_X)$ and $\mathcal{Y} = (Y, I_Y, F_Y, \lambda_Y)$. We have

$$\ell(x) = (\mathtt{EV}(x), \dashrightarrow_x, \lambda_{\mathtt{ev},X})$$

$$\ell \circ f^n(x) = (\mathtt{EV} \circ f^n(x), \dashrightarrow_y, \lambda_{\mathtt{ev},Y})$$

with

$$\mathtt{EV}(x) = \left(\mathtt{ev}_X^1 \circ \delta^n_{\nu,A_1^n}(x) \dashrightarrow \mathtt{ev}_X^1 \circ \delta^n_{\nu,A_2^n}(x) \dashrightarrow \ldots \dashrightarrow \mathtt{ev}_X^1 \circ \delta^n_{\nu,A_n^n}(x)\right)$$

for any $\nu \in \{0,1\}$. Here we will simply refer to $\delta^n_{\nu,A_t^n}(x)$ as x_t for all $1 \leq t \leq n$. Since we have $\delta^n_{\nu,A_t^n} \circ f^n(x) = f^1 \circ \delta^n_{\nu,A_t^n}(x)$ this gives us

$$\mathtt{EV}(x) = \left(\mathtt{ev}_X^1\left(x_1\right) \dashrightarrow_x \mathtt{ev}_X^1\left(x_2\right) \dashrightarrow_x \ldots \dashrightarrow_x \mathtt{ev}_X^1\left(x_n\right)\right)$$

$$\mathtt{EV} \circ f^n(x) = \left(\mathtt{ev}_Y^1 \circ f^1\left(x_1\right) \dashrightarrow_y \mathtt{ev}_Y^1 \circ f^1\left(x_2\right) \dashrightarrow_y \ldots \dashrightarrow_y \mathtt{ev}_Y^1 \circ f^1\left(x_n\right)\right)$$

Since $\mathtt{EV}(x)$ and $\mathtt{EV} \circ f^n(x)$ are both linear posets of the same size there exists a unique bijection $g : \mathtt{EV}(x) \to \mathtt{EV} \circ f^n(x)$ with $g \circ \mathtt{ev}_X^1\left(x_t\right) = \mathtt{ev}_Y^1 \circ f^1\left(x_t\right)$ for all $1 \leq t \leq n$ that preserves the $\dashrightarrow_x$ relation as $\dashrightarrow_y$.
Recall from theorem 6.25 that $\lambda_{\mathtt{ev},X} : !!E_X \to !\Sigma$ is the unique precubical map such that $\lambda_X = \lambda_{\mathtt{ev},X} \circ \mathtt{ev}_X$ with $\mathtt{ev}_X : X \to !!E_X$. Similarly we also have $\lambda_Y = \lambda_{\mathtt{ev},Y} \circ \mathtt{ev}_Y$ with $\mathtt{ev}_Y : Y \to !!E_Y$. Therefore for all $1 \leq t \leq n$ we have

$$\lambda^1_{\mathtt{ev},Y} \circ \mathtt{ev}_Y^1 \circ f^1\left(x_t\right) = \lambda_Y^1 \circ f^1\left(x_t\right) = \lambda_X^1\left(x_t\right) = \lambda^1_{\mathtt{ev},X} \circ \mathtt{ev}_X^1\left(x_t\right)$$

which shows that $g : \mathtt{EV}(x) \to \mathtt{EV} \circ f^n(x)$ preserves the labelling function as well. Therefore this makes $g : \ell(x) \to \ell \circ f^n(x)$ a pomset isomorphism which means that we have $\ell(x) \cong \ell \circ f^n(x)$. $\square$

Theorem 6.31. *Let $\mathcal{X}$ and $\mathcal{Y}$ be HDA, let $f : \mathcal{X} \to \mathcal{Y}$ be a HDA map and let ρ be a track in $\mathcal{X}$. Then we have $\ell(\rho) = \ell \circ f(\rho)$.*

Proof. We have

$$\rho = (x_1, x_2, \ldots, x_m) = (x_1, x_2) * (x_2, x_3) * \ldots * (x_{m-1}, x_m)$$

for a certain $m \in \mathbb{N}$, $m \geq 1$. From theorem 6.9 it follows that

$$f\left(\rho\right) = \left(f\left(x_1\right), f\left(x_2\right), \ldots, f\left(x_m\right)\right)$$

$$= \left(f\left(x_1\right), f\left(x_2\right)\right) * \left(f\left(x_2\right), f\left(x_3\right)\right) * \ldots * \left(f\left(x_{m-1}\right), f\left(x_m\right)\right)$$

For ease of notation we define

$$f(\rho) = \tau = (y_1, y_2, \ldots, y_m) = (y_1, y_2) * (y_2, y_3) * \ldots * (y_{m-1}, y_m)$$

with $y_t = f\left(x_t\right)$ for all $1 \leq t \leq m$. For all $t \in \mathbb{N}$, $1 \leq t \leq m-1$ we have

$$\ell\left(x_t, x_{t+1}\right) = \left(\xi_a \circ \mathtt{EV}\left(x_a\right), \emptyset, \dashrightarrow_X, \lambda_X, \xi_a \circ \mathtt{EV}\left(x_t\right), \xi_a \circ \mathtt{EV}\left(x_{t+1}\right)\right)$$

$$\ell\left(y_t, y_{t+1}\right)=\left(\zeta_a \circ \mathtt{EV}\left(y_a\right), \emptyset, \dashrightarrow_Y, \lambda_Y, \zeta_a \circ \mathtt{EV}\left(y_t\right), \zeta_a \circ \mathtt{EV}\left(y_{t+1}\right)\right)$$

for a certain $a \in \{t, t+1\}$ and certain bijections $\xi_a : \mathtt{EV}\left(x_a\right) \to [0, n_a - 1]$ and $\zeta_a : \mathtt{EV}\left(y_a\right) \to [0, n_a - 1]$ with $n_a = \left|\mathtt{EV}\left(x_a\right)\right| = \left|\mathtt{EV}\left(y_a\right)\right|$ as defined in theorem 6.28. Note that as a result of theorem 6.9 we have $x_t \lhd^* x_{t+1} \iff y_t \lhd^* y_{t+1}$ and $x_t \rhd^* x_{t+1} \iff y_t \rhd^* y_{t+1}$.
Theorem 6.30 gives us the bijection $g : \mathtt{EV}\left(x_a\right) \to \mathtt{EV}\left(y_a\right)$ which preserves the $\dashrightarrow$-relation and the labelling function. This makes $\zeta_z \circ g : \mathtt{EV}\left(x_a\right) \to [0, n-1]$ a bijection that preserves the $\dashrightarrow$-relation. Theorem 5.9 then gives us that $\zeta_a \circ g = \xi_a$, which means that we have $\ell\left(x_t, x_{t+1}\right) = \ell\left(y_t, y_{t+1}\right)$.
Since therefore for all $1 \leq t \leq n-1$ we have $\ell\left(x_t, x_{t+1}\right) = \ell\left(y_t, y_{t+1}\right)$ this also gives us $\ell(\rho) = \ell(\tau) = \ell \circ f(\rho)$, which proves the theorem. $\square$

7 Languages of Higher-Dimensional Automata

In this section we introduce the languages of higher-dimensional automata and their maps. After that we show that coproducts and colimits of HDA are equivalent with the union of the languages of the HDA in the diagrams.
Throughout this section the set of labels Σ is fixed. We can only describe the relation between languages that use the same labelling set Σ.

7.1 HDA languages

Definition 7.1. *Let $\mathcal{X}$ be a HDA. We define the language of $\mathcal{X}$ as*

$$L\left(\mathcal{X}\right) = \{\ell(\rho) \mid \rho \text{ is an accepting track in } \mathcal{X}\}$$

We then define the maps between languages as inclusion maps.

Definition 7.2. *Let L_1 and L_2 be languages. If we have $L_1 \subseteq L_2$ then we define the unique language map $F : L_1 \to L_2$ as the inclusion map such that for all $P \in L\left(\mathcal{X}\right)$ we have $P = F(P)$.*

Remark 7.2.1. *From the reflexivity and transitivity of the inclusion relation $\subseteq$ we get that we have all identity language maps and compositions of language maps.*

Definition 7.3. *We define `HLang` as the category of languages generated by HDA with the morphisms being language maps.*

One can also define a subcategory of `Lang` of which the objects are languages that are generated by finite HDA. The category of languages of HDA is part of a broader category of interval ipomset languages, which are covered by the paper [FJSZ21] (we describe interval ipomsets in appendix B). While we won't cover these languages specifically, it is important to consider them when discussing colimits of languages.

Definition 7.4. *The category `Lang` is the category of interval ipomset languages, where the morphisms send interval ipomsets in the source language to the same ipomsets in the target language.*

By definition we have that `HLang` is a full subcategory of `Lang`.

Theorem 7.5. *Let $\mathcal{X}$ and $\mathcal{Y}$ be HDA and let $f : \mathcal{X} \to \mathcal{Y}$ be a HDA map. Then we have $L(\mathcal{X}) \subseteq L(\mathcal{Y})$.*
Let $F : L(\mathcal{X}) \to L(\mathcal{Y})$ be the unique language map as in definition 7.2. Then for every accepting track ρ in $\mathcal{X}$ we have $F \circ \ell(\rho) = \ell \circ f(\rho)$.

Proof. Theorem 6.31 gives us that for every accepting track ρ in $\mathcal{X}$ we have $\ell(\rho) = \ell \circ f(\rho)$. Since $f : \mathcal{X} \to \mathcal{Y}$ is a HDA map it follows from theorem 6.11 that $f(\rho)$ is an accepting track in $\mathcal{Y}$. Since every element $P \in L(\mathcal{X})$ is by definition generated by an accepting track in $\mathcal{X}$ this gives us that $L(\mathcal{X}) \subseteq L(\mathcal{Y})$.
Let $F : L(\mathcal{X}) \to L(\mathcal{Y})$ be the unique language map as in definition 7.2. For every accepting track ρ in $\mathcal{X}$ we have $F \circ \ell(\rho) = \ell(\rho) = \ell \circ f(\rho)$. ☐

Theorem 7.6. *The language operator L is a functor $L : \mathtt{HDA} \to \mathtt{Lang}$ that sends HDA to their languages and HDA maps to the respective language maps.*

Proof. This follows from definition 7.1, definition 7.2 and theorem 7.5. ☐

7.2 Colimits of languages of HDA

In this subsection we will cover the diagrams of languages of HDA and their colimits and describe the connection to diagrams of HDA.

Theorem 7.7. *Let $L : J \to \mathtt{Lang}$ be a small diagram of languages. For every $i \in J$ define the language map $\Theta_i : L_i \to \bigcup_{i \in J} L_i$ as the inclusion. Then $\left(\bigcup_{i \in J} L_i, \Theta\right)$ is the colimit of the diagram L.*

Proof. Note that $\bigcup_{i \in J} L_i$ is defined if and only if $\mathrm{obj}(J)$ is a set (which is true if J is small).
For all $i \in J$ the maps $\Theta_i : L_i \to \bigcup_{i \in J} L_i$ clearly exist as defined in definition 7.2. Let $i, j \in J$ be such that there exists a morphism $f : i \to j$. Then there exists a language map $L_f : L_i \to L_j$. By definition for all $P \in L_i$ we have $P = L_f(P)$ and therefore $\Theta_i(P) = P = L_f(P) = \Theta_j \circ L_f(P)$. This makes $\left(\bigcup_{i \in J} L_i, \Theta\right)$ a co-cone of L.
Suppose that (L_N, Ψ) is a co-cone of L. Then for all $i \in J$ we have to have $L_i \subseteq L_N$ and therefore $\bigcup_{i \in J} L_i \subseteq L_N$. Then definition 7.2 gives us that there exists a unique language map $F : \bigcup_{i \in J} L_i \to L_N$. Since by definition for all $P \in \bigcup_{i \in J} L_i$ we have $P = F(P)$ and therefore $F \circ \Theta_i(P) = F(P) = P = \Psi_i(P)$ this gives us that F satisfies the requirements for a unique language map with $F \circ \Theta_i = \Psi_i$ for all $i \in J$.
This proves that $\left(\bigcup_{i \in J} L_i, \Theta\right)$ is a colimit of the diagram L. ☐

Theorem 7.8. *Let $\mathcal{X} : J \to \mathtt{HDA}$ be a small diagram of HDA. Then $L(\mathcal{X}) : J \to \mathtt{Lang}$ is a small diagram of languages. If $\mathcal{X}$ is a discrete or filtered diagram then $L(\mathcal{X})$ is discrete or filtered as well.*

Proof. Theorem 7.5 gives us that for every $i, j \in J$ such that there exists a morphism $f : i \to j$ the HDA map $\mathcal{X}_f : \mathcal{X}_i \to \mathcal{X}_j$ generates a unique language map $L_f : L(\mathcal{X}_i) \to L(\mathcal{X}_j)$. Due to the uniqueness of language maps if there exists a $k \in J$ and a morphism $g : j \to k$ then $L_g : L(\mathcal{X}_j) \to L(\mathcal{X}_k)$ and $L_{g \circ f} : L(\mathcal{X}_i) \to L(\mathcal{X}_k)$ are language maps such that $L_g \circ L_f = L_{g \circ f}$. Therefore $L(\mathcal{X}) : J \to \mathtt{Lang}$ is a small diagram of languages.
In the case that $\mathcal{X}$ is a discrete or filtered diagram we have that J is a discrete or filtered category which automatically gives us that $L(\mathcal{X})$ is a discrete or filtered diagram. ☐

Theorem 7.9. *Let $\mathcal{X} : J \to \texttt{HDA}$ be a small diagram of HDA and let $(\mathcal{N}, \psi)$ be a co-cone of $\mathcal{X}$. For all $i \in J$ define $\Psi_i : L(\mathcal{X}_i) \to L(\mathcal{N})$ such that for every accepting track ρ in $\mathcal{X}_i$ we have $\Psi_i \circ \ell(\rho) = \ell \circ \phi_i(\rho)$. Then $(L(\mathcal{N}), \Psi)$ is a co-cone of the diagram $L(\mathcal{X})$ and we have $\bigcup_{i\in J} L(\mathcal{X}_i) \subseteq L(\mathcal{N})$.*

Proof. Suppose that we have $i, j \in J$ with a morphism $f : i \to j$. Then we have the language maps $\Psi_i : L(\mathcal{X}_i) \to L(\mathcal{N})$, $\Psi_j : L(\mathcal{X}_j) \to L(\mathcal{N})$ and $L_f : L(\mathcal{X}_i) \to L(\mathcal{X}_j)$. Remark 7.2.1 gives us that $\Psi_j \circ L_f$ is also a language map and definition 7.2 then gives us that $\Psi_i = \Psi_j \circ L_f$. This then gives us that $(L(\mathcal{N}), \Psi)$ is a co-cone of the diagram $L(\mathcal{X})$ and since $\left(\bigcup_{i\in J} L(\mathcal{X}_i), \Theta\right)$ is a colimit of $L(\mathcal{X})$ there exists a unique language map $\bigcup_{i\in J} L(\mathcal{X}_i) \to L(\mathcal{N})$. Definition 7.2 then gives us that $\bigcup_{i\in J} L(\mathcal{X}_i) \subseteq L(\mathcal{N})$. □

Theorem 7.10. *Let $\mathcal{X} : J \to \texttt{HDA}$ be a small discrete diagram of HDA and let $(\mathcal{N}, \psi)$ be a coproduct of $\mathcal{X}$. Then $(L(\mathcal{L}), \Phi)$ is a coproduct of $L(\mathcal{X})$ with $L(\mathcal{L}) = \bigcup_{i\in J} L(\mathcal{X}_i)$.*

Proof. Theorem 7.9 gives us that $(L(\mathcal{L}), \Phi)$ is a co-cone of $L(\mathcal{X})$ and that we have $\bigcup_{i\in J} L(\mathcal{X}_i) \subseteq L(\mathcal{L})$. From theorem 6.13 it follows that for every accepting track ρ in $\mathcal{L}$ there exists a $i \in J$ and an accepting track τ in $\mathcal{X}_i$ such that $\phi_i(\tau) = \rho$. Since by definition every element of $L(\mathcal{L})$ is the labelling of an accepting track of $\mathcal{L}$ and since theorem 6.31 and theorem 7.5 give us that $\Phi_i \circ \ell(\tau) = \ell \circ \phi_i(\tau) = \ell(\rho)$ this means that for every $P \in L(\mathcal{L})$ there exists a $i \in J$ and a $Q \in L(\mathcal{X}_i)$ such that $\Phi_i(Q) = P$. This gives us $L(\mathcal{L}) = \bigcup_{i\in J} L(\mathcal{X}_i)$. Since definition 7.2 gives us that the language map $L(\mathcal{X}_i) \to \bigcup_{i\in J} L(\mathcal{X}_i)$ is unique this gives us $\Phi_i = \Theta_i$ for all $i \in J$ and therefore $(L(\mathcal{L}), \Phi)$ □

Theorem 7.11. *Let $\mathcal{X} : J \to \texttt{HDA}$ be a small filtered diagram of HDA and let $(\mathcal{N}, \psi)$ be a filtered colimit of $\mathcal{X}$. Then $(L(\mathcal{L}), \Phi)$ is a filtered colimit of $L(\mathcal{X})$ with $L(\mathcal{L}) = \bigcup_{i\in J} L(\mathcal{X}_i)$.*

Proof. This proof is analogous to the proof of theorem 7.10 with the reference to theorem 6.13 replaced with theorem 6.16. □

Theorem 7.12. *There exists a small diagram of HDA $\mathcal{X} : J \to \texttt{HDA}$ with the colimit $(\mathcal{L}, \phi)$ such that $(L(\mathcal{L}), \Phi)$ is not a colimit of $L(\mathcal{X})$ and $L(\mathcal{L}) \supsetneq \bigcup_{i\in J} L(\mathcal{X}_i)$.*

Proof. This follows from theorem 6.17. Note that in the example given the languages $L(\mathcal{X}_1)$, $L(\mathcal{X}_2)$ and $L(\mathcal{X}_3)$ are all empty since each of the HDA $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$ has no initial or final cells. Since $\rho = \left(\delta^1_{0,1}(y_1), y_1, y_2, y_3, \delta^1_{1,1}(y_3)\right)$ is an accepting track in $\mathcal{L}$ the language $L(\mathcal{L})$ is not empty which means that we have $L(\mathcal{L}) \neq \bigcup_{i\in J} L(\mathcal{X}_i)$. □

Theorem 7.13. *Every HDA language is the union of languages of finite HDA.*

Proof. This follows from theorem 4.20 and theorem 7.11. □

For the proof of the theorem above one could also show that for every interval ipomset there exists a finite HDA that generates it. However with this approach the original HDA might not be the colimit of the diagram of finite HDA that generates its language.

Theorem 7.14. *The functor $L : \texttt{HDA} \to \texttt{Lang}$ preserves small coproducts and small filtered colimits.*

Proof. This follows from theorem 7.10 and theorem 7.11. □

8 Tensor Product

We have defined the higher-dimensional automata and their languages. Similarly to ordinary automata there are a few operations that can be applied to the HDA and the HDA languages. In the previous sections we have seen the coproduct, with its equivalent on languages being the union. In this section we will cover the tensor product, equivalently the parallel composition on languages. In essence this operation will represent executing two or more HDA in parallel.

8.1 Tensor product definition

First we define the tensor product on precubical sets. To avoid confusion we will sometimes denote the face maps on a precubical set X with δ_X.

Definition 8.1. *Let X and Y be precubical sets. We define tensor product $X \otimes Y = Z$ as the family of sets $Z = (Z^n)_{n\in\mathbb{N}}$ with*

$$Z^n = \bigsqcup_{k+l=n} X^k \times Y^l$$

For all $k, l, n \in \mathbb{N}$ with $k + l = n$ and for all $x \in X^k$, $y \in Y^l$ we define the face maps on Z as

$$\delta^n_{\nu,a}((x,y)) = \begin{cases} \left((\delta_X)^k_{\nu,a}(x), y\right) & \text{if } a \leq k \\ \left(x, (\delta_Y)^l_{\nu,a-k}(y)\right) & \text{if } a > k \end{cases}$$

for all $\nu \in \{0,1\}$ and $a \in \mathbb{N}$, $1 \leq a \leq n$.

Theorem 8.2. *Let X and Y be precubical sets. Then $X \otimes Y$ is a precubical set as well.*

Proof. Let $Z = X \otimes Y$ and recall definition 2.2. For all $k, l \in \mathbb{N}$ with $k + l = n$ we have that X^k and Y^l are sets and therefore $X^k \times Y^l$ and $\bigsqcup_{k+l=n} X^k \times Y^l = Z^n$ are sets as well.
We now want to prove that for all $n \in \mathbb{N}$, $z \in Z^n$, $\nu, \mu \in \{0,1\}$ and $a, b \in \mathbb{N}$ with $1 \leq a < b \leq n$ we have

$$\delta^{n-1}_{\nu,a} \circ \delta^n_{\mu,b}(z) = \delta^{n-1}_{\mu,b-1} \circ \delta^n_{\nu,a}(z)$$

Let $z \in Z^n$ for a certain $n \in \mathbb{N}$. By definition there exist unique $x \in X^k$ and $y \in Y^l$ for certain $k, l \in \mathbb{N}$, $k + l = n$ such that $z = (x, y)$. Suppose that we have $a < b \leq k$. Then we have

$$\delta^{n-1}_{\nu,a} \circ \delta^n_{\mu,b}((x,y)) = \delta^{n-1}_{\nu,a}\left((\delta_X)^k_{\mu,b}(x), y\right) = \left((\delta_X)^{k-1}_{\nu,a} \circ (\delta_X)^k_{\mu,b}(x), y\right)$$

because $a < b \leq k$ implies that $a \leq k - 1$. This gives us

$$\left((\delta_X)^{k-1}_{\nu,a} \circ (\delta_X)^k_{\mu,b}(x), y\right) = \left((\delta_X)^{k-1}_{\mu,b-1} \circ (\delta_X)^k_{\nu,a}(x), y\right)$$

We have

$$\delta^{n-1}_{\mu,b-1} \circ \delta^n_{\nu,a}((x,y)) = \left((\delta_X)^{k-1}_{\mu,b-1} \circ (\delta_X)^k_{\nu,a}(x), y\right)$$

because $a \leq k$ and $b - 1 \leq k - 1$. This proves the statement for $a < b \leq k$.

Suppose that $a \leq k < b$. Then we have

$$\delta_{\nu,a}^{n-1} \circ \delta_{\mu,b}^{n}\left((x,y)\right) = \delta_{\nu,a}^{n-1}\left(x, (\delta_Y)_{\mu,b-k}^{l}(y)\right) = \left((\delta_X)_{\nu,a}^{k}(x), (\delta_Y)_{\mu,b-k}^{l}(y)\right)$$

Here the first step is because $k < b$, and since this only reduces the dimension of the Y part by 1 (and not the X part, we get $k + (l-1) = n-1$) we still have $a \leq k$ which gives us the second step.

$$\delta_{\mu,b-1}^{n-1} \circ \delta_{\nu,a}^{n}\left((x,y)\right) = \delta_{\mu,b-1}^{n-1}\left((\delta_X)_{\mu,a}^{k}(x), y\right) = \left((\delta_X)_{\nu,a}^{k}(x), (\delta_Y)_{\mu,b-1-(k-1)}^{l}(y)\right)$$

$$= \left((\delta_X)_{\nu,a}^{k}(x), (\delta_Y)_{\mu,b-k}^{l}(y)\right) = \delta_{\mu,b-1}^{n-1} \circ \delta_{\nu,a}^{n}\left((x,y)\right)$$

which proves the statement.
Suppose that $k < a < b$. This gives us

$$\delta_{\nu,a}^{n-1} \circ \delta_{\mu,b}^{n}\left((x,y)\right) = \delta_{\nu,a}^{n-1}\left(x, (\delta_Y)_{\mu,b-k}^{l}(y)\right) = \left(x, (\delta_Y)_{\nu,a-k}^{l-1} \circ (\delta_Y)_{\mu,b-k}^{l}(y)\right)$$

and

$$\delta_{\mu,b-1}^{n-1} \circ \delta_{\nu,a}^{n}\left((x,y)\right) = \delta_{\mu,b-1}^{n-1}\left(x, (\delta_Y)_{\nu,a-k}^{l}(y)\right)$$

$$= \left(x, (\delta_Y)_{\mu,b-1-k}^{l-1} \circ (\delta_Y)_{\nu,a-k}^{l}(y)\right) = \left(x, (\delta_Y)_{\nu,a-k}^{l-1} \circ (\delta_Y)_{\mu,b-k}^{l}(y)\right)$$

because Y is a precubical set. This proves the statement for $k < a < b$, and therefore the statement is true for all $a, b \in \mathbb{N}$ with $1 \leq a < b \leq n$, which makes $Z = X \otimes Y$ a precubical set. ☐

Theorem 8.3. *Let X and Y be event consistent precubical sets. Then $X \otimes Y$ is an event consistent precubical set as well.*

Proof. Theorem 8.2 gives us that $X \otimes Y$ is a precubical set.
We define $\equiv_X$ and $\equiv_Y$ as the event relations relations on X^1 and Y^1 as defined in definition 3.2.
We define $\equiv$ on Z^1 as the transitive closure of

$$\left\{\left(\delta_{\nu,a}^{2}(z), \delta_{\mu,a}^{2}(z)\right) \mid z \in Z^2,\ \nu, \mu \in \{0,1\},\ a \in \{1,2\}\right\}$$

Note that every element of Z^2 is an element of $X^2 \times Y^0$, $X^1 \times Y^1$ or $X^0 \times Y^2$. This gives us

$$\delta_{\nu,a}^{2}(z) = \delta_{\nu,a}^{2}\left((x,y)\right) = \begin{cases} \left(\delta_{\nu,a}^{2}(x), y\right) & \text{if} \quad (x,y) \in X^2 \times Y^0 \\ \left(\delta_{\nu,1}^{1}(x), y\right) & \text{if} \quad (x,y) \in X^1 \times Y^1 \quad \text{and} \quad a = 1 \\ \left(x, \delta_{\nu,1}^{1}(y)\right) & \text{if} \quad (x,y) \in X^1 \times Y^1 \quad \text{and} \quad a = 2 \\ \left(x, \delta_{\nu,a}^{2}(y)\right) & \text{if} \quad (x,y) \in X^0 \times Y^2 \end{cases}$$

for all $z \in Z^2$, $z = (x,y)$, $a \in \{1,2\}$ and $\nu \in \{0,1\}$.
It's important to note here that $\nu \in \{0,1\}$ does not influence which of the four cases above $\delta_{\nu,a}^{2}(z)$ falls under. Therefore in the pairs $\left(\delta_{\nu,a}^{2}(z), \delta_{\mu,a}^{2}(z)\right)$ both elements will always fall under the same case, meaning that both are elements of either $X^1 \times Y^0$ or $X^0 \times Y^1$. This means that the same is true for elements of the transitive closure as well.
A result of this is that for all $z \in Z^2$, $z = (x,y)$, $a \in \{1,2\}$ and $\nu, \mu \in \{0,1\}$ the elements $\delta_{\nu,a}^{2}(z) = (x_\nu, y_\nu)$ and $\delta_{\mu,a}^{2}(z) = (x_\mu, y_\mu)$ we must have $x_\nu = x_\mu = x$ or $y_\nu = y_\mu = y$. The same is

therefore true for elements of the transitive closure $\equiv$. This then gives us that for all $z_1, z_2 \in Z^1$ with $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ we have $z_1 \equiv z_2$ if and only if $x_1 = x_2$ and $y_1 \equiv_Y y_2$ or $x_1 \equiv_X x_2$ and $y_1 = y_2$. Note that in the first case we have $(x_1, y_1), (x_2, y_2) \in X^0 \times Y^1$ and in the second case we have $(x_1, y_1), (x_2, y_2) \in X^1 \times Y^0$.
Recall definition 3.1. Suppose that there exist $z \in Z^2$, $z = (x, y)$, $\nu, \mu \in \{0, 1\}$ and $a, b \in \{1, 2\}$ with $a \neq b$ such that $\delta^2_{\nu,a}(z) \equiv \delta^2_{\mu,b}(z)$. Let $\delta^2_{\nu,a}(z) = (x_1, y_1)$ and $\delta^2_{\mu,b}(z) = (x_2, y_2)$.
The first thing we want to do is figure out which of the four cases $\delta^2_{\nu,a}(z)$ and $\delta^2_{\mu,b}(z)$ can fall into. If $(x, y) \in X^2 \times Y^0$ or $(x, y) \in X^0 \times Y^2$ then the both fall under the same case, which can be the first or fourth. In the first case we have $y_1 = y_2 = y$ which means that $x_1 \equiv_X x_2$. However since $x_1 = \delta^2_{\nu,a}(x)$ and $x_2 = \delta^2_{\mu,b}(x)$ with $a \neq b$ this is in contradiction with $\equiv_X$ being the event relation on X and X being event consistent. Analogously the fourth case results in a contradiction as well, since it would require $\delta^2_{\nu,a}(y) \equiv_Y \delta^2_{\mu,b}(y)$ with $a \neq b$.
Now suppose that $(x, y) \in X^1 \times Y^1$. Without loss of generality we assume that $a = 1$ and $b = 2$. Then we have $(x_1, y_1) = \left(\delta^1_{\nu,1}(x), y\right)$ and $(x_2, y_2) = \left(x, \delta^1_{\nu,1}(y)\right)$. However we cannot have $\delta^1_{\nu,1}(x) = x$ or $y = \delta^1_{\nu,1}(y)$, which means we cannot have $\left(\delta^1_{\nu,1}(x), y\right) \equiv \left(x, \delta^1_{\nu,1}(y)\right)$.
From our definition of $\equiv$ it is clear that for all $z \in Z^2$, $z = (x, y)$, $\nu, \mu \in \{0, 1\}$ and $a, b \in \{1, 2\}$ if $a = b$ then we have $\delta^2_{\nu,a}(z) \equiv \delta^2_{\mu,b}(z)$ and if $a \neq b$ then we have $\delta^2_{\nu,a}(z) \not\equiv \delta^2_{\mu,b}(z)$. This proves that $\equiv$ is an equivalence relation as in definition 3.1 which makes $X \otimes Y$ an event consistent precubical set. $\square$

Remark 8.3.1. *Due to the way we defined $\equiv$ in the previous theorem it is equal to the event relation as defined in definition 3.2.*

Theorem 8.4. *Let X and Y be precubical sets. Then for all $z \in (X \otimes Y)^n$, $n \in \mathbb{N}$, $n \geq 2$, $z = (x, y)$, $x \in X^k$, $y \in Y^l$, $k + l = n$, $k, l \geq 1$ we have*

$$\delta^n_{\nu,A^n_t}((x, y)) = \begin{cases} \left(\delta^k_{\nu,A^k_t}(x), \delta^l_{\nu,A^l}(y)\right) & \text{if } t \leq k \\ \left(\delta^k_{\nu,A^k}(x), \delta^l_{\nu,A^l_{t-k}}(y)\right) & \text{if } t > k \end{cases}$$

for all $\nu \in \{0, 1\}$ and $1 \leq t \leq n$, where $A^k = (1, 2, ..., k)$ and $A^l = (1, 2, ..., l)$.

Proof. For all $1 \leq t \leq n$ we have

$$\delta^n_{\nu,A^n_t}(z) = \delta^2_{\nu,1} \circ \delta^3_{\nu,2} \circ ... \circ \delta^t_{\nu,t-1} \circ \delta^{t+1}_{\nu,t+1} \circ ... \circ \delta^{n-1}_{\nu,n-1} \circ \delta^n_{\nu,n}(z)$$

with $k + l = n$ and $k, l \geq 1$.
Suppose that $t \leq k$. Then we have

$$\delta^n_{\nu,A^n_t}(z) = \delta^k_{\nu,A^k_t} \circ \delta^{k+1}_{\nu,k+1} \circ ... \circ \delta^{n-1}_{\nu,n-1} \circ \delta^n_{\nu,n}(z)$$

We know that an elementary face map $\delta^n_{\nu,a}$ on an element $z \in Z^n$ with $z = (x, y)$, $x \in X^k$, $y \in Y^l$, $n = k + l$ applies to the y part if $a > k$, or the dimension of the x part. In this case it decreases the dimension of the y part by one, meaning that if $k < b < a$ then $\delta^{n-1}_{\nu,b}$ will also apply to the y part of $\delta^n_{\nu,a}(z)$. Therefore all of the face maps $\delta^{k+1}_{\nu,k+1}, ..., \delta^{n-1}_{\nu,n-1}, \delta^n_{\nu,n}$ apply to the y part which gives us

$$\delta^k_{\nu,A^k_t} \circ \delta^{k+1}_{\nu,k+1} \circ ... \circ \delta^{n-1}_{\nu,n-1} \circ \delta^n_{\nu,n}((x, y)) = \delta^k_{\nu,A^k_t}\left(\left(x, \delta^1_{\nu,1} \circ ... \circ \delta^{l-1}_{\nu,l-1} \circ \delta^l_{\nu,l}(y)\right)\right)$$

$$= \delta^k_{\nu,A^k_t}\left(\left(x, \delta^l_{\nu,A^l}(y)\right)\right) = \left(\delta^k_{\nu,A^k_t}(x), \delta^l_{\nu,A^l}(y)\right)$$

Analogously for $t > k$ we get

$$\delta^n_{\nu,A^n_t}(z) = \delta^{k+1}_{\nu,A^k} \circ ... \circ \delta^t_{\nu,t-1} \circ \delta^{t+1}_{\nu,t+1} \circ ... \circ \delta^{n-1}_{\nu,n-1} \circ \delta^n_{\nu,n}\left((x, y)\right)$$

$$= \delta^{k+1}_{\nu,A^k}\left(\left(x, \delta^{k+2}_{\nu,k+1} \circ ... \circ \delta^{t-k}_{\nu,t-1-k} \circ \delta^{t+1-k}_{\nu,t+1-k} \circ ... \circ \delta^{l-1}_{\nu,l-1} \circ \delta^l_{\nu,l}(y)\right)\right)$$

$$= \delta^{k+1}_{\nu,A^k}\left(\left(x, \delta^l_{\nu,A^l_{t-k}}(y)\right)\right) = \left(\delta^{k+1}_{\nu,A^k}(x), \delta^l_{\nu,A^l_{t-k}}(y)\right)$$

which proves the statement. ☐

Theorem 8.5. *Let (X, λ_X) and (Y, λ_Y) be labelled precubical sets with $\lambda_X : X \to !\Sigma$ and $\lambda_Y : Y \to !\Sigma$. We define the labelling function $\lambda : X \otimes Y \to !\Sigma$ as*

$$\lambda^n\left((x, y)\right) = \begin{array}{l}\Big(\lambda^1_X \circ \delta^k_{\nu,A^k_1}(x), ..., \lambda^1_X \circ \delta^k_{\nu,A^k_k}(x), \\ \lambda^1_Y \circ \delta^l_{\nu,A^l_1}(y), ..., \lambda^1_Y \circ \delta^l_{\nu,A^l_l}(y)\Big)\end{array}$$

for all $(x, y) \in X^k \times Y^l$, $k, l \in \mathbb{N}$, $n = k + l$ and any $\nu \in \{0, 1\}$. Then $(X \otimes Y, \lambda)$ with λ as defined above is a labelled precubical set as well.

Proof. From theorem 8.3 it follows that $X \otimes Y$ is an event consistent precubical set. Theorem 4.2 gives us that we can define the labelling $\lambda : X \otimes Y \to !\Sigma$ using the function $\lambda^1 : (X \otimes Y)^1 \to \Sigma$. We define

$$\lambda^1\left((x, y)\right) = \begin{cases} \lambda^1_X(x) & \text{if} \quad (x, y) \in X^1 \otimes Y^0 \\ \lambda^1_Y(y) & \text{if} \quad (x, y) \in X^0 \otimes Y^1 \end{cases}$$

for all $z \in (X \otimes Y)^1$, $z = (x, y)$. For all $n \in \mathbb{N}$, $n \geq 2$ and $z \in Z^n$ with $z = (x, y)$, $x \in X^k$, $y \in Y^l$ such that $k + l = n$ we get

$$\lambda^n(z) = \left(\lambda^1 \circ \delta^n_{\nu,A^n_1}(z), ..., \lambda^1 \circ \delta^n_{\nu,A^n_k}(z), \lambda^1 \circ \delta^n_{\nu,A^n_{k+1}}(z), ..., \lambda^1 \circ \delta^n_{\nu,A^n_n}(z)\right)$$

for a any $\nu \in \{0, 1\}$. As a result of theorem 8.4 for all $1 \leq t \leq k$ we have

$$\lambda^1 \circ \delta^n_{\nu,A^n_t}(z) = \lambda^1 \circ \left(\delta^k_{\nu,A^k_t}(x), \delta^l_{\nu,A^l}(y)\right) = \lambda^1_X \circ \delta^k_{\nu,A^k_t}(x)$$

and for all $k + 1 \leq t \leq n$ we have

$$\lambda^1 \circ \delta^n_{\nu,A^n_t}(z) = \lambda^1 \circ \left(\delta^k_{\nu,A^k}(x), \delta^l_{\nu,A^l_{t-k}}(y)\right) = \lambda^1_Y \circ \delta^l_{\nu,A^l_{t-k}}(y)$$

This shows that our original definition for $\lambda : X \otimes Y \to !\Sigma$ defines a labelling function, making $(X \otimes Y, \lambda)$ a labelled precubical set. ☐

Depending on the context we might use $\lambda_{X \otimes Y}$ as the notation for the labelling function on $X \otimes Y$.

Theorem 8.6. *Let $\mathcal{X} = (X, I_X, F_X, \lambda_X)$ and $\mathcal{Y} = (Y, I_Y, F_Y, \lambda_Y)$ be HDA. Let $\lambda_{X \otimes Y} : X \otimes Y \to !\Sigma$ be the labelling function as defined in theorem 8.5. Then $\mathcal{X} \otimes \mathcal{Y} = (X \otimes Y, I_X \times I_Y, F_X \times F_Y, \lambda_{X \otimes Y})$ is a HDA as well.*

Proof. From theorem 8.5 we get that $(X \otimes Y, \lambda_{X \otimes Y})$ is a labelled precubical set. For all $x \in X^k$ and $y \in Y_l$ if $x \in I_X$ and $y \in I_Y$ then $(x, y) \in X^k \times Y^l$ and therefore $(x, y) \in (X \otimes Y)^{k+l}$. Analogously the same is true for all $x \in X^k$ and $y \in Y_l$ with $x \in F_X$ and $y \in F_Y$. This gives us that every element of $I_X \times I_Y$ and $F_X \times F_Y$ is an element of $X \otimes Y$. Therefore $\mathcal{X} \otimes \mathcal{Y}$ is a HDA. $\square$

The tensor product on HDA has the property that there exists something similar to an identity. However due to the way HDA are defined an actual identity is not possible, but we do have something similar:

Theorem 8.7. *Let S be any precubical set with $|S^0| = 1$ and $S^n = \emptyset$ for all $n \in \mathbb{N}$, $n \geq 1$. Since S contains no elements of dimension 0 there exists a unique labelling function $\lambda_{id} : S \to !\Sigma$ which sends the unique element of S^0 to the empty vector $\varepsilon = ()$.*
We define the parallel identity HDA as $\mathcal{X}_{id} = (S, I_{id}, F_{id}, \lambda_{id})$ with $I_{id} = S = F_{id}$. Then for every HDA $\mathcal{X} = (X, I_X, F_X, \lambda_X)$ we have $\mathcal{X} \otimes \mathcal{X}_{id} \cong \mathcal{X} \cong \mathcal{X}_{id} \otimes \mathcal{X}$.

Proof. Let $S^0 = \{s\}$ and let $\mathcal{Z} = \mathcal{X} \otimes \mathcal{X}_{id}$. For all $n \in \mathbb{N}$ we have $Z^n = X^n \times S^0$. Since $|S^0| = 1$ it is then clear that $X \cong Z$. We can define a HDA map $f : \mathcal{X} \to \mathcal{X} \otimes \mathcal{X}_{id}$ as the following: for all $n \in \mathbb{N}$, $x \in X^n$ we have $f^n(x) = (x, s)$. It is clear that f is a precubical map. If $x \in I_X$ since $s \in I_{id}$ we have $(x, s) \in I_X \times I_{id}$. Analogously the same is true for the final cells. From the definition of $\lambda_X \otimes \lambda_{id}$ it also follows that f preserves the labelling function, making it a HDA map. It is clear that $f : \mathcal{X} \to \mathcal{X} \otimes \mathcal{X}_{id}$ is an isomorphism, proving the statement. $\square$

Again this is not an actual identity, though it functions similarly.

8.1.1 Intuitive explanation of the tensor product

To understand how the tensor product works in practice let's look at two examples:

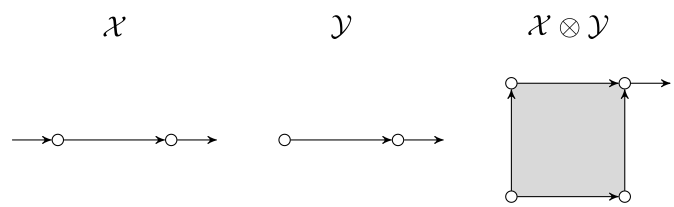


Figure 13: Here the HDA $\mathcal{X}$ has an initial node and a final node, but the HDA $\mathcal{Y}$ only has a final node. Their tensor product $\mathcal{X} \otimes \mathcal{Y}$ will therefore also only have a final node and no initial cells.

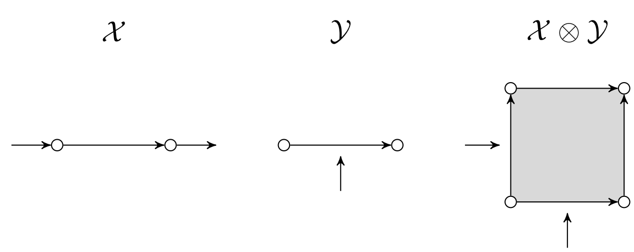


Figure 14: Here the HDA $\mathcal{Y}$ has an edge as initial cell. Because $\mathcal{X}$ has an initial node the tensor product $\mathcal{X} \otimes \mathcal{Y}$ has two initial edges. Note that if $x \in I_X$ with $x \in X^0$ and $y \in I_Y$ with $y \in Y^1$ then we have $(x, y) \in X^0 \times Y^1 = (X \otimes Y)^1$.

8.2 Tensor product of maps

In this subsection we will define the tensor product of precubical maps and HDA maps.

Theorem 8.8. *Let U, V, X and Y be precubical sets and let $f : U \to X$ and $g : V \to Y$ be precubical maps. Then $h : U \otimes V \to X \otimes Y$ defined as*

$$h^n\left((u,v)\right) = \left(f^k(u), g^l(v)\right)$$

for all $k,l,n \in \mathbb{N}$ with $k+l=n$, $u \in U^k$ and $v \in V^l$ is a precubical map.

Proof. Recall definition 2.3. If (u,v) is an element in $U^k \times V^l$ then $\left(f^k(u), g^l(v)\right)$ is an element in $X^k \times Y^l$, making it clear that h^n exists and is well defined for all $n \in \mathbb{N}$, $k,l \in \mathbb{N}$ with $k+l=n$ and $(u,v) \in U^k \times V^l$.
We want to prove that for all $n,k,l \in \mathbb{N}$ with $k+l=n$, $(u,v) \in U^k \times V^l$, $\nu \in \{0,1\}$ and $a \in \mathbb{N}$, $1 \le a \le n$ we have

$$h^{n-1} \circ \delta^n_{\nu,a}\left((u,v)\right) = \delta^n_{\nu,a} \circ h^n\left((u,v)\right)$$

Suppose that $a \le k$. Then we have

$$h^{n-1} \circ \delta^n_{\nu,a}\left((u,v)\right) = h^{n-1}\left(\left(\delta_U\right)^k_{\nu,a}(u), v\right) = \left(f^{k-1} \circ \left(\delta_U\right)^k_{\nu,a}(u), g^l(v)\right)$$

$$\delta^n_{\nu,a} \circ h^n\left((u,v)\right) = \delta^n_{\nu,a}\left(f^k(u), g^l(v)\right) = \left(\left(\delta_U\right)^k_{\nu,a} \circ f^k(u), g^l(v)\right)$$

and because f is a precubical map we get

$$\left(f^{k-1} \circ \left(\delta_U\right)^k_{\nu,a}(u), g^l(v)\right) = \left(\left(\delta_U\right)^k_{\nu,a} \circ f^k(u), g^l(v)\right)$$

which proves the statement for $a \le k$.
Suppose that $a > k$. Then we have

$$h^{n-1} \circ \delta^n_{\nu,a}\left((u,v)\right) = h^{n-1}\left(u, \left(\delta_V\right)^l_{\nu,a}(v)\right) = \left(f^k(u), g^{l-1} \circ \left(\delta_V\right)^l_{\nu,a}(v)\right)$$

$$\delta^n_{\nu,a} \circ h^n\left((u,v)\right) = \delta^n_{\nu,a}\left(f^k(u), g^l(v)\right) = \left(f^k(u), \left(\delta_V\right)^l_{\nu,a} \circ f^l(v)\right)$$

and because g is a precubical map we get

$$\left(f^k(u), g^{l-1} \circ \left(\delta_V\right)^k_{\nu,a}(v)\right) = \left(f^k(u), \left(\delta_V\right)^l_{\nu,a} \circ g^l(v)\right)$$

which proves the statement for $a > k$ which gives us that the statement is true for all $a \in \mathbb{N}$, $1 \le a \le n$. Therefore $h : U \otimes V \to X \otimes Y$ is a precubical map. □

The precubical map $h : U \otimes V \to X \otimes Y$ as defined above is denoted with $h = f \otimes g$.

Theorem 8.9. *Let $\mathcal{X}_U$, $\mathcal{X}_V$, $\mathcal{X}$ and $\mathcal{Y}$ be HDA and let $f : \mathcal{X}_U \to \mathcal{X}$ and $g : \mathcal{X}_V \to \mathcal{Y}$ be HDA maps. Then the precubical map $h : U \otimes V \to X \otimes Y$ as defined in theorem 8.8 constructs a HDA map $h : \mathcal{X}_U \otimes \mathcal{X}_V \to \mathcal{X} \otimes \mathcal{Y}$.*

Proof. Theorem 8.8 gives us that $h : U \otimes V \to X \otimes Y$ is a precubical map with

$$h^n\left((u,v)\right) = \left(f^k(u), g^l(v)\right)$$

for all $k, l, n \in \mathbb{N}$ with $k + l = n$, $u \in U^k$ and $v \in V^l$.
Suppose that $u \in I_U$ and $v \in I_V$. Since f and g are HDA maps this gives us that $f^k(u) \in I_X$ and $g^l(v) \in I_Y$, which means that $h^n\left((u,v)\right) \in I_X \times I_Y = I_{X\otimes Y}$. Analogously the same is true for the final cells. This means that $h = f \otimes g$ preserves initial and final cells. We also have

$$\lambda^n_{U\otimes V}\left((u,v)\right) = \begin{array}{l}\Big(\lambda^1_U \circ \delta^k_{\nu,A^k_1}(u), ..., \lambda^1_U \circ \delta^k_{\nu,A^k_k}(u),\\ \quad \lambda^1_V \circ \delta^l_{\nu,A^l_1}(v), ..., \lambda^1_V \circ \delta^l_{\nu,A^l_l}(v)\Big)\end{array}$$

Let $x = f^k(u)$ and $y = g^l(v)$. Then we have

$$\begin{aligned}\lambda^n_{X\otimes Y}\left((x,y)\right) &= \begin{array}{l}\Big(\lambda^1_X \circ \delta^k_{\nu,A^k_1}(x), ..., \lambda^1_X \circ \delta^k_{\nu,A^k_k}(x),\\ \quad \lambda^1_Y \circ \delta^l_{\nu,A^l_1}(y), ..., \lambda^1_Y \circ \delta^l_{\nu,A^l_l}(y)\Big)\end{array}\\
&= \begin{array}{l}\Big(\lambda^1_X \circ \delta^k_{\nu,A^k_1} \circ f^k(u), ..., \lambda^1_X \circ \delta^k_{\nu,A^k_k} \circ f^k(u),\\ \quad \lambda^1_Y \circ \delta^l_{\nu,A^l_1} \circ g^l(v), ..., \lambda^1_Y \circ \delta^l_{\nu,A^l_l} \circ g^l(v)\Big)\end{array}\\
&= \begin{array}{l}\Big(\lambda^1_X \circ f^1 \circ \delta^k_{\nu,A^k_1}(u), ..., \lambda^1_X \circ f^1 \circ \delta^k_{\nu,A^k_k}(u),\\ \quad \lambda^1_Y \circ g^1 \circ \delta^l_{\nu,A^l_1}(v), ..., \lambda^1_Y \circ g^1 \circ \delta^l_{\nu,A^l_l}(v)\Big)\end{array}\end{aligned}$$

Since f and g are HDA maps we get $\lambda_X \circ f = \lambda_U$ and $\lambda_Y \circ g = \lambda_V$ which gives us $\lambda^n_{U\otimes V}\left((u,v)\right) = \lambda^n_{X\otimes Y}\left(\left(f^k(u), g^l(v)\right)\right)$. This means that $h = f \otimes g$ also preserves the labelling function, therefore making it a HDA map. $\square$

Theorem 8.10. *Let X_1, X_2, Y_1, Y_2, Z_1 and Z_2 be precubical sets and let $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$, $g_1 : Y_1 \to Z_1$ and $g_2 : Y_2 \to Z_2$ be precubical maps. Then we have $(g_1 \otimes g_2) \circ (f_1 \otimes f_2) = (g_1 \circ f_1) \otimes (g_2 \circ f_2)$.*

Proof. Let $(x_1, x_2) \in X^k_1 \times X^l_2$ for certain $k, l \in \mathbb{N}$, $n = k + l$. We have

$$(f_1 \otimes f_2)(x_1, x_2) = \left(f^k_1(x_1), f^l_2(x_2)\right)$$

$$\begin{aligned}(g_1 \otimes g_2) \circ (f_1 \otimes f_2)(x_1, x_2) &= (g_1 \otimes g_2)\left(f^k_1(x_1), f^l_2(x_2)\right)\\
= \left(g^k_1 \circ f^k_1(x_1), g^l_2 \circ f^l_2(x_2)\right) &= \left((g_1 \circ f_1) \otimes (g_2 \circ f_2)\right)(x_1, x_2)\end{aligned}$$

which proves the theorem. $\square$

The above theorem automatically works for HDA and HDA maps as well, since if HDA maps are equal as precubical maps then they are equal as HDA maps as well.

8.3 Tensor product of diagrams

Definition 8.11. *Let J and K be small categories. We define $J \times K$ as the product category of J and K, which means that we have*

$$obj(J \times K) = \{(j,k) \mid j \in obj(J),\ k \in obj(K)\}$$

$$mor(J \times K) = \left\{ (f,g) : (j_1,k_1) \to (j_2,k_2) \,\middle|\, \begin{matrix} f \in mor(J), & f : j_1 \to j_2 \\ g \in mor(K), & g : k_1 \to k_2 \end{matrix} \right\}$$

Theorem 8.12. *The product $J \times K$ as defined in definition 8.11 is a small category.*

Proof. We know that $\mathrm{obj}\,(J \times K)$ has as many elements as $\mathrm{obj}\,(J) \times \mathrm{obj}\,(K)$ and $\mathrm{mor}\,(J \times K)$ has as many elements as $\mathrm{mor}\,(J) \times \mathrm{mor}\,(K)$, making $J \times K$ a small category. □

Theorem 8.13. *Let J and K be small discrete categories. Then $J \times K$ is a small discrete category as well.*

Proof. Let $(j,k) \in J \times K$ be an object. The only morphism with j as source or target is the identity morphism i_j and the only morphism with k as source or target is the identity morphism i_k. Therefore the only morphism with (j,k) as source or target is (i_j, i_k), which is the identity morphism. This means that the category $J \times K$ only has identity morphisms, making it a discrete category. □

Theorem 8.14. *Let J and K be small filtered categories. Then $J \times K$ is a small filtered category as well.*

Proof. Because J and K are filtered and therefore not empty $J \times K$ is not empty as well.
Suppose that (j_1,k_1) and (j_2,k_2) are objects of $J \times K$. Then j_1 and j_2 are objects of J and k_1 and k_2 are objects of K. Therefore there exist objects j_3 in J and k_3 in K and morphisms $f_1 : j_1 \to j_3$, $f_2 : j_2 \to J_3$ in J and morphisms $g_1 : k_1 \to k_3$, $g_2 : k_2 \to k_3$ in K. This gives us the morphisms $(f_1,g_1) : (j_1,k_1) \to (j_3,k_3)$ and $(f_2,g_2) : (j_2,k_2) \to (j_3,k_3)$.
Suppose that (j_1,k_1) and (j_2,k_2) are objects of $J \times K$ and suppose that there are parallel morphisms $(f_1,g_1) : (j_1,k_1) \to (j_2,k_2)$ and $(f_2,g_2) : (j_1,k_1) \to (j_2,k_2)$. Then there exist objects j_3 in J and $k_3 \in K$ and morphisms $f_3 : j_2 \to j_3$ in J and $g_3 : k_2 \to k_3$ in K such that $f_3 f_1 = f_3 f_2$ and $g_3 g_1 = g_3 g_1$. This gives us that there exists a morphism $(f_3,g_3) : (j_2,k_2) \to (j_3,k_3)$ with $(f_3,g_3)\,(f_1,g_1) = (f_3,g_3)\,(f_2,g_2)$.
Therefore $J \times K$ is a filtered category. □

Theorem 8.15. *Let $X : J \to \mathtt{Set}^{\square^{op}}$ and $Y : K \to \mathtt{Set}^{\square^{op}}$ be small diagrams of precubical sets. Then $Z : J \times K \to \mathtt{Set}^{\square^{op}}$ with $Z_{(j,k)} = X_j \otimes Y_k$ for all $j \in J$ and $k \in K$ and $Z_{f \otimes g} = X_f \otimes Y_g$ for all morphisms $f : j_1 \to j_2$ and $g : k_1 \to k_2$ is a small diagram of precubical sets.*

Proof. For every $i \in J \times K$ the precubical set Z_i is well-defined. For every $i_1, i_2 \in J \times K$ with a morphism $h : i_1 \to i_2$ the precubical map $Z_f : Z_{i_1} \to Z_{i_2}$ is also well-defined. For all $j \in J$, $k \in K$ and the identity morphisms $id_j : j \to j$ and $id_k : k \to k$ the precubical maps $X_{id_j} : X_j \to X_j$ and $Y_{id_k} : Y_k \to Y_k$ are well-defined and identity maps. This them automatically makes $Z_{(id_j, id_k)} : X_j \otimes Y_k \to X_j \otimes Y_k$ an identity map as well. Theorem 8.10 gives us that the compositions of precubical maps are well-defined as well and theorem 8.12 gives us that $J \times K$ is a small category. This makes $Z : J \times K \to \mathtt{Set}^{\square^{op}}$ a small diagram. □

Theorem 8.16. *Let $\mathcal{X} : J \to \texttt{HDA}$ and $\mathcal{Y} : K \to \texttt{HDA}$ be small diagrams of HDA. Then $\mathcal{X} \otimes \mathcal{Y} : J \times K \to \texttt{HDA}$ with $(\mathcal{X} \otimes \mathcal{Y})_{(j,k)} = \mathcal{X}_i \otimes \mathcal{Y}_k$ for all $j \in J$ and $k \in K$ and $(\mathcal{X} \otimes \mathcal{Y})_{f \otimes g} = \mathcal{X}_f \otimes \mathcal{Y}_g$ for all morphisms $f : j_1 \to j_2$ and $g : k_1 \to k_2$ is a small diagram of HDA.*

Proof. This follows from the same reasoning as used for theorem 8.15. $\square$

We will now just prove everything for HDA immediately, instead of proving things for precubical sets first.

Theorem 8.17. *Let $\mathcal{X} : J \to \texttt{HDA}$ and $\mathcal{Y} : K \to \texttt{HDA}$ be small diagrams of HDA. Suppose that $(\mathcal{L}, \phi)$ is a co-cone of $\mathcal{X}$ and $(\mathcal{N}, \psi)$ is a co-cone of $\mathcal{Y}$. Then $(\mathcal{L} \otimes \mathcal{N}, \phi \otimes \psi)$ with $(\phi \otimes \psi)_{(j,k)} = \phi_j \otimes \psi_k$ for all $(j, k) \in J \times K$ is a co-cone of $\mathcal{X} \otimes \mathcal{Y}$.*

Proof. From theorem 8.16 it follows that $\mathcal{X} \otimes \mathcal{Y} : J \times K \to \texttt{HDA}$ is a diagram of HDA. From theorem 8.6 it follows that $\mathcal{L} \otimes \mathcal{N}$ is a HDA and from theorem 8.9 it follows that $\phi_j \otimes \psi_k = (\phi_j, \psi_k)$ is a HDA map for all objects (j, k) of $J \times K$.
Suppose that j_1, j_2 are two objects of J with the morphism $f : j_1 \to j_2$ and suppose that k_1, k_2 are two objects of K with the morphism $g : k_1 \to k_2$. Then we have $\phi_{j_2} \circ \mathcal{X}_f = \phi_{j_1}$ and $\psi_{k_2} \circ \mathcal{Y}_g = \psi_{k_1}$ which gives us $(\phi_{j_2}, \psi_{k_2}) \circ (\mathcal{X}_f, \mathcal{Y}_g) = (\phi_{j_1}, \psi_{k_1})$.
Therefore $(\mathcal{L} \otimes \mathcal{N}, \phi \otimes \psi)$ is a co-cone of $\mathcal{X} \otimes \mathcal{Y}$. $\square$

Theorem 8.18. *Let $\mathcal{X} : J \to \texttt{HDA}$ and $\mathcal{Y} : K \to \texttt{HDA}$ be small diagrams of HDA. Suppose that $(\mathcal{L}, \phi)$ is a colimit of $\mathcal{X}$ and $(\mathcal{N}, \psi)$ is a colimit of $\mathcal{Y}$. Then $(\mathcal{L} \otimes \mathcal{N}, \phi \otimes \psi)$ with $(\phi \otimes \psi)_{(j,k)} = \phi_j \otimes \psi_k$ for all $(j, k) \in J \times K$ is a colimit of $\mathcal{X} \otimes \mathcal{Y}$.*

Proof. From theorem 8.17 it follows that $(\mathcal{L} \otimes \mathcal{N}, \phi \otimes \psi)$ is a co-cone.
Theorem 2.25 states that $(\mathcal{L} \otimes \mathcal{N}, \phi \otimes \psi)$ is a colimit if for all $(j_1, k_1), (j_2, k_2) \in J \times K$, $m, l, n \in \mathbb{N}$ with $m + l = n$, $(x_1, y_1) \in X^m_{j_1} \otimes Y^l_{k_1}$ and $(x_2, y_2) \in X^m_{j_2} \otimes Y^l_{k_2}$ we have

$$(x_1, y_1) \sim (x_2, y_2) \iff \left(\phi^m_{j_1}(x_1), \psi^l_{k_1}(y_1)\right) = \left(\phi^m_{j_2}(x_2), \psi^l_{k_2}(y_2)\right)$$

and for all $n, m, l \in \mathbb{N}$ with $m + l = n$, $(x_1, y_1) \in L^m \otimes N^l$ there exists a $(j, k) \in J \otimes K$, $(x_2, y_2) \in X^m_j \otimes Y^l_k$ with $\left(\phi^m_j(x_2), \psi^l_k(y_2)\right) = (x_1, y_1)$.
Suppose that we have $(x, y) \in L^m \times N^l$ for a certain $m, l \in \mathbb{N}$. Then from theorem 2.24 it follows that there exist $j \in J$, $k \in K$, $x_j \in X^m_j$ and $y_k \in Y^l_k$ such that $\phi^m_j(x_j) = x$ and $\psi^l_k(y_k) = y$. This gives us $(\phi_j \otimes \psi_k)^{m+l}(x_j, y_k) = (x, y)$.
Theorem 2.22 and theorem 2.21 gives us

$$x_1 \sim x_2 \iff \phi^m_{j_1}(x_1) = \phi^m_{j_2}(x_2)$$

$$y_1 \sim y_2 \iff \psi^l_{k_1}(y_1) = \psi^l_{k_2}(y_2)$$

$$(x_1, y_1) \sim (x_2, y_2) \implies \left(\phi^m_{j_1}(x_1), \psi^l_{k_1}(y_1)\right) = \left(\phi^m_{j_2}(x_2), \psi^l_{k_2}(y_2)\right)$$

Here $\sim$ means the equivalence relation on X, Y or $X \otimes Y$ depending on the context. This gives us

$$x_1 \sim x_2,\ y_1 \sim y_2 \iff \left(\phi^m_{j_1}(x_1), \psi^l_{k_1}(y_1)\right) = \left(\phi^m_{j_2}(x_2), \psi^l_{k_2}(y_2)\right)$$

$$x_1 \sim x_2,\ y_1 \sim y_2 \Longleftarrow (x_1, y_1) \sim (x_2, y_2)$$

Recall the construction of $\sim$ in definition 2.14. From this it follows that $x_1 \sim x_2 \iff (x_1, y) \sim (x_2, y)$ and $y_1 \sim y_2 \iff (x, y_1) \sim (x, y_2)$. This gives us

$$x_1 \sim x_2,\, y_1 \sim y_2 \iff (x_1, y_1) \sim (x_2, y_2)$$

and therefore

$$(x_1, y_1) \sim (x_2, y_2) \iff \left(\phi^m_{j_1}(x_1), \psi^l_{k_1}(y_1)\right) = \left(\phi^m_{j_2}(x_2), \psi^l_{k_2}(y_2)\right)$$

which shows that $(\mathcal{L} \otimes \mathcal{N}, \phi \otimes \psi)$ satisfies the conditions of theorem 2.25, making it a colimit of the diagram $\mathcal{X} \otimes \mathcal{Y} : J \times K \to \texttt{HDA}$. $\square$

It is obvious that the above theorem applies to diagrams of precubical sets as well. Because of theorem 8.13 and theorem 8.14 the tensor product of two coproducts is a coproduct and the tensor product of two filtered colimits is a filtered colimit as well.

Theorem 8.19. *Let $\mathcal{X} : J \to \texttt{HDA}$ and $\mathcal{Y} : K \to \texttt{HDA}$ be small diagrams of HDA, let $(\mathcal{L}, \phi)$ be a colimit of $\mathcal{X}$ and let $(\mathcal{N}, \psi)$ be a colimit of $\mathcal{Y}$. Suppose that J and K are both discrete or both filtered diagrams. Then we have*

$$L(\mathcal{L} \otimes \mathcal{N}) = \bigcup_{(j,k) \in J \times K} L(\mathcal{X}_j \otimes \mathcal{Y}_k)$$

Proof. From theorem 8.13, theorem 8.14 and theorem 8.18 it follows that $(\mathcal{L} \otimes \mathcal{N}, \phi \otimes \psi)$ is a coproduct or filtered colimit of $\mathcal{X} \otimes \mathcal{Y}$. The statement then follows from theorem 7.10 and theorem 7.11. $\square$

8.4 Tensor product and languages

For this subsection we will mostly just refer to the paper [FJSZ21]. Specifically the operation parallel composition on HDA languages as defined in definition 106 on page 35. Actually defining this parallel composition ourselves is outside the scope of this thesis, so we will just make do with three of its properties:

1. For all HDA $\mathcal{X}$ and $\mathcal{Y}$ we have

$$L(\mathcal{X} \otimes \mathcal{Y}) = L(\mathcal{X}) \parallel L(\mathcal{Y})$$

2. Let $(L_i)_{i \in I}$ and $(M_j)_{j \in J}$ be families of languages. Then we have

$$\left(\bigcup_{i \in I} L_i\right) \parallel \left(\bigcup_{j \in J} M_j\right) = \bigcup_{(i,j) \in I \times J} L_i \parallel M_j$$

3. The language $L_\varepsilon = \{P_\varepsilon\}$, with P_ε being the empty ipomset, is the identity of the parallel composition such that for all languages L we have

$$L \parallel L_\varepsilon = L = L_\varepsilon \parallel L$$

The first property follows from theorem 108 on page 36 and the second and third properties follow from the definition itself. For the actual definition of this operation one must refer to [FJSZ21]. Here we will simply assume it exists and that it is well-defined and works as described above.
Now it's important to note why we need all three of these properties for the parallel composition on the languages to be equivalent to the tensor product on the HDA.
It is obvious why we need the first property. The need for the second property is highlighted with the following remark:

Remark 8.19.1. *Let $\mathcal{X} : J \to \texttt{HDA}$ and $\mathcal{Y} : K \to \texttt{HDA}$ be small diagrams of HDA, let $(\mathcal{L}, \phi)$ be a colimit of $\mathcal{X}$ and let $(\mathcal{N}, \psi)$ be a colimit of $\mathcal{Y}$. Suppose that J and K are both discrete or both filtered diagrams. From theorem 8.19 and the properties described above it follows that we have*

$$L(\mathcal{L}) \parallel L(\mathcal{N}) = L(\mathcal{L} \otimes \mathcal{N}) = \bigcup_{(j,k)\in J\times K} L(\mathcal{X}_j \otimes \mathcal{Y}_k) = \bigcup_{(j,k)\in J\times K} L(\mathcal{X}_j) \parallel L(\mathcal{Y}_k)$$

$$= \left(\bigcup_{j\in J} L(\mathcal{X}_j)\right) \parallel \left(\bigcup_{k\in K} L(\mathcal{Y}_k)\right) = L(\mathcal{L}) \parallel L(\mathcal{N})$$

If the second property wasn't there then there could be a case in which we have $\bigcup_{(j,k)\in J\times K} L(\mathcal{X}_j) \parallel L(\mathcal{Y}_k) \neq \left(\bigcup_{j\in J} L(\mathcal{X}_j)\right) \parallel \left(\bigcup_{k\in K} L(\mathcal{Y}_k)\right)$, which would lead to a contradiction as it would mean that $L(\mathcal{L}) \parallel L(\mathcal{N}) \neq L(\mathcal{L}) \parallel L(\mathcal{N})$.
The third property we need for the following definition:

Definition 8.20. *Let L be a language. Then the parallel Kleene star of L is defined as*

$$L^{(*)} = \bigcup_{i\in\mathbb{N}} L_i$$

with $L_0 = L_\varepsilon$ and $L_i = L_{i-1} \parallel L$ for all $i \in \mathbb{N}$, $i \geq 1$.

Recall theorem 8.7, which defines something similar to the identity for the tensor product on HDA. It is clear that we have $L(\mathcal{X}_{id}) = L_\varepsilon$. This then gives us the following:

Theorem 8.21. *Let $\mathcal{X}$ be a HDA. Then we have*

$$L(\mathcal{X})^{(*)} = \bigcup_{i\in\mathbb{N}} L\left(\bigotimes_{0<k\leq i} \mathcal{X}\right) = L\left(\bigsqcup_{i\in\mathbb{N}}\left(\bigotimes_{0<k\leq i} \mathcal{X}\right)\right)$$

where we define $\bigotimes_{0<k\leq i} \mathcal{X} = \left(\bigotimes_{0<k\leq i-1} \mathcal{X}\right) \otimes \mathcal{X}$ for all $i \in \mathbb{N}$, $i \geq 1$ and $\bigotimes_{0<k\leq 0} \mathcal{X} = \mathcal{X}_{id}$.

Proof. The HDA $\mathcal{X}_{id}$ is defined in theorem 8.7. From theorem 8.6 it then follows that $\bigotimes_{0<k\leq i} \mathcal{X}$ is a HDA for all $i \geq 0$. From theorem 4.18 it follows that $\bigsqcup_{i\in\mathbb{N}} \left(\bigotimes_{0<k\leq i} \mathcal{X}\right)$ is a HDA as well. The rightmost equality then follows from theorem 7.10.
Let $L = L(\mathcal{X})$ and define the languages L_i as in definition 8.20. Suppose that for all $j \leq i-1$ we have $L_j = L\left(\bigotimes_{0<k\leq j} \mathcal{X}\right)$. Then we have

$$L_i = L_{i-1} \parallel L = L\left(\bigotimes_{0<k\leq i-1} \mathcal{X}\right) \parallel L(\mathcal{X}) = L\left(\bigotimes_{0<k\leq i} \mathcal{X}\right)$$

which shows that for all $i \in \mathbb{N}$ we have $L_i = L\left(\bigotimes_{0<k\leq i} \mathcal{X}\right)$. This therefore proves the leftmost equality to be true as well. $\square$

This gives us a HDA equivalent to the parallel Kleene star on languages. However for this we used the coproduct, while we might want to use the filtered colimit instead. The question of how to do this we leave open for now.

9 Conclusion

For the precubical sets, event consistent precubical sets, higher-dimensional automata and the languages of higher-dimensional automata we were able to contribute some structural theorems mainly regarding colimits. The category of HDA is finitely accessible. Every HDA can be canonically expressed as the filtered colimit of a diagram of finite HDA, and its language can be expressed as the union of the languages of the HDA in this diagram.

We discussed the relation between colimits and the coproduct and tensor product. We did not cover serial composition of ipomset languages as defined in [FJSZ22]. Further research could be done on the relation between the serial composition (and serial Kleene star) and colimits. Some other research could also be done into limits of HDA and their languages, and whether the category of HDA is complete or not.

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A Face map theorems

This section contains some miscellaneous theorems specific to the face maps. They were omitted from the main sections for having proofs that are too long and results that are uninteresting. Recall definition 2.7 and definition 2.10 for notation.

Theorem A.1. *Let X be an event consistent precubical set. For all $n \in \mathbb{N}$, $x \in X^n$, $\nu \in \{0,1\}$ and for all n-dimensional vectors A and B with elements $a_i, b_i \in \mathbb{N}_{\geq 1}$ such that for all $1 \leq i < j \leq n$ we have $1 \leq a_i < a_j \leq n$ and $1 \leq b_i < b_j \leq n$ the following statement is true:*

$$\delta^n_{\nu,A}(x) = \delta^n_{\nu,B}(x) \iff A = B$$

Proof. This follows from lemma 22 from [FJSZ21]. □

Theorem A.2. *Let X be a precubical set, $n \in \mathbb{N}_{\geq 3}$, $\nu, \mu \in \{0,1\}$, $B \subseteq A^n_1$, $|B| = m \leq n-1$ and $1 \leq a \leq n$ such that $a < min(B)$. Then we have*

$$\delta^{n-m}_{\nu,a} \circ \delta^n_{\mu,B}(x) = \delta^{n-1}_{\nu,B'} \circ \delta^n_{\nu,a}(x)$$

where B' is the m-dimensional vector with $b'_i = b_i - 1$ for all $1 \leq i \leq m$.

Proof. We take $B = \{b_1 \to ... \to b_m\}$ and $B^{-1} = \{b_1 - 1 \to ... \to b_m - 1\}$ which gives us

$$\delta^{n-m}_{\nu,a} \circ \delta^n_{\mu,B}(x) = \delta^{n-m}_{\nu,a} \circ \delta^{n-m+1}_{\mu,b_1} \circ ... \circ \delta^n_{\mu,b_m}(x)$$

Since $a < \min(B)$ we have $a < b_t$ for all $t \in \mathbb{N}$, $1 \leq t \leq m$. This gives us

$$\delta^{n-m}_{\nu,a} \circ \delta^{n-m+1}_{\mu,b_1} \circ ... \circ \delta^n_{\mu,b_m}(x) = \delta^{n-m}_{\mu,b_1-1} \circ \delta^{n-m+1}_{\nu,a} \circ \delta^{n-m+2}_{\mu,b_2} \circ ... \circ \delta^n_{\mu,b_m}(x) = ...$$

$$= \delta^{n-m}_{\mu,b_1-1} \circ ... \circ \delta^{n-1}_{\mu,b_m-1} \circ \delta^n_{\nu,a}(x) = \delta^{n-1}_{\mu,B^{-1}} \circ \delta^n_{\nu,a}(x)$$

which proves the statement. □

Theorem A.3. *Let X be a precubical set, $n \in \mathbb{N}_{\geq 3}$, $\nu, \mu \in \{0,1\}$, $A \subseteq A^n_n$, $|A| = m \leq n-1$ and $1 \leq b \leq n$ such that $b > max(A)$. Then we have*

$$\delta^{n-1}_{\nu,A} \circ \delta^n_{\mu,b}(x) = \delta^{n-m}_{\mu,b-m} \circ \delta^n_{\nu,A}(x)$$

Proof. We take $A = \{a_1 \to ... \to a_m\}$. Therefore we have $b < a_t$ for all $t \in \mathbb{N}$, $1 \leq t \leq m$. Since A is increasing we also have $a_t < a_{t+1}$ for all $t \in \mathbb{N}$, $1 \leq t \leq m-1$. Therefore we have $a_{m-1} < b-1$, $a_{m-2} < b-2$... or $a_{m-t} < b-t$ for all $1 \leq t \leq m-1$. This gives us

$$\delta^{n-1}_{\nu,A} \circ \delta^n_{\mu,b}(x) = \delta^{n-m}_{\nu,a_1} \circ ... \circ \delta^{n-1}_{\nu,a_m} \circ \delta^n_{\mu,b}(x) = \delta^{n-m}_{\nu,a_1} \circ ... \circ \delta^{n-2}_{\nu,a_{m-1}} \circ \delta^{n-1}_{\mu,b-1} \circ \delta^n_{\nu,a_m}(x)$$

$$= \delta^{n-m}_{\mu,b-m} \circ \delta^{n-m+1}_{\nu,a_1} \circ ... \circ \delta^n_{\nu,a_m}(x) = \delta^{n-m}_{\mu,b-m} \circ \delta^n_{\nu,A}(x)$$

which proves the statement. □

Theorem A.4. *Let X be a precubical set, $n \in \mathbb{N}_0$, $\nu, \mu, \tau \in \{0,1\}$, $A, B \subseteq A^n_n$, $|A| = l$, $|B| = m \leq n-2$, $m + l \leq n-1$ and $1 \leq c \leq n$ such that $max(A) < c < min(B)$. Then we have*

$$\delta^{n-m-1}_{\nu,A} \circ \delta^{n-1}_{\mu,B} \circ \delta^n_{\tau,c}(x) = \delta^{n-m-1}_{\nu,A} \circ \delta^{n-m}_{\tau,c} \circ \delta^n_{\mu,B^+}(x) = \delta^{n-m-l}_{\tau,c-l} \circ \delta^{n-m}_{\nu,A} \circ \delta^n_{\mu,B^+}(x)$$

where B^+ is the m-dimensional vector with $b^+_i = b_i + 1$ for all $1 \leq i \leq m$.

Proof. This follows from the inversion of theorem A.2 and theorem A.3. □

Theorem A.5. *Let X be a precubical set, $n \in \mathbb{N}_0$, $\nu, \mu, \tau, \sigma \in \{0,1\}$, $A, B \subseteq A_n^n$, $|A| = l$, $|B| = m$, $1 \le c \le n$ such that $max(A) < c < min(B)$. Then we have*

$$\delta_{\nu,A}^{n-m-2} \circ \delta_{\sigma,c}^{n-m-1} \circ \delta_{\mu,B}^{n-1} \circ \delta_{\tau,c}^{n}(x) = \delta_{\nu,A}^{n-m-2} \circ \delta_{\tau,c}^{n-m-1} \circ \delta_{\tau,c}^{n-m} \circ \delta_{\mu,B^+}^{n}(x)$$

$$= \delta_{\nu,A}^{n-m-2} \circ \delta_{\tau,c}^{n-m-1} \circ \delta_{\tau,c+1}^{n-m} \circ \delta_{\mu,B^+}^{n}(x) = \delta_{\tau,c-l}^{n-m-l-1} \circ \delta_{\nu,A}^{n-m-1} \circ \delta_{\tau,c+1}^{n-m} \circ \delta_{\mu,B^+}^{n}(x)$$

where B^+ is the m-dimensional vector with $b_i^+ = b_i + 1$ for all $1 \le i \le m$.

Proof. The first equality follows from the inversion of theorem A.2. The second equality follows from the definition of the face maps and the third equality follows from theorem A.3. □

Theorem A.6. *Let X be a precubical set. Suppose that we have $n, t, a \in \mathbb{N}$, $n \ge 2$, $1 \le t \le n-1$, $1 \le a \le n$, $\nu \in \{0,1\}$ and $x \in X^n$. Then we have*

$$\delta_{\nu,A_t^{n-1}}^{n-1} \circ \delta_{\nu,a}^{n}(x) = \begin{cases} \delta_{\nu,A_t^n}^{n}(x) & \text{for all } a > t \\ \delta_{\nu,A_{t+1}^n}^{n}(x) & \text{for all } a \le t \end{cases}$$

Proof. If $a = t$ the result follows from theorem A.4. If $a < t$ or $a > t$ the result follows from theorem A.5. □

Theorem A.7. *Let X be an event consistent precubical set and let $\equiv_X$ be an equivalence relation as defined in definition 3.1. Suppose that we have $n \in \mathbb{N}_{\ge 2}$, $x \in X^n$. Then for all $\nu \in \{0,1\}$ and $s, t \in \mathbb{N}_{\ge 1}$ with $s, t \le n$ we have*

$$\delta_{\nu,A_s^n}^{n}(x) \equiv_X \delta_{\nu,A_t^n}^{n}(x) \iff s = t$$

Proof. The statement is trivial for $n = 2$. Suppose that $n = 3$. Then all of the possible subsequences of $\{1 \to 2 \to 3\}$ of size 2 are $\{1 \to 2\}$, $\{1 \to 3\}$ and $\{2 \to 3\}$. Then for all $x \in X^3$ and a certain $\nu \in \{0,1\}$ we get

$$\delta_{\nu,1}^2 \circ \delta_{\nu,2}^3(x) = \delta_{\nu,1}^2 \circ \delta_{\nu,1}^3(x)$$
$$\delta_{\nu,1}^2 \circ \delta_{\nu,3}^3(x) = \delta_{\nu,2}^2 \circ \delta_{\nu,1}^3(x)$$
$$\delta_{\nu,2}^2 \circ \delta_{\nu,3}^3(x) = \delta_{\nu,2}^2 \circ \delta_{\nu,2}^3(x)$$

By definition we have $\delta_{\nu,1}^2 \circ \delta_{\nu,1}^3(x) \not\equiv_X \delta_{\nu,2}^2 \circ \delta_{\nu,1}^3(x)$, $\delta_{\nu,1}^2 \circ \delta_{\nu,2}^3(x) \not\equiv_X \delta_{\nu,2}^2 \circ \delta_{\nu,2}^3(x)$ and $\delta_{\nu,1}^2 \circ \delta_{\nu,3}^3(x) \not\equiv_X \delta_{\nu,2}^2 \circ \delta_{\nu,3}^3(x)$. Therefore the statement is true for $n = 3$.

Suppose that the statement is true for all $n - r$, $n \in \mathbb{N}_{\ge 2}$, $r \in \mathbb{N}$ with $1 \le r \le n-2$. Suppose that we have $s, t \in \mathbb{N}_{\ge 1}$, $s, t \le n$ and suppose that $\delta_{\nu,A_s^n}^{n}(x) \equiv_X \delta_{\nu,A_t^n}^{n}(x)$ for a certain $x \in X^n$.

Suppose that $s \ne t$. Without loss of generality we assume that $s < t$. Let $a \in \mathbb{N}$ be a number such that $1 \le a \le n$ and $a \ne s$ and $a \ne t$. Such a number exists for all $n \ge 3$. Using theorem A.6, if $a < s < t$ we get

$$\delta_{\nu,A_s^n}^{n}(x) = \delta_{\nu,A_{s-1}^{n-1}}^{n-1} \circ \delta_{\nu,a}^{n}(x)$$

$$\delta_{\nu,A_t^n}^{n}(x) = \delta_{\nu,A_{t-1}^{n-1}}^{n-1} \circ \delta_{\nu,a}^{n}(x)$$

if $s < a < t$ we get

$$\delta^n_{\nu,A^n_s}(x) = \delta^{n-1}_{\nu,A^{n-1}_s} \circ \delta^n_{\nu,a}(x)$$

$$\delta^n_{\nu,A^n_t}(x) = \delta^{n-1}_{\nu,A^{n-1}_{t-1}} \circ \delta^n_{\nu,a}(x)$$

and if $s < t < a$ we get

$$\delta^n_{\nu,A^n_s}(x) = \delta^{n-1}_{\nu,A^{n-1}_s} \circ \delta^n_{\nu,a}(x)$$

$$\delta^n_{\nu,A^n_t}(x) = \delta^{n-1}_{\nu,A^{n-1}_t} \circ \delta^n_{\nu,a}(x)$$

For all of these cases since the statement applies to all elements of X^{n-1} and since $\delta^n_{\nu,a}(x) \in X^{n-1}$. This gives us a contradiction for the cases $a < s < t$ and $s < t < a$. In the case of $s < a < t$ it is only possible if $s = t - 1$, which is in contradiction with a, s and t being integers and $s < a < t$. This means that $\delta^n_{\nu,A^n_s}(x) \equiv_X \delta^n_{\nu,A^n_t}(x)$ if and only if $s = t$. Therefore the statement is true for the case n if it is true for all $n - r$, $n \in \mathbb{N}_{\geq 2}$, $r \in \mathbb{N}$ with $1 \leq r \leq n - 2$. Since the statement is true for $n = 2$ and $n = 3$, it is therefore true for all $n \in \mathbb{N}$. $\square$

Theorem A.8. *Let X be a precubical set, let Σ be a set and let $f : X \to \Sigma$ be a function such that for all $x \in X^2$, $a \in \{1, 2\}$ we have $f \circ \delta^2_{0,a}(x) = f \circ \delta^2_{1,a}(x)$. Then for all $n \in \mathbb{N}$, $x \in X^n$ and $t \in \mathbb{N}_{\geq 1}$ such that $1 \leq n$ we have*

$$f \circ \delta^n_{0,A^n_t}(x) = f \circ \delta^n_{1,A^n_t}(x)$$

Proof. The statement is trivial for $n = 2$. Suppose that $n = 3$. Then all of the possible subsequences of $\{1 \to 2 \to 3\}$ of size 2 are $\{1 \to 2\}$, $\{1 \to 3\}$ and $\{2 \to 3\}$. Then for all $x \in X^3$ we get

$$f \circ \delta^2_{0,1} \circ \delta^3_{0,2}(x) = f \circ \delta^2_{1,1} \circ \delta^3_{0,2}(x) = f \circ \delta^2_{0,1} \circ \delta^3_{1,1}(x)$$

$$= f \circ \delta^2_{1,1} \circ \delta^3_{1,1}(x) = f \circ \delta^2_{1,1} \circ \delta^3_{1,2}(x)$$

$$f \circ \delta^2_{0,1} \circ \delta^3_{0,3}(x) = f \circ \delta^2_{1,1} \circ \delta^3_{0,3}(x) = f \circ \delta^2_{0,2} \circ \delta^3_{1,1}(x)$$

$$= f \circ \delta^2_{1,2} \circ \delta^3_{1,1}(x) = f \circ \delta^2_{1,1} \circ \delta^3_{1,3}(x)$$

$$f \circ \delta^2_{0,2} \circ \delta^3_{0,3}(x) = f \circ \delta^2_{1,2} \circ \delta^3_{0,3}(x) = f \circ \delta^2_{0,2} \circ \delta^3_{1,2}(x)$$

$$= f \circ \delta^2_{1,2} \circ \delta^3_{1,2}(x) = f \circ \delta^2_{1,2} \circ \delta^3_{1,3}(x)$$

Therefore the statement is true for $n = 3$. Suppose that the statement is true for all $n - r$, $n \in \mathbb{N}_{\geq 2}$, $r \in \mathbb{N}$ with $1 \leq r \leq n - 2$. Suppose that we have $x \in X^n$ and $t \in \mathbb{N}$ with $1 \leq t \leq n$.
Suppose that $t = n$. Then we have

$$f \circ \delta^n_{0,A^n_t}(x) = f \circ \delta^n_{0,A^n_n}(x) = f \circ \delta^2_{0,1} \circ \delta^3_{0,2} \circ ... \circ \delta^n_{0,n-1}(x)$$

$$= f \circ \delta^2_{1,1} \circ \delta^3_{0,2} \circ ... \circ \delta^n_{0,n-1}(x)$$

since $\delta^3_{0,2} \circ ... \circ \delta^n_{0,n-1}(x) \in X^2$.

$$= f \circ \delta^2_{0,1} \circ \delta^3_{1,1} \circ ... \circ \delta^n_{0,n-1}(x) = f \circ \delta^2_{0,1} \circ ... \circ \delta^{n-3}_{0,n-3} \circ \delta^{n-2}_{1,1} \circ \delta^n_{0,n-1}(x)$$

$$= f \circ \delta^2_{0,1} \circ ... \circ \delta^{n-1}_{0,n-2} \circ \delta^n_{1,1}(x) = f \circ \delta^{n-1}_{0,A^{n-1}_{n-1}} \circ \delta^n_{1,1}(x) = f \circ \delta^{n-1}_{1,A^{n-1}_{n-1}} \circ \delta^n_{1,1}(x)$$

$$= f \circ \delta^n_{1,A^n_n}(x) = f \circ \delta^n_{1,A^n_t}(x)$$

as a result of theorem A.6. Suppose that $t > 1$. Then we have

$$f \circ \delta^n_{0,A^n_t}(x) = f \circ \delta^2_{0,1} \circ \ldots \circ \delta^t_{0,t-1} \circ \delta^{t+1}_{0,t+1} \circ \ldots \circ \delta^n_{0,n}(x)$$

$$= f \circ \delta^2_{1,1} \circ \ldots \circ \delta^t_{0,t-1} \circ \delta^{t+1}_{0,t+1} \circ \ldots \circ \delta^n_{0,n}(x) = f \circ \delta^2_{0,1} \circ \ldots \circ \delta^{t-1}_{0,t-2} \circ \delta^t_{1,1} \circ \delta^{t+1}_{0,t+1} \circ \ldots \circ \delta^n_{0,n}(x)$$

$$= f \circ \delta^2_{0,1} \circ \ldots \circ \delta^{t-1}_{0,t-2} \circ \delta^t_{0,t} \circ \delta^{t+1}_{1,1} \circ \ldots \circ \delta^n_{0,n}(x)$$

$$= f \circ \delta^2_{0,1} \circ \ldots \circ \delta^{t-1}_{0,t-2} \circ \delta^t_{0,t} \circ \ldots \circ \delta^{n-1}_{0,n-1} \circ \delta^n_{1,1}(x) = f \circ \delta^{n-1}_{0,A^{n-1}_{t-1}} \circ \delta^n_{1,1}(x)$$

$$= f \circ \delta^{n-1}_{1,A^{n-1}_{t-1}} \circ \delta^n_{1,1}(x) = f \circ \delta^n_{1,A^n_t}(x)$$

as a result of theorem A.6. Suppose that $t = 1$. Then we have

$$f \circ \delta^n_{0,A^n_1}(x) = f \circ \delta^2_{0,2} \circ \ldots \circ \delta^n_{0,n}(x) = f \circ \delta^2_{1,2} \circ \ldots \circ \delta^n_{0,n}(x)$$

$$= f \circ \delta^2_{0,2} \circ \ldots \circ \delta^{n-1}_{0,n-1} \circ \delta^n_{1,2}(x) = f \circ \delta^{n-1}_{0,A^{n-1}_1} \circ \delta^n_{1,2}(x) = f \circ \delta^{n-1}_{1,A^{n-1}_1} \circ \delta^n_{1,2}(x)$$

$$= f \circ \delta^n_{1,A^n_1}(x)$$

Therefore the statement is true for all $n \in \mathbb{N}$. □

B Interval ipomsets

In this appendix section we will define interval ipomsets. These are relatively important to the languages of higher-dimensional automata, since as we will see at the end of this section for any track its labelling generates an interval ipomset. We don't really use the interval ipomsets in this thesis, but they are used extensively in [FJSZ21] from which we use many results. They are especially necessary for defining the parallel composition of languages. Interval ipomsets are often referred to as iipomsets.

Definition B.1. *An ipomset $(P, \prec, \dashrightarrow, S_P, T_P)$ is called an interval ipomset if for all unique elements $p_1, p_2, q_1, q_2 \in P$ with $p_1 \prec p_2$ and $q_1 \prec q_2$ there exist $a, b \in \{1, 2\}$ for which we either have $p_a \prec q_b$ or $q_b \prec p_a$.*

Theorem B.2. *Let $\mathcal{P} = (P, \prec_P, \dashrightarrow_P, S_P, T_P)$ and $\mathcal{Q} = (Q, \prec_Q, \dashrightarrow_Q, S_Q, T_Q)$ be interval ipomsets such that $\mathcal{P} * \mathcal{Q}$ exists. Then $\mathcal{P} * \mathcal{Q}$ is an interval ipomset as well.*

Proof. Suppose that we have $P = T_P$ or $Q = S_Q$. Then we have

$$\mathcal{P} * \mathcal{Q} = (Q, \prec_Q, \dashrightarrow_Q, \lambda_Q, S_P, T_Q)$$

or

$$\mathcal{P} * \mathcal{Q} = (P, \prec_P, \dashrightarrow_P, \lambda_P, S_P, T_Q)$$

The first ipomset is the same as $\mathcal{P}$ with a different target set. The second ipomset is the same as $\mathcal{Q}$ with a different source set. Since P and Q are both interval ipomsets and since the source and

target sets have no effect on whether or not an ipomset is an interval ipomset in both cases we have that $\mathcal{P} * \mathcal{Q}$ is an interval ipomset as well.
Suppose that $P \neq T_P$ and $Q \neq S_Q$. Then we have

$$\mathcal{R} = \mathcal{P} * \mathcal{Q} = (R, \prec, \dashrightarrow, S, T)$$

Recall definition 5.17. Let $x_1, x_2, y_1, y_2 \in R$ with $x_1 \prec x_2$ and $y_1 \prec y_2$ such that for all $a, b \in \{1, 2\}$ we have $x_a \not\prec y_b$ and $y_b \not\prec x_a$. Let $f : P \to R$, $g : S_Q \to R$ and $h : Q \backslash S_Q \to R$ be the maps as defined in definition 5.17.
For each pair $z_1, z_2 \in R$ with $z_1 \prec z_2$ there are four cases:

$$\begin{array}{ccccc} z_1, z_2 \in \operatorname{im}(f) & z_1 \prec z_2 & \iff & f^{-1}(z_1) \prec_P f^{-1}(z_2) \\ z_1 \in \operatorname{im}(f),\ z_1 \notin \operatorname{im}(g),\ z_2 \in \operatorname{im}(h) & z_1 \prec z_2 & \iff & \text{always} \\ z_1 \in \operatorname{im}(g),\ z_2 \in \operatorname{im}(h) & z_1 \prec z_2 & \iff & g^{-1}(z_1) \prec_Q h^{-1}(z_2) \\ z_1, z_2 \in \operatorname{im}(h) & z_1 \prec z_2 & \iff & h^{-1}(z_1) \prec_Q h^{-1}(z_2) \end{array}$$

Suppose that $x_1, x_2, y_1, y_2 \in \operatorname{im}(f)$. Then we have $f^{-1}(x_1) \prec_P f^{-1}(x_2)$ and $f^{-1}(y_1) \prec_P f^{-1}(y_2)$ and since $\mathcal{P}$ is an interval ipomset there must be a $a, b \in \{1, 2\}$ such that $f^{-1}(x_a) \prec_P f^{-1}(y_b)$ or $f^{-1}(y_b) \prec_P f^{-1}(x_a)$, which results in a contradiction.
Suppose that $x_1, x_2 \in \operatorname{im}(f)$ and $y_1 \in \operatorname{im}(f)$, $y_1 \notin \operatorname{im}(g)$, $y_2 \in \operatorname{im}(h)$. Since we cannot have $x_1 \in \operatorname{im}(g)$ and $x_2 \in \operatorname{im}(g)$ (since S_Q contains only $\prec_Q$-minimal elements) there exists a $a \in \{1, 2\}$ such that $x_a \notin \operatorname{im}(g)$ and therefore $x_a \prec y_2$, which results in a contradiction.
Suppose that $x_1, x_2 \in \operatorname{im}(f)$ and $y_1, y_2 \in \operatorname{im}(h)$. As explained above there must exist a $a \in \{1, 2\}$ such that $x_a \notin \operatorname{im}(g)$ and therefore $x_a \prec y_1 \prec y_2$, which results in a contradiction.
Suppose that $x_1, y_1 \in \operatorname{im}(f)$, $y_1 \notin \operatorname{im}(g)$, $x_2, y_2 \in \operatorname{im}(h)$. Then we automatically have $x_1 \prec y_2$ and $y_1 \prec x_2$ as well, which results in a contradiction.
Suppose that $x_1 \in \operatorname{im}(f)$, $y_1 \notin \operatorname{im}(g)$, $x_2 \in \operatorname{im}(h)$ and $y_1, y_2 \in \operatorname{im}(h)$. Then we automatically have $x_1 \prec y_2, y_2$, which results in a contradiction.
Suppose that $x_1, x_2, y_1, y_2 \in \operatorname{im}(h)$. Then we have $h^{-1}(x_1) \prec_Q h^{-1}(x_2)$ and $h^{-1}(y_1) \prec_Q h^{-1}(y_2)$ and since $\mathcal{Q}$ is an interval ipomset there must be a $a, b \in \{1, 2\}$ such that $h^{-1}(x_a) \prec_Q h^{-1}(y_b)$ or $h^{-1}(y_b) \prec_Q h^{-1}(x_a)$, which results in a contradiction.
Analogously the cases where x_1, x_2 and y_1, y_2 are switched all result in contradictions as well.
Therefore the statement is false, which makes $\mathcal{P} * \mathcal{Q}$ an interval ipomset. □

Theorem B.3. *Let (X, λ) be a labelled precubical set and let ρ be a track in X. Then $\ell(\rho)$ is an interval ipomset.*

Proof. Suppose that $\rho = (x_1)$ is a track of size 1. Then the $\prec$-relation on $\ell(\rho)$ must be empty making $\ell(\rho)$ an interval ipomset. Similarly if $\rho = (x_1, x_2)$ is a track of size 2 then the $\prec$-relation on $\ell(\rho)$ must be empty as well making it an interval ipomset.
Let $\rho = (x_1, x_2, ..., x_m)$ be a track with $m \geq 2$. Then we have

$$\ell(\rho) = \ell(x_1, x_2) * \ell(x_2, x_3) * ... * \ell(x_{m-1}, x_m)$$

The fact that the labels of basic tracks are always interval ipomset combined with theorem B.2 gives us that $\ell(\rho)$ is an interval ipomset. □